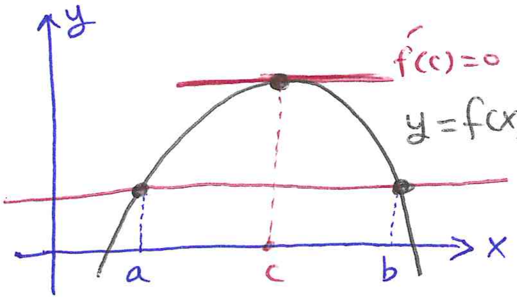
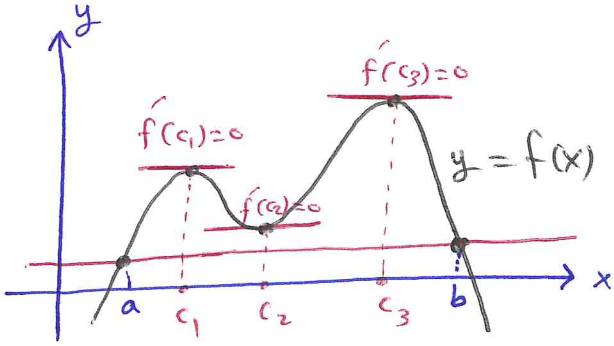
4.2 / 4.3 The Mean Value Theorem

Th3 (Rolle's Theorem)

Suppose $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$





Proof: Let f be a continuous function on $[a, b]$.

By Th1, f has max and min values on $[a, b]$.

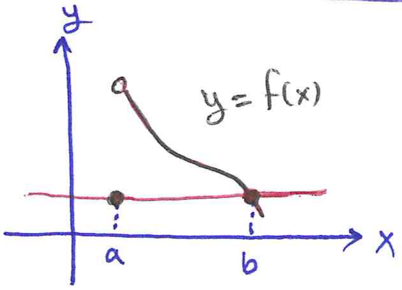
These values can occur only at

1) interior point $c \in (a, b)$ where $f'(c) = 0$
2) interior point $c \in (a, b)$ where $f'(c)$ DNE ✗ since f is differentiable on (a, b)
3) the endpoints a and b.

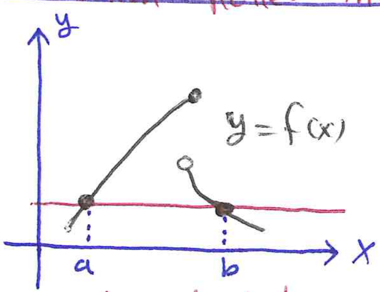
→ If the case is 1), then we are done.

→ If the max and the min occur at the endpoints, then f must be a constant function because we are given $f(a) = f(b)$. Thus, $f(x) = C$ and so $f'(x) = 0$. Hence, c can be taken any value in (a, b).

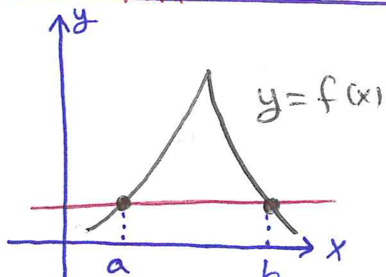
"When Rolle's Th. does not hold"



Dis. cont. at endpoints



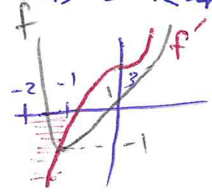
dis. cont. at interior



Continuous on $[a, b]$ but not diff. at interior point.

Example: show that the function $f(x) = x^4 + 3x + 1$ has exactly one real solution on $[-2,-1]$

- Note that f is continuous and
$f(-2) = (-2)^4 + 3(-2) + 1 = 16 - 6 + 1 = 11 > 0$
$f(-1) = (-1)^4 + 3(-1) + 1 = 1 - 3 + 1 = -1 < 0$
} By the IVT, f crosses x-axis somewhere in $(-2,-1)$



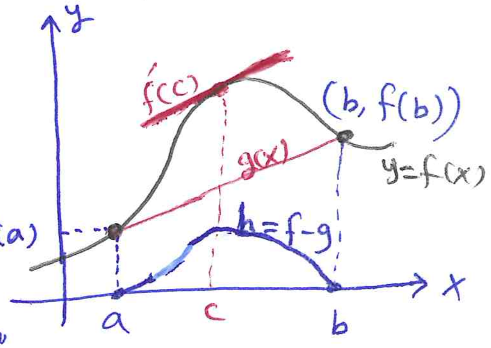
- $f'(x) = 4x^3 + 3$ is never zero on $(-2,-1)$ because it is always negative. There is no $c \in (-2,-1)$ where $f'(c) = 0$. Therefore, f has no more than one zero.

Th (The Mean Value Theorem)

Suppose $y = f(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

Then, there is at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Proof: $g(x) = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$

Let $h(x) = f(x) - g(x)$ "vertical distance"

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$h(x)$ satisfies Rolle's Th: since
1) $h(x)$ is continous on $[a,b]$
2) differentiable on (a,b)
3) $h(a) = h(b) = 0$

$\Rightarrow$ there is at least one point $c \in (a,b)$ such that $h'(c) = 0$

$$\Leftrightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Corollary 1 If $f'(x)=0$ for every $x \in (a,b)$, then $f(x)=c$ for all $x \in (a,b)$, where c is constant.

Proof: we need to show that f is constant value on (a,b).

Let x_1 and x_2 be two points in (a,b) with $x_1 < x_2$.

we need to show that $f(x_1)=f(x_2)$.

Note that f satisfies the MVT on $[x_1, x_2]$. Hence, there is at least one point $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2)-f(x_1)}{x_2 - x_1} = 0$

$\Leftrightarrow f(x_2) = f(x_1)$.

Corollary 2 If $f'(x) = g'(x)$ for every $x \in (a,b)$, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a,b)$. That is $f - g = C$.

Proof Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$

Thus, $h(x) = C$ on (a,b) by Corollary 1. That is $f - g = C$.

Example: Find the function $f(x)$ whose derivative is $\sin x$ and passes through the point $(0,2)$.

$f(x) = -\cos x + C$

$f(0) = -\cos 0 + C = 2 \Leftrightarrow -1 + C = 2 \Leftrightarrow \boxed{C=3}$

$f(x) = -\cos x + 3$

Find the body's position if the body's velocity is $v = 32t - 2$ and the body passes through $(\frac{1}{2}, 4)$

$s(t) = 16t^2 - 2t + C$

$s(\frac{1}{2}) = 16(\frac{1}{4}) - 2(\frac{1}{2}) + C = 4 \Leftrightarrow 4 - 1 + C = 4 \Leftrightarrow \boxed{C=1}$

$\boxed{s(t) = 16t^2 - 2t + 1}$