5.3 The Definite Integral

Let $f(x)$ be a function defined on a closed interval $[a,b]$.

We say a number J is the definite integral of f over $[a,b]$ and we write $J = \lim_{\|P\|\to 0} \underbrace{\sum_{k=1}^{n} f(c_k)\Delta x_k}_{\text{Riemann sum}}$ if

Given $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have $\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \epsilon$

So, we say the Riemann sums of f on $[a,b]$ converge to the definite integral J. That is

$$\lim_{\|P\|\to 0} S_n = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J = \int_a^b f(x)\,dx$$

b: upper limit of integration

$f(x)$: the integrand

dx: the variable of integration "dummy variable"

$\int$: integral

a: lower limit of integration

- When the partition P has n equal subintervals each of width $\Delta x = \frac{b-a}{n}$, then $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(\underbrace{c_k}_{a+k\left(\frac{b-a}{n}\right)})\left(\frac{b-a}{n}\right) = J = \int_a^b f(x)\,dx$

- Note that $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx = \int_a^b f(u)\,du$

Example Express the following limits as definite integrals:

1. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \Delta x_k$, where P is a partition of $[0,2]$ $= \int_0^2 x^2\,dx$

2. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x_k$, where P is a partition of $[-7,5]$ $= \int_{-7}^{5} (x^2 - 3x)\,dx$

Th (Integrability of Continuous Functions)

If f is continuous on $[a,b]$ or
If f has finitely many jump discontinuities on $[a,b]$,
then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable on $[a,b]$.

Th If f and g are integrable on $[a,b]$, then

$\boxed{1}$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\boxed{2}$ $\int_a^a f(x)\,dx = 0$

$\boxed{3}$ $\int_a^b K\,f(x)\,dx = K\int_a^b f(x)\,dx$

$\boxed{4}$ $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\boxed{5}$ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\boxed{6}$ If f has max value M and min value m on $[a,b]$, then

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$$

$\boxed{7}$ If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

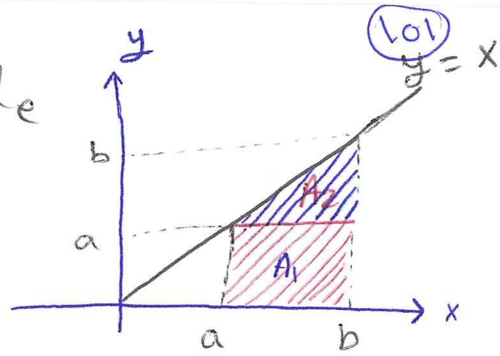
If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

Example show that $\int_0^1 \sqrt{1+\sin x}\,dx \le \sqrt{2}$

$$\int_0^1 \sqrt{1+\sin x}\,dx \le \int_0^1 \sqrt{1+1}\,dx = \int_0^1 \sqrt{2}\,dx \le \sqrt{2}(1-0) = \sqrt{2}$$

Def If $y = f(x)$ is nonnegative and integrable on $[a,b]$, then the area under the curve $y=f(x)$ on $[a,b]$ is the integral of f from a to b: $A = \int_a^b f(x)\,dx$.

Example: Use areas to evaluate



① $\int_a^b x\,dx = A_1 + A_2$

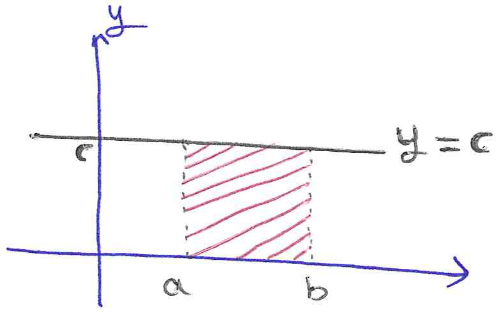
$= (b-a)\,a + \frac{1}{2}(b-a)(b-a)$

شبه المنحرف or $A = \frac{(b+a)(b-a)}{2} = \frac{b^2-a^2}{2}$ ✓

$$\int_a^b x\,dx = ab - a^2 + \frac{1}{2}(b^2 - 2ab + a^2)$$

$$= \cancel{ab} - a^2 + \frac{b^2}{2} - \cancel{ab} + \frac{a^2}{2}$$

$$\boxed{\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}}$$



② $\int_a^b c\,dx = (b-a)c$

③ $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ $\quad a < b$

Example: Use Riemann sum to calculate $\int_0^b x\,dx$

- Consider the partition P that divids $[0,b]$ into n subintervals of eqval width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$
- $P = \{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n} = b\}$

  $[0, \frac{b}{n}], [\frac{b}{n}, \frac{2b}{n}], [\frac{2b}{n}, \frac{3b}{n}], \dots [\frac{(n-1)b}{n}, b]$
- choose c_k to be the right endpoint $c_k = \frac{kb}{n}$

$$\int_0^b x\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{kb}{n}\right)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb}{n}\,\frac{b}{n}$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{b^2}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right)$$

$$= \frac{b^2}{2}$$

Def If f is integrable on $[a, b]$, then the average value of f on $[a, b]$ "Mean" is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Example Find the average value of $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$.

$$av(f) = \frac{1}{\sqrt{3}-0} \int_0^{\sqrt{3}} (x^2 - 1)\, dx = \frac{1}{\sqrt{3}} \left[\int_0^{\sqrt{3}} x^2\, dx - \int_0^{\sqrt{3}} dx \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{(\sqrt{3})^3}{3} - \frac{(0)^3}{3} - (\sqrt{3} - 0) \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{3\sqrt{3}}{3} - \sqrt{3} \right]$$

$$= \frac{1}{\sqrt{3}} \left[\sqrt{3} - \sqrt{3} \right]$$

$$= 0$$