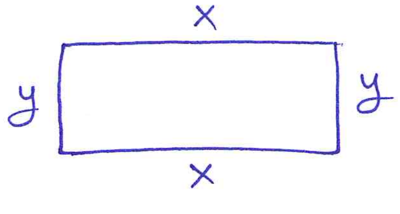
4.5 Applied Optimization

Example: What is the smallest perimeter possible for a rectangle whose area is $100\ cm^2$, and what are its dimensions?

$P = 2x + 2y \quad , \quad A = xy = 100$

$\boxed{y = \frac{100}{x}}$



$P = 2x + \frac{200}{x}$

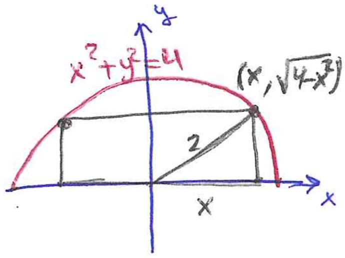
$\frac{dP}{dx} = 2 - \frac{200}{x^2} = 0 \iff x = \pm 10 \implies x = 10\ cm$ critical point

$P'' = \frac{400}{x^3}$ which is always positive (concave up) for all $x > 0$. Thus, at $x = 10$ we have min (actually abs. min).

$\implies y = \frac{100}{10} = 10\ cm$, so the perimeter becomes

$P = 2x + 2y = 2(10) + 2(10) = 20 + 20 = 40\ cm.$

Example: What is the largest area and dimensions that a rectangle can be inscribed in a semicircle of radius 2.

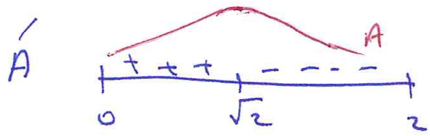


$A = 2x\sqrt{4-x^2} \quad , \quad 0 \le x \le 2$

$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$

$= \frac{8 - 4x^2}{\sqrt{4-x^2}} = 0 \iff x = \pm\sqrt{2}$

$\boxed{\implies x = \sqrt{2} \text{ critical point}}$



$A(0) = A(2) = 0$

$A(\sqrt{2}) = 2\sqrt{2}\sqrt{2} = 4$

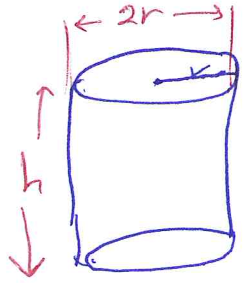
Max area is at $x = \sqrt{2}$

$\implies$ length is $2x = 2\sqrt{2}$

height is $\sqrt{4-x^2} = \sqrt{2}$

Example: what dimensions will you use for least material can be used to design a one liter can of cylinder shape.

- Let r be the radius and h be the height.



$$V = r^2 \pi h = 1000 \text{ cm}^3 \iff \boxed{h = \frac{1000}{\pi r^2}}$$

- least Materials:

surface area: $A = 2r^2\pi + 2r\pi h$

$$A = 2r^2\pi + 2r\pi\left(\frac{1000}{\pi r^2}\right)$$

$$A = 2r^2\pi + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \iff r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

critical point

$A'' = 4\pi + \frac{4000}{r^3}$ which is positive (concave up)

$\Rightarrow$ the value of A at $r = \sqrt[3]{\frac{500}{\pi}}$ is abs. min.

- The corresponding height is: $h = \frac{1000}{\pi (5.42)^2} \approx 10.84$ cm

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To solve an Applied Optimization problem:

1) Read the problem: what is given?
   what needs to be optimize?
2) Draw a picture: introduce variables, see if there is relation.
3) Write an equation for the unknown quantity in terms of single variable.
4) Find the critical points and test them, together with the endpoints in the Domain of the unknown, using the first and the second derivatives.

Examples from Economics

- Suppose that $r(x)$ = the revenue from selling x items

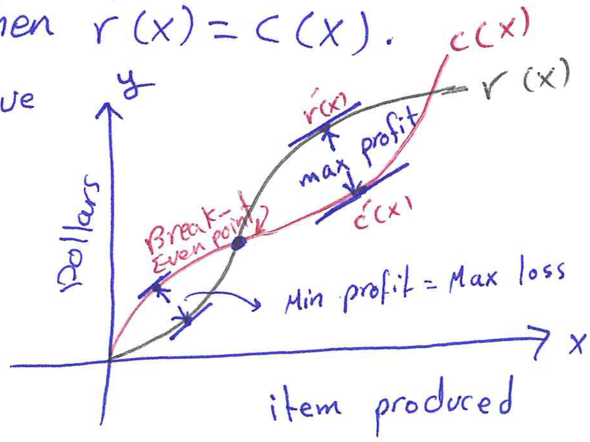
  $c(x)$ = the cost of producing the x items

  $p(x) = r(x) - c(x)$ is the profit from producing and selling x items.

- The Maximum profit occurs when $r'(x) = c'(x)$.

  where $r'(x)$ = marginal revenue

  $c'(x)$ = marginal cost



Ex **Example** Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 18x$ where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

$r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 18$

$p(x) = r(x) - c(x)$. Thus $p'(x) = r'(x) - c'(x) = 0$

$r'(x) = c'(x) \iff 3x^2 - 12x + 18 = 9 \iff x^2 - 4x + 3 = 0$

$\iff (x-1)(x-3) = 0 \iff x = 1$ or $x = 3$ critical points.

- $p''(x) = r''(x) - c''(x)$

  $= 0 - 6x + 12$

  $\iff p''(x) = 12 - 6x$

  $p''(1) = 12 - 6 = 6 > 0$ "concave up" local min

  $p''(3) = 12 - 18 = -6 < 0$ "concave down" local max.

- Max profit at level of production $x = 3$

  $p(3) = r(3) - c(3)$

  $= 9(3) - [(3)^3 - 6(3)^2 + 18(3)]$

  $= 27 - [27 - \cancel{54} + \cancel{54}]$

  $= 0$