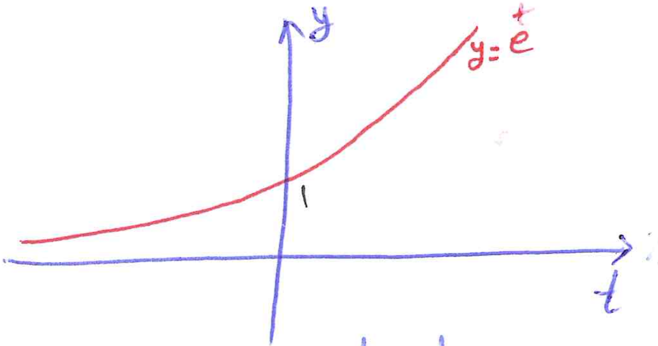
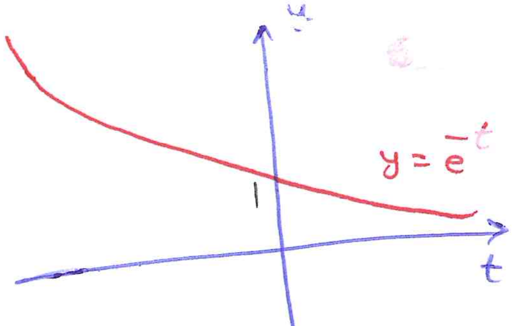
$\boxed{7.4}$ Exponential change and Separable Differential Equations

- Recall that the exponential function is





e^t describes growth natural

e^{-t} describes natural decay

- Recall that
  - Differential Equations are equations (relations) with derivatives (rates)
  - DE's that describes growth or decay may have the form

The changes in the amount of $y(t)$ with time is proportional to the amount present

$$\frac{dy}{dt} = k\,y \quad \text{----} \; *^1$$

  - DE's may have initial condition:

$$y(0) = y_0 \quad \text{----} \; *^2$$

- IVP is a DE together with initial condition

$$\frac{dy}{dt} = ky, \quad y(0) = y_0 \quad \text{----} \; *$$

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* How to solve the IVP $*$?

we use the method of calculus to find the solution $y(t)$ of $*$:

$$\frac{1}{y}\frac{dy}{dt} = K \iff \int \frac{y'}{y} = \int K$$

$$\ln|y| = kt + c \iff |y| = e^{kt+c}$$

$$|y| = e^{c} e^{kt} \iff y = \pm e^{c} e^{kt}$$

$$y(t) = D e^{kt} \quad \text{where } D = \pm e^{c}$$

To find D, we use the initial condition:

$$y(0) = D e^{k(0)} = y_0 \iff D = y_0$$

Thus, the solution becomes

$$\boxed{y(t) = y_0 e^{kt}} \text{ ---- } *^3$$

* Note that if $K > 0$, the solution grows exponentially
if $K < 0$, the solution decays exponentially

The growth equation is $y(t) = y_0 e^{kt}$ / The decay eq. is $y(t) = y_0 e^{-kt}$

* Check $*^3$ is a solution for $*$

* Note that $\boxed{\text{Half-life} = \frac{\ln 2}{K}}$ because $\frac{y_0}{2} = y_0 e^{-kt}$

$$\frac{1}{2} = e^{-kt} \iff -\ln 2 = -kt \iff \boxed{t = \frac{\ln 2}{K}}$$

Separable Differential Equation

Example: Solve the DE $2\sqrt{xy}\,\frac{dy}{dx} = 1$, $x, y > 0$.

$$2\sqrt{x}\sqrt{y}\,dy = dx \iff 2\,y^{\frac{1}{2}}\,dy = x^{-\frac{1}{2}}\,dx$$

$$2\,\frac{2}{3}\,y^{\frac{3}{2}} = \frac{2}{1}\,x^{\frac{1}{2}} + c \iff \frac{4}{3}\,y^{\frac{3}{2}} - 2\sqrt{x} = c$$

implicit solution

$$\frac{4}{3}\,y^{\frac{3}{2}} = 2\sqrt{x} + c \iff y = \left[\frac{3}{4}\left(2\sqrt{x} + c\right)\right]^{\frac{2}{3}}$$

explicit solution

Example: Solve the IVP $\frac{dy}{dx} = \frac{y\cos x}{1+3y^3}$, $y(0) = 1$

$$(1+3y^3)\,dy = y\cos x\,dx$$

$$\left(\frac{1}{y} + 3y^2\right)dy = \cos x\,dx$$

$$\ln|y| + y^3 = \sin x + C$$

$$\ln 1 + (1)^3 = \sin(0) + C \iff \boxed{1 = C}$$

The solution becomes $\ln|y| + y^3 = \sin x + 1$ implicit solution

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Example (Radioactive) The half-life of the plutonium is 24,360 years. If 10g of plutonium is relased into the atmosphere, how many years will it take for 80% of the plutonium to decay.

Equation for decay is $P(t) = P_0\,e^{-kt}$ $\quad P_0 = 10\,g$

Half-life $= \frac{\ln 2}{k} \iff k = \frac{\ln 2}{24{,}360} \simeq 0.00002845$

We need to find the time t when $P(t) = 20\%\,P_0$

$$0.2 P_0 = P_0 e^{-kt} \Leftrightarrow -kt = \ln 0.2 = -1.61$$

$$\Leftrightarrow t = \frac{1.61}{0.00002845} \approx 56690 \text{ year}$$

Heat Transfer "Newton's law of Cooling"

- If $H(t) = H$ is the temperature of the object at time t,
  H_s is the constant surrounding tempreature

then the DE that describes the heat transfer is

$$\frac{dH}{dt} = -k(H - H_s) \quad \cdots \quad *$$

* To solve * Let $y = H - H_s$

$$\frac{dy}{dt} = \frac{dH}{dt} = -k(H - H_s)$$

$$\frac{dy}{dt} = -ky \quad \Leftrightarrow \quad y(t) = y_0 e^{-kt}$$

$$\Leftrightarrow H - H_s = (H_0 - H_s) e^{-kt}$$

$$H(t) = H_s + (H_0 - H_s) e^{-kt}$$

Example A boiled egg at 98°C is put in a sink of 18°C water. After 5 min the egg's temperature is 38°C. How long will it take the egg to reach 20°C?

$H(0) = H_0 = 98°C$, $H_s = 18°C$, $H(5) = 38°C$

We need to find t such that $H(t) = 20°C$.

$$H(t) = 18 + (98 - 18) e^{-kt}$$

$$\boxed{H(t) = 18 + 80 e^{-kt}}$$

$$H(5) = 18 + 80 e^{-k(5)} = 38 \quad \Leftrightarrow 80 e^{-k(5)} = 20$$

$$e^{-5k} = \frac{1}{4} \iff -5k = -\ln 4 \iff k = \frac{\ln 4}{5} \approx 0.28$$

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$$H(t) = 18 + 80\, e^{-0.28t}$$

Thus,

$$20 = 18 + 80\, e^{-0.28t}$$

$$2 = 80\, e^{-0.28t}$$

$$e^{-0.28t} = \frac{1}{40} \iff -0.28t = -\ln 40$$

$$\iff t = \frac{\ln 40}{0.28} \approx 13 \text{ min.}$$