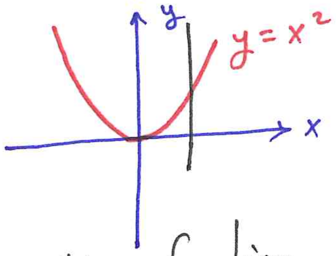
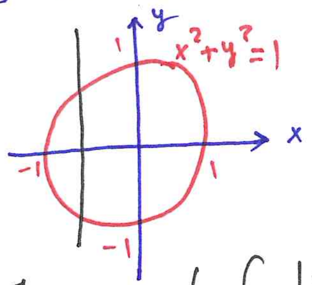
7.1 Inverse Functions and Their Derivatives

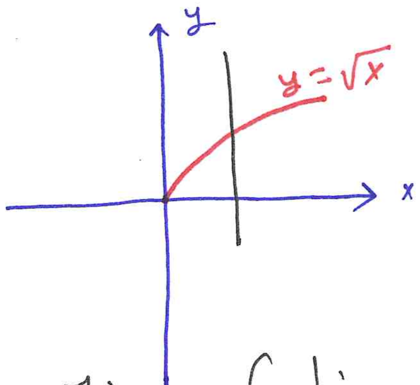
* A function is a rule that assigns a value from its range to each element in its domain.



This is function



This is not a function



This is a function

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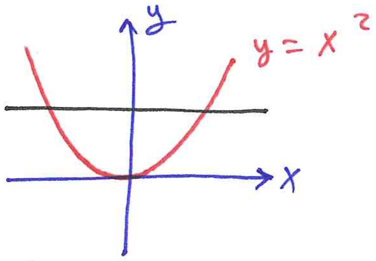
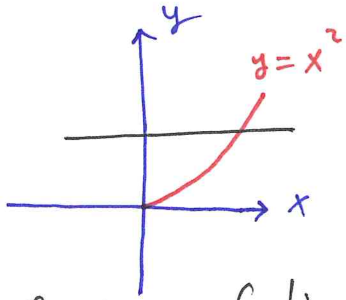
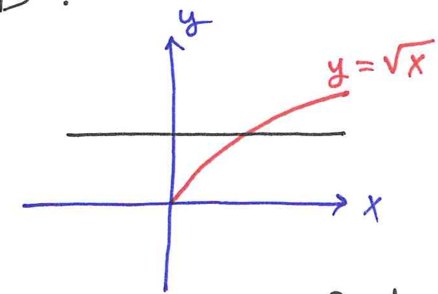
* A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.



Not one-to-one function



One-to-One function on $D=[0,\infty)$



one-to-one function for any Domain

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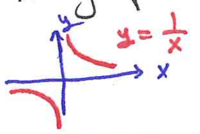
* A function $f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



is one-to-one function

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* Def: Let $f: D \to R$ be one-to-one. The inverse function $f^{-1}: R \to D$ is defined by $f^{-1}(b)=a$ if $f(a)=b$.

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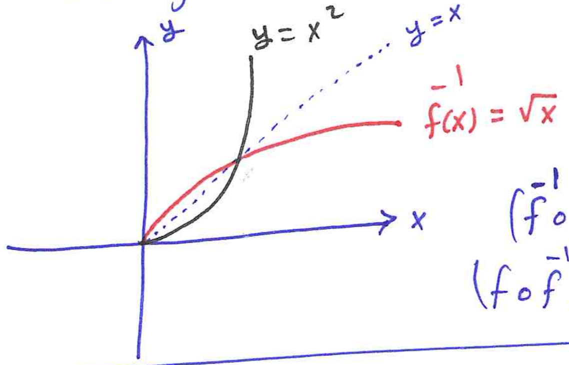
* The domain of f^{-1} is R and the range is D
* $f^{-1}(x) \neq \dfrac{1}{f(x)}$ but $[f(x)]^{-1} = \dfrac{1}{f(x)}$.
* Only one-to-one functions have inverse.
* $(f^{-1} \circ f)(x) = x$ for all $x \in D(f)$
* $(f \circ f^{-1})(y) = y$ for all $y \in D(f^{-1}) = R(f)$

Example Find the inverse of the function $y = x^2$, $x \geq 0$

$\sqrt{y} = \sqrt{x^2} = |x| = x$ since $x \geq 0$

interchange x and y to obtain $y = \sqrt{x} \Rightarrow f^{-1}(x) = \sqrt{x}$



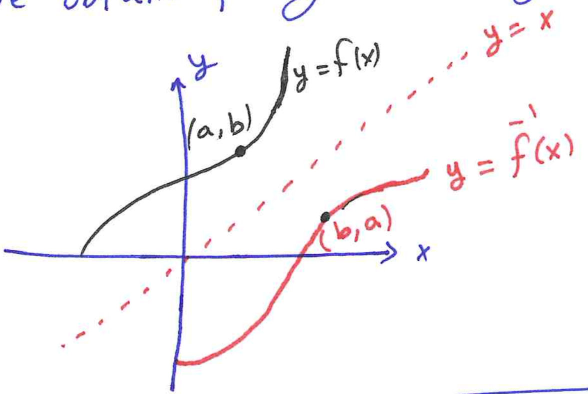
y: $D = [0, \infty)$, $R = [0, \infty)$

f^{-1}: $D = [0, \infty)$, $R = [0, \infty)$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

* We obtain f^{-1} by reflecting the graph of f about the line $y = x$.



x	-2	0	1	5
$y = f(x)$	4	3	2	6

y	4	3	2	6
$f^{-1}(y)$	-2	0	1	5

Example Find the inverse of the function $f(x) = x^2 - 2x$, $x \leq 1$

$f(x) = (x-1)^2 - 1 \Leftrightarrow y = (x-1)^2 - 1 \Leftrightarrow (x-1)^2 = y + 1$

$|x - 1| = \sqrt{y+1} \Leftrightarrow 1 - x = \sqrt{y+1} \Leftrightarrow x = 1 - \sqrt{y+1}$

$y = 1 - \sqrt{x+1} \Leftrightarrow f^{-1}(x) = 1 - \sqrt{x+1}$ with $D = [-1, \infty)$

$R = (-\infty, 1]$

Th 1 If $f: D \to R$ is one-to-one with f' exists and never zero on D then $f^{-1}: R \to D$ is differentiable on R and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \text{or} \quad \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x = f^{-1}(b)}}$$

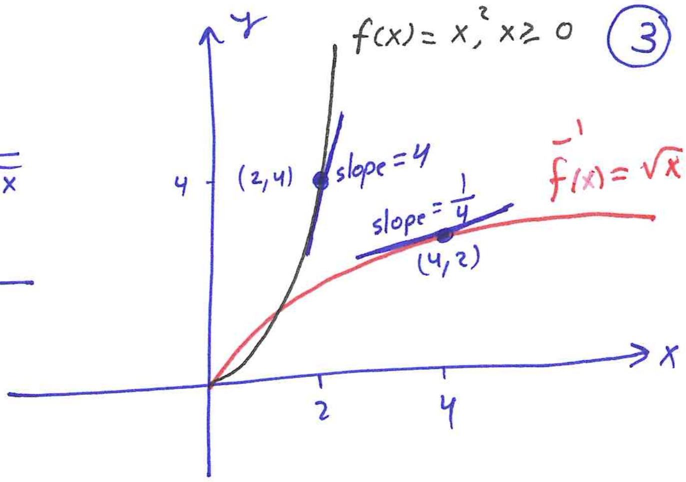
$f(x) = x^2, \quad x \geq 0$

$f'(x) = 2x$

$f'(2) = 2(2) = 4$

$f^{-1}(x) = \sqrt{x}$

$(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$



$$(f^{-1})'(4) = \frac{1}{f'\left(f^{-1}(4)\right)}$$

$$= \frac{1}{f'(\sqrt{4})} = \frac{1}{f'(2)} = \frac{1}{2(2)} = \frac{1}{4}$$

Example Let $f'(x) = 3x^2$. Find $\left.\dfrac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})}$

$$\frac{df^{-1}}{dx}(f(\sqrt{2})) = \frac{1}{f'\left(f^{-1}(f(\sqrt{2}))\right)} = \frac{1}{f'(\sqrt{2})} = \frac{1}{3(2)} = \frac{1}{6}$$