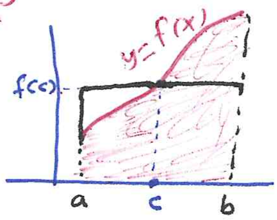
$\boxed{5.4}$ The Fundamental Theorem of Calculus + $\boxed{5.5}$

Theorem (The Mean Value Theorem for Definite Integrals)

If f is continuous on $[a,b]$, then at some point $c \in [a,b]$, we have: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$



* $f(c)$ is the average height "Mean" of f on $[a,b]$

The area of the rectangle = area under f

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

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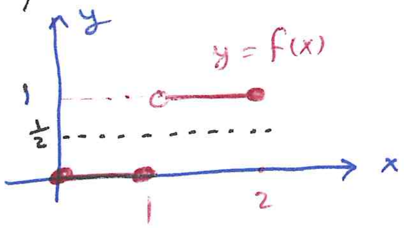
* If f is discontinuous, then f may never equals its average value

$$av(f) = \frac{1}{2-0}\int_0^2 f(x) = \frac{1}{2}\int_1^2 f(x) = \frac{1}{2}(1) = \frac{1}{2}$$

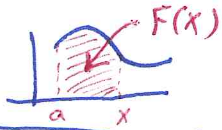
but $\nexists\, c \in [0,2]$ s.t $f(c) = \frac{1}{2}$



Th① (The Fundamental Theorem of Calculus)

If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) with

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



Example: Find $\frac{dy}{dx}$ for

① $y = \int_3^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

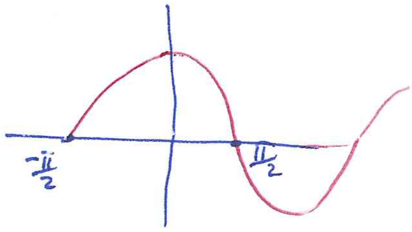
② $y = \int_5^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

③ $y = \int_a^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

④ $y = \int_x^5 (t^2 - 5t)\,dt \Rightarrow -\int_5^x (t^2 - 5t)\,dt = -(x^2 - 5x) = 5x - x^2$

⑤ $y = \int_x^0 \sqrt{1+t^2}\,dt \Rightarrow -\sqrt{1+x^2}$

⑥ $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$, $|x| < \frac{\pi}{2}$



$y' = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{\cos x} = 1$

⑦ $y = \int_3^{x^3} \sin t\,dt \Rightarrow 3x^2 \sin x^3$

$u = x^3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \int_{27}^{u} \sin t\,dt \iff y' = \left(\int_{27}^{u} \sin t\,dt\right)' u' = \sin u\,(3x^2) = 3x^2 \sin x^3$

Th ② (The Fundamental Theorem of Calculs)

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Example: Evaluate the integrals:

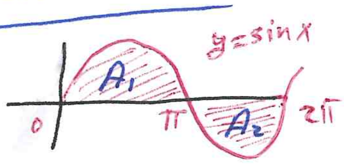
① $\int_{-2}^{0} (2x+5)\,dx = x^2 + 5x \Big|_{-2}^{0} = -[4-10] = 6$

② $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan x \Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1$

* To Find the area between the graph of $y=f(x)$ and the x-axis over the interval $[a,b]$:

- Find the zeros of f on $[a,b]$
- Divid $[a,b]$ at the zeros of f.
- Integrate f over each subinterval in absolute value.

Example: let f be as defined in the graph:



[a] Find the definit integral of f on $[0, 2\pi]$

$\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1-1] = 0$

[b] Find the area between f and the x-axis on $[0, 2\pi]$

Area $= |A_1| + |A_2|$

where $A_1 = \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$ (105)

$$A_2 = \int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1+1] = -2$$

$$\text{Area} = |A_1| + |A_2| = |2| + |-2| = 4$$

Th (The Net Change Theorem)

The net change in the Function $F(x)$ on $[a,b]$ is the integral of its rate of change: $F(b) - F(a) = \int_a^b F'(x)\,dx$

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* If $c(x)$ is the cost for producing x units, then

$c'(x)$ is the marginal cost and

$\int_{x_1}^{x_2} c'(x)\,dx = c(x_2) - c(x_1)$ is the cost of increasing the production from x_1 units to x_2 units.

* If $s(t)$ is the position of an object, then

$s'(t) = v(t)$ is its velocity and

$\int_{t_1}^{t_2} \underbrace{v(t)}_{\text{velocity}}\,dt = s(t_2) - s(t_1)$ is the displacement over $[t_1, t_2]$ and

$\int_{t_1}^{t_2} \overbrace{|v(t)|}^{\text{speed}}\,dt$ is the total distance traveled on $[t_1, t_2]$

* $$\underset{\substack{\downarrow \\ \text{Final value}}}{F(b)} = \underset{\substack{\downarrow \\ \text{initial value}}}{F(a)} + \underbrace{\int_a^b F'(x)\,dx}_{\text{Net change}}$$