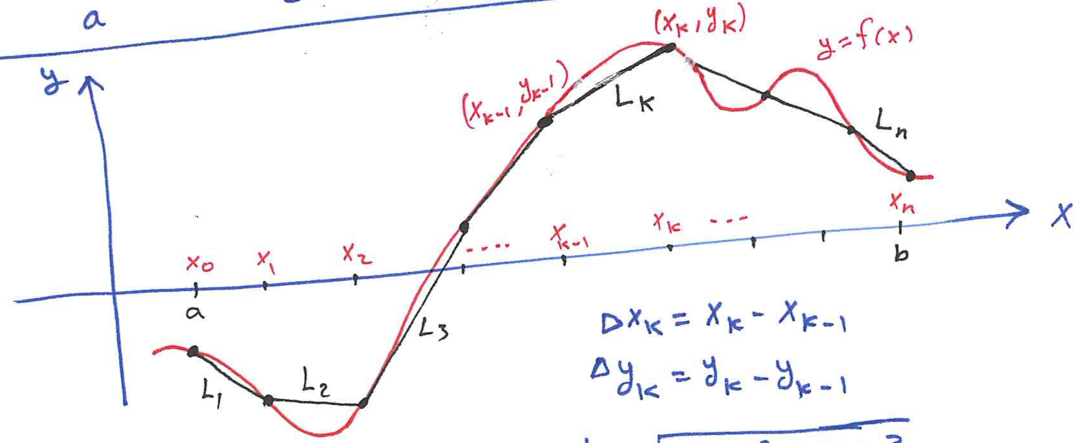
6.3 Arc Length

Def: If f' is continuous on $[a, b]$, then the arc length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$\Delta x_k = x_k - x_{k-1}$$
$$\Delta y_k = y_k - y_{k-1}$$
$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

- The Length of the curve is approximated by:

$$L = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

- By Mean Value Theorem, there is a point c_k with $x_{k-1} < c_k < x_k$ such that

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k} \iff \Delta y_k = f'(c_k)\, \Delta x_k$$

- Therefore, $L = \sum_{k=1}^{n} \boxed{\sqrt{1 + [f'(c_k)]^2}}\, \Delta x_k$ This is continuous on $[a, b]$ $\Rightarrow$

- The approximation L improves as the partition of $[a,b]$ becomes finner:

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example: Find the length of the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 4$.

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow 1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$
$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$L = \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12}$$
$$= 6.$$

Example: Find the length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x=0$ to $x=2$.

$$y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\frac{1}{2} = \frac{1}{3}\left(\frac{2}{x}\right)^{\frac{1}{3}}$$

"y' is not defined at $x=0$"

$\Rightarrow$ we write x interms of y

$$L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\Rightarrow y^{\frac{3}{2}} = \frac{x}{2} \Rightarrow x = 2y^{\frac{3}{2}}$$

$x=0 \Rightarrow y=0$

$x=2 \Rightarrow y=1$

$$\Rightarrow x' = 3y^{\frac{1}{2}}$$

$$\Rightarrow (x')^2 = 9y$$

$$\Rightarrow L = \int_0^1 \sqrt{1+9y}\, dy$$

$u = 1+9y$

$du = 9\,dy$

$y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=10$

$$= \int_1^{10} \frac{1}{9} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{9} u^{\frac{3}{2}} \frac{2}{3}\Big|_1^{10} = \frac{2}{27}\left[(10)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{27}\left[10\sqrt{10}-1\right]$$

$$\approx 2.27.$$

Example Find the length of the curve $y = \int_0^x \sqrt{\cos 2t}\, dt$ from $x=0$ to $x=\frac{\pi}{4}$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$y' = \sqrt{\cos 2x} \Rightarrow (y')^2 = \cos 2x$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 2x}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1+2\cos^2 x - 1}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos x\, dx$$

$$= \sqrt{2}\sin x\Big|_0^{\frac{\pi}{4}} = \sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right] = 1$$
