5.2 Sigma Notation and Limits of Finite Sums

Finite Sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n .$$

n ← The index k ends at $k=n$

Sigma "Sum"

$k=1$ ← The index k starts at $k=1$

a_k is the formula for the k^{th} term.

Example ① $\sum_{k=1}^{4} k = 1+2+3+4 = 10$

② write $\sum_{k=0}^{2} \frac{6k}{k+1}$ without sigma notation

$$\sum_{k=0}^{2} \frac{6k}{k+1} = 0 + \frac{6}{2} + \frac{12}{3} = 3 + 4 = 7$$

③ $\sum_{k=1}^{2} (-1)^{k+1} \sin\frac{\pi}{k} = \sin\pi - \sin\frac{\pi}{2} = 0 - 1 = -1$

Example: Express the sum $\frac{2}{3} + \frac{4}{3} + \frac{6}{3} + \frac{8}{3} + \frac{10}{3}$ in sigma notation

$$\frac{1}{3}\left[2+4+6+8+10\right] = \frac{1}{3}\sum_{k=1}^{5} 2k$$

Algebra Rules for Finite Sums:

1. Sum Rule: $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
2. Difference Rule: $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
3. Constant Multiple Rule: $\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$
4. Constant Value Rule: $\sum_{k=1}^{n} c = nc$

$\boxed{5}$ $$\sum_{K=1}^{n} K = \frac{n(n+1)}{2}$$

$\boxed{6}$ $$\sum_{K=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{7}$ $$\sum_{K=1}^{n} K^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Example: Evaluate:

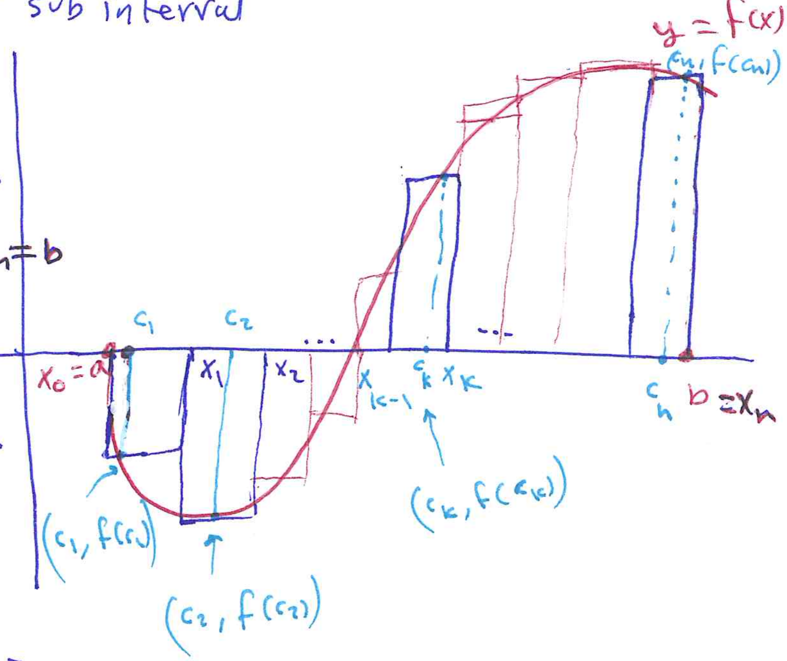
① $$\sum_{K=1}^{7}(-2K) = -2\sum_{K=1}^{7} K = -2\,\frac{7(7+1)}{2} = -7(8) = -56$$

② $$\sum_{K=1}^{6}(3-K^2) = \sum_{K=1}^{6} 3 - \sum_{K=1}^{6} K^2 = (6)(3) - \frac{6(7)(13)}{6}$$
$$= 18 - 91 = -73$$

Riemann Sums

* Consider a bounded function f defined on $[a,b]$
* Divide $[a,b]$ into n closed sub interval by choosing $n-1$ points $\{x_1, x_2, \dots, x_{n-1}\}$ between a and b.

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

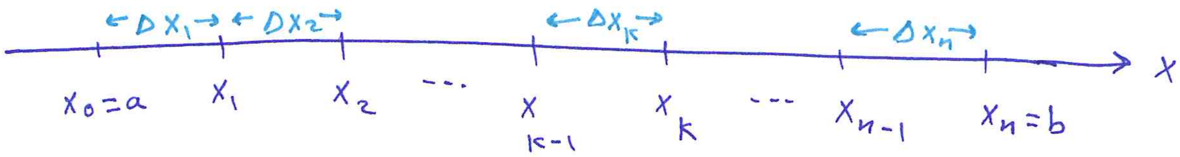


* The set $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ is called a partition of $[a,b]$.
* The partition P divides $[a,b]$ into n closed subintervals:

$$[x_0,x_1], [x_1,x_2], \dots, \underbrace{[x_{k-1}, x_k]}_{k^{th}\text{ subinterval}}, \dots [x_{n-1}, x_n]$$

* $\Delta x_k = x_k - x_{k-1}$ is the width of the k^{th} subinterval.
* $\text{Max}\,|\Delta x_k|$ is the norm of the partition P denoted by $\|P\|$.

* If all subintervals have equal width, then the width of any subinterval is $\Delta x = \frac{b-a}{n}$



* select a point c_k in Δx_k for all $k = 1, 2, \dots, n$
* On each subinterval we find $f(c_k)\,\Delta x_k$. This product is
  - positive when $f(c_k)$ is positive. This gives area above x
  - negative when $f(c_k)$ is negative. This gives area below x
* The Riemann sum S_p for f on the interval $[a, b]$ is

$$S_p = \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

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* If we choose n subintervals all having equal width, $\Delta x = \frac{b-a}{n}$, and we choose the point c_k to be the right hand endpoint of each subinterval, Then Riemann Sum formula becomes

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$$

__Example__ Let $f(x) = 1 - x^2$ on $[0, 1]$

[a] Find formula for the Riemann sum by dividing $[0,1]$ into n equal subintervals and using c_k to be the right hand endpoint. $a = 0$, $b = 1$

$$\Delta x_k = \frac{1-0}{n} = \frac{1}{n}$$ the width of the k^{th} subinterval $k = 1, 2, \dots, n$

$$S_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n}\left[1 - \left(\frac{k}{n}\right)^2\right]\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= \frac{n}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 = 1 - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$S_n = 1 - \frac{2n^3+3n^2+n}{6n^3} = 1-\left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right]$$

$$S_n = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

[b] Calculat the area under the curve over [0,1]
"Take limit for S_n as $n \to \infty$"

$$\lim_{n\to\infty} S_n = \boxed{\frac{2}{3}}$$

[c] Find $\int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$

Example: Find the norm of the partition
$P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$

The subintervals are: $[0, 1.2]$, $[1.2, 1.5]$, $[1.5, 2.3]$, $[2.3, 2.6]$, $[2.6, 3]$

$\Delta x_1 = 1.2$, $\Delta x_2 = 0.3$, $\Delta x_3 = 0.8$, $\Delta x_4 = 0.3$, $\Delta x_5 = 0.4$

The norm of the partition P is $\|P\| = 1.2$
