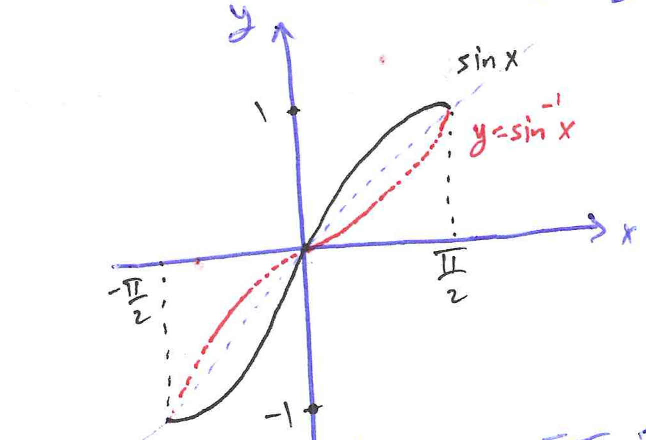
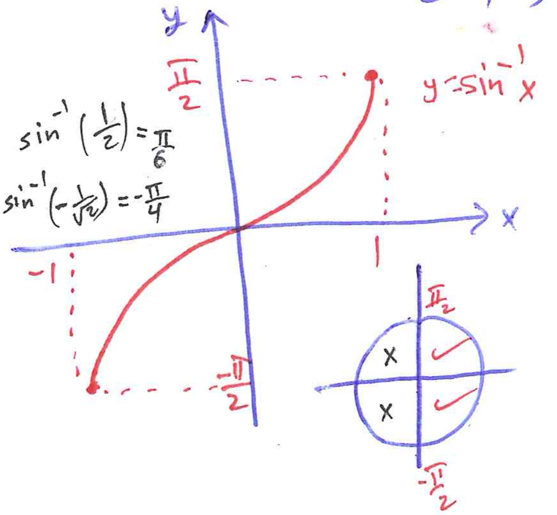
7.6 Inverse Trigonometric Functions

If $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $f^{-1}(x) = y = \sin^{-1} x$ on $[-1, 1]$
$= \arcsin x$



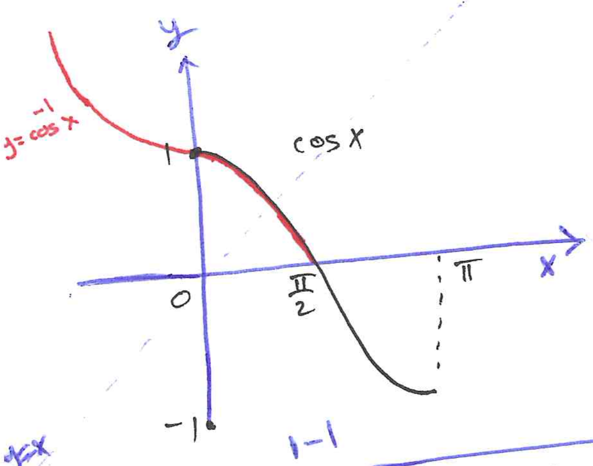
1-1 function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$



$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

If $f(x) = \cos x$ on $[0, \pi]$, then $f^{-1}(x) = y = \cos^{-1} x$ on $[-1, 1]$
$= \arccos x$



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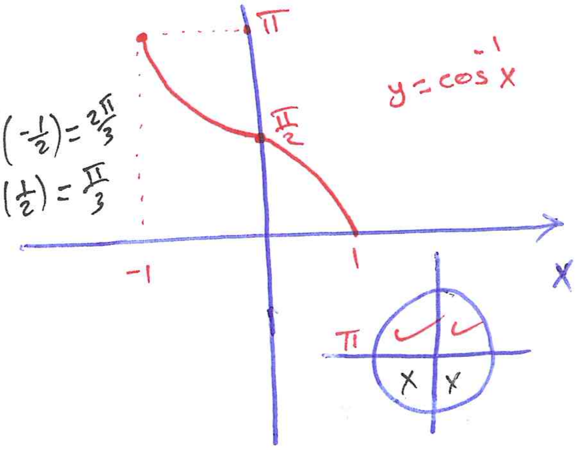
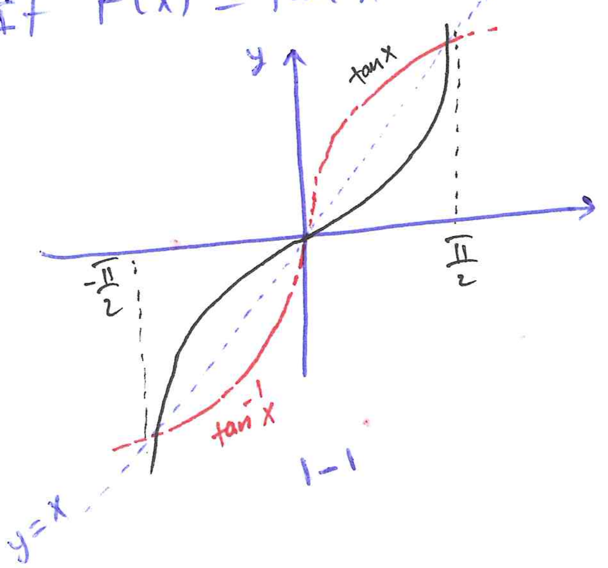


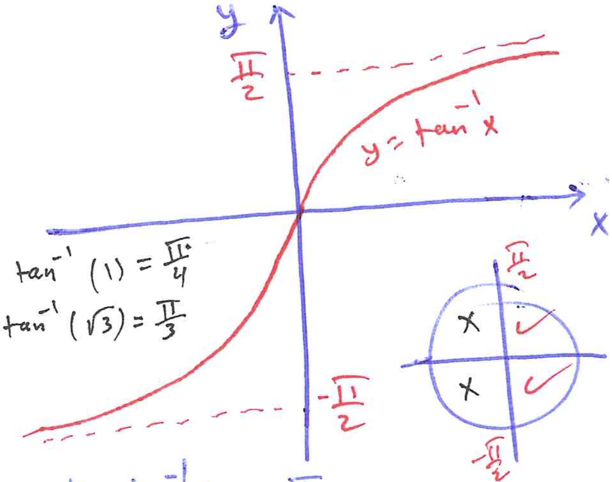
$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

If $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, then $f^{-1}(x) = y = \tan^{-1} x$ on $\mathbb{R}$
$= \arctan x$



1-1



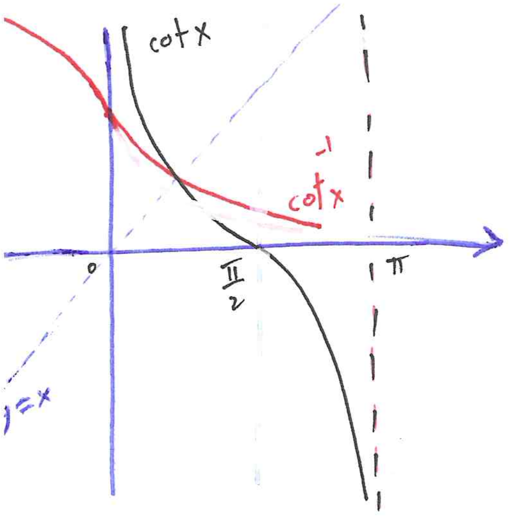
$\tan^{-1}(1) = \frac{\pi}{4}$

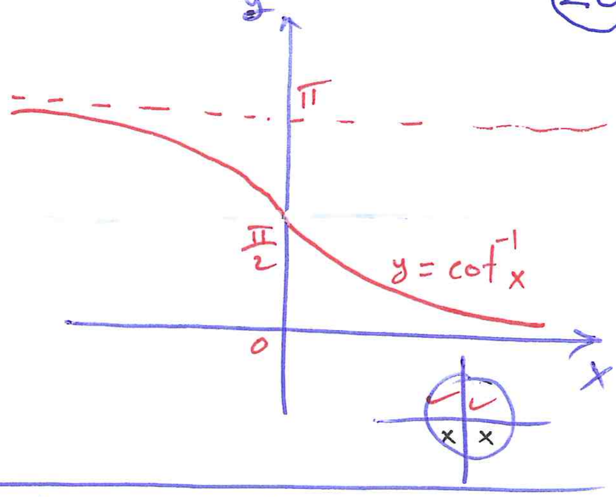
$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

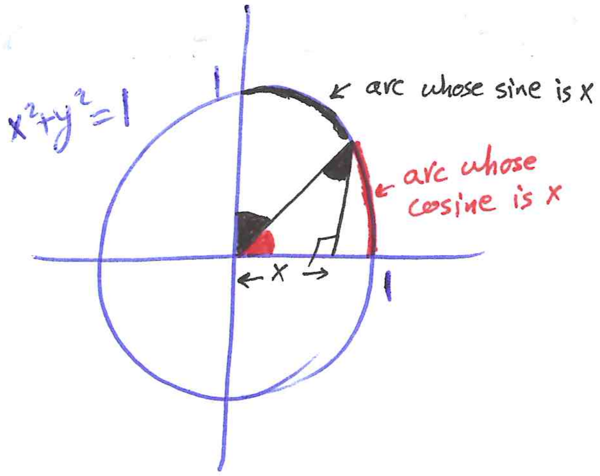
$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

If $f(x) = \cot x$ on $(0, \pi)$, then $f^{-1}(x) = y = \cot^{-1} x$ on $\mathbb{R}$

$= \text{arc} \cot x$

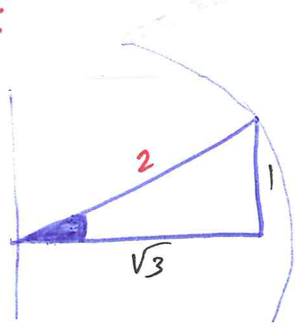




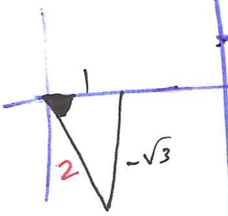


$$s = r\theta = \theta$$

$$\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$



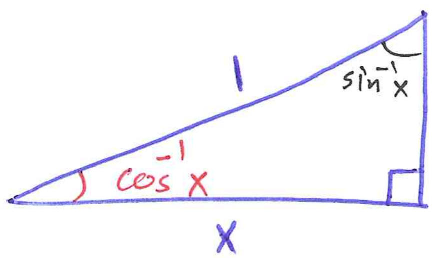
- $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

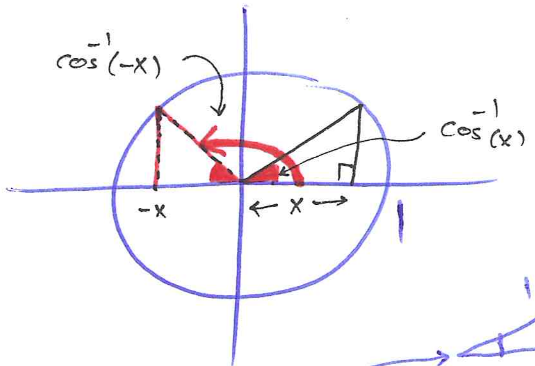


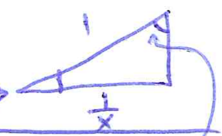
Note that

$$\cos^{-1} x + \cos^{-1}(-x) = \pi$$

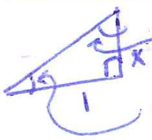
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$







$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$



$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

* If $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$, then

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

$$= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}$$

$\sin(\sin^{-1} x) = x$

$\sin^2(\sin^{-1} x) = \sin(\sin^{-1} x)\,\sin(\sin^{-1} x)$
$= x \quad x = x^2$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

Example $\frac{d}{dx}(\sin^{-1}\sqrt{2}\,x) = \frac{1}{\sqrt{1 - (\sqrt{2}x)^2}}\sqrt{2} = \frac{\sqrt{2}}{\sqrt{1 - 2x^2}}$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$

$f(x) = \tan x$ with $f^{-1}(x) = \tan^{-1} x$

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1} x)}$$

$$= \frac{1}{1+x^2}$$

$\tan(\tan^{-1} x) = x$

Example $y = \ln \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\tan^{-1} x}\left(\frac{1}{1+x^2}\right)$

Similarly

$$\frac{d}{dx}\left(\cos^{-1}u\right) = \frac{-1}{\sqrt{1-u^2}}\,\frac{du}{dx}, \quad |u|<1$$

$$\frac{d}{dx}\left(\cot^{-1}u\right) = \frac{-1}{1+u^2}\,\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec^{-1}u\right) = \frac{1}{|u|\sqrt{u^2-1}}\,\frac{du}{dx}, \quad |u|>1$$

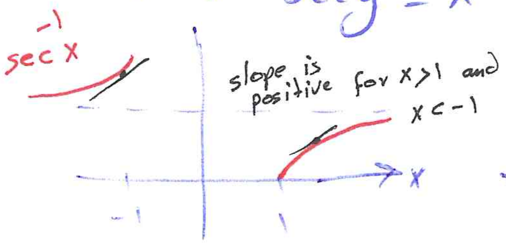
Let $y = \sec^{-1}x \implies \sec y = x$

$$\implies \sec y \tan y \frac{dy}{dx} = 1$$

$$\implies \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

But $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1}$

$$= \pm\sqrt{x^2-1}$$



$$\implies \frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

Example $y = \sec^{-1}(2x+1) \implies \dfrac{dy}{dx} = \dfrac{2}{|2x+1|\sqrt{(2x+1)^2-1}} = \dfrac{1}{|2x+1|\sqrt{x^2+x}}$

Similarly

$$\frac{d}{dx}\left(\csc^{-1}u\right) = \frac{-1}{|u|\sqrt{u^2-1}}\,\frac{du}{dx}, \quad |u|>1$$

Now for any constant $a \neq 0$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \quad , \quad |u| > a > 0$$

Exp. $$\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}$$
$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Exp $$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$$

$$4x - x^2 = -(x^2 - 4x)$$
$$= -(x-2)^2 + 4$$
$$= 4 - (x-2)^2$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 2$
$u = x - 2$
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$
