7.5 Indeterminate Forms and L'Hopital Rule

- Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0, 1^∞, ∞^0

Th (L'Hopital Rule): suppose f and g are differentiable on an open interval I s.t
$\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$f(a) = g(a) = 0$ where $a \in I$

and $g'(x) \neq 0$ on I if $x \neq a$

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$ assuming this limit exists.

Example ① $$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4} \qquad \left(\frac{0}{0}\right)$$

② $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1-\cos x}{3x^2} = \lim_{x \to 0} \frac{\sin x}{6x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$

③ $$\lim_{x \to 0} \frac{1-\cos x}{x + x^2} = \lim_{x \to 0} \frac{\sin x}{1+2x} = \frac{0}{1} = 0 \qquad \left(\frac{0}{0}\right)$$

④ $$\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} = \frac{1}{\text{very small}^+} = \infty$$

⑤ $$\lim_{x \to 0^-} \frac{\sin x}{x^2} = \lim_{x \to 0^-} \frac{\cos x}{2x} = \frac{1}{\text{very small}^-} = -\infty$$

⑥ $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$ $\left(\frac{\infty}{\infty}\right)$

⑦ $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$ $\left(\frac{\infty}{\infty}\right)$

⑧ $$\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$ $(\infty \cdot 0)$

⑨ $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}}$$ $(\infty \cdot 0)$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$

⑩ $$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x}$$ $(\infty - \infty)$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\frac{x-1}{x} + \ln x} = \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x}$$

$$= \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a} \ln f(x)}$$

Example ⑪ $$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln (1+x)^{\frac{1}{x}}}$$ (1^∞)

$$= e^{\lim_{x\to 0^+} \frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}} = e^1 = e$$

(12) $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim_{x\to\infty} \frac{\ln x}{x}}$ (18) (∞^0)

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = e^{\lim_{x\to\infty} \frac{1}{x}} = e^0 = 1$$

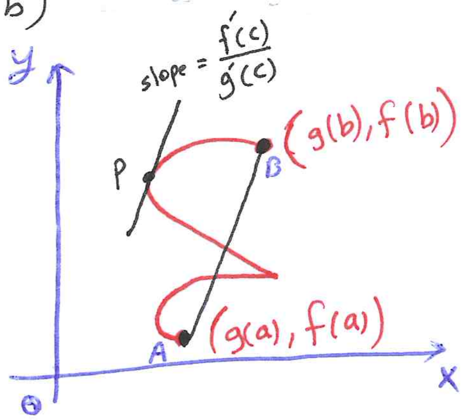
Th (Cauchy's Mean Value Theorem) CMVT

Suppose
- the functions f and g are continuous on $[a, b]$
- " " " " " " differentiable on (a, b)
- $g'(x) \neq 0$ on (a, b)

Then there exists a number $c \in (a, b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

slope of tangent = slope of secant



<u>Example</u>: Find values of c that satisfy CMVT for the function $f(x) = x$, $g(x) = x^2$ on $(-2, 0)$

$a = -2$
$b = 0$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{1}{2c} = \frac{f(0) - f(-2)}{g(0) - g(-2)}$$

$$\frac{1}{2c} = \frac{0 - -2}{0 - 4}$$

$$2c = \frac{-4}{2} = -2$$

$$\boxed{c = -1}$$

$f(0) = 0$
$f(-2) = -2$

$g(0) = 0$
$g(-2) = 4$

$f'(x) = 1$
$g'(x) = 2x$