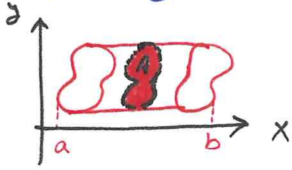
$\boxed{6.1}$ Volumes Using Cross - Sections

* The volume of cylindrical solid is $V =$ base area $\times$ height $= Ah$

* To Find Volume of a Solid:

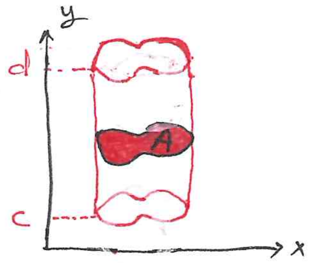


1- Graph the solid

2- Determine a cross-section of the solid

(a) • If the cross-sectional is perpendicular to the x-axis, then the volume is

$$V = \int_a^b A(x)\,dx$$

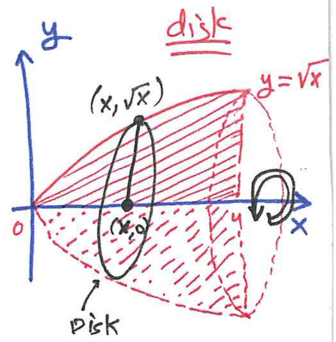


(b) • If the cross-sectional is perpendicular to the y-axis, then the volume is

$$V = \int_c^d A(y)\,dy$$

Disk Method (c) • If the cross-sectional is perpendicular to the x-axis results by rotation about x-axis, then

$$V = \int_a^b \underbrace{A(x)}_{\text{Disk}}\,dx = \int_a^b \pi \,\underbrace{[R(x)]}_{\text{radius}}^2\,dx$$



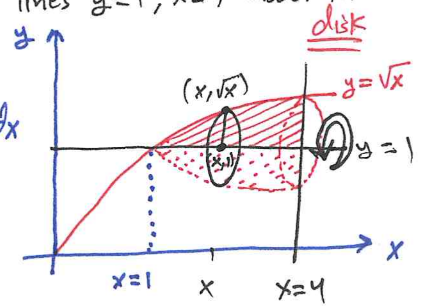
Example: Find the volume of the region between the curve $y=\sqrt{x}$, $0 \le x \le 4$ and the x-axis that is revolved about the x-axis.

$$V = \int_0^4 \pi (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$$

Example: Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$.

$$V = \int_1^4 \pi (\sqrt{x}-1)^2\,dx = \pi \int_1^4 (x - 2x^{\frac{1}{2}} + 1)\,dx$$

$$= \pi \left[\frac{x^2}{2} - 2\,\frac{2}{3}\,x^{\frac{3}{2}} + x\right]_1^4 = \frac{7\pi}{6}$$



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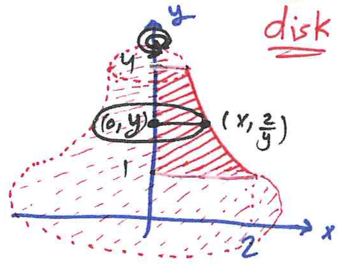
Disk Method (d) • If the cross-sectional is perpendicular to the y-axis results by rotation about y-axis, then

disk



$$V = \int_c^d \underbrace{A(y)}_{\text{Disk}}\, dy = \int_c^d \pi [R(y)]^2\, dy$$

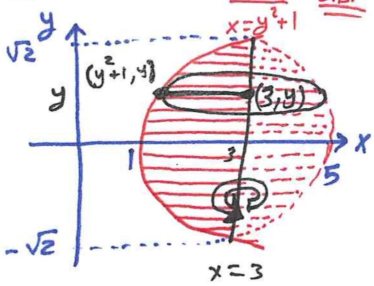
Example: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

when $y = 1 \Rightarrow x = 2$

$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left.\frac{y^{-1}}{-1}\right|_1^4 = 3\pi$$

Example: Find the volume of the solid generated by revolving the region between $x = y^2+1$ and the line $x=3$ about the line $x=3$.

disk



$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi [3-y^2-1]^2\, dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)^2 dy$$

$$= \pi \left(4 - 4y^2 + y^4\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\pi\sqrt{2}}{15}$$

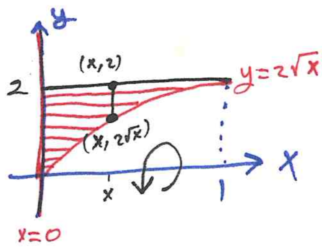
$y^2+1 = 3 \Leftrightarrow y^2 = 2 \Leftrightarrow y = \pm\sqrt{2}$

Washer Method (e) • If the cross-sectional is perpendicular to x-axis results by rotation about x-axis with outer radius $R(x)$ and inner radius $r(x)$, then

$$V = \int_a^b A(x)\, dx = \int_a^b \pi [R^2(x) - r^2(x)]\, dx$$

Example: Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$, $y = 2$ and $x = 0$ about the x-axis.

$$V = \pi \int_0^1 \boxed{(2)^2 - (2\sqrt{x})^2}\, dx = \pi \int_0^1 [4 - 4x]\, dx$$



$$= 4\pi \int_0^1 (1-x)\, dx = 4\pi \left[x - \frac{x^2}{2}\right]_0^1$$

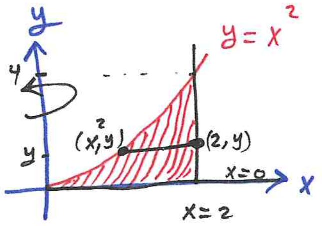
$$= 2\pi$$

no disk

Washer Method (f) If the cross-sectional is perpendicular to y-axis results by rotation about y-axis with outer radius $R(y)$ and inner radius $r(y)$, then (111)

$$V=\int_c^d A(y)\,dy=\int_c^d \pi\left[R^2(y)-r^2(y)\right]dy$$

<u>Example</u>: Find the volume of the solid generated by <u>revolving</u> the region, bounded by $y=x^2$, x-axis, and $x=2$ about the <u>y-axis</u>.
in the first quadrant



$R(y)=2$, $r(y)=\sqrt{y}$

$$V=\pi\int_0^4\left[2^2-(\sqrt{y})^2\right]dy$$

$$=\pi\int_0^4(4-y)\,dy=\pi\left[4y-\frac{y^2}{2}\right]\Big|_0^4=8\pi$$