7.7 Hyperbolic Functions

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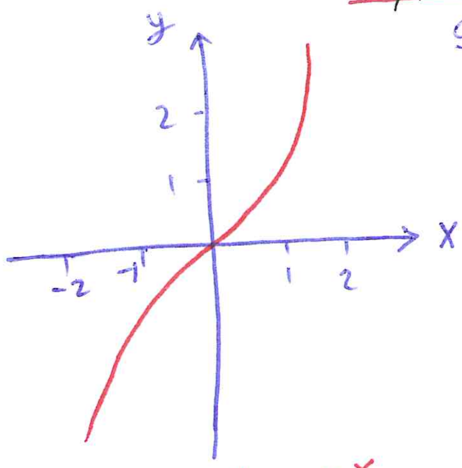
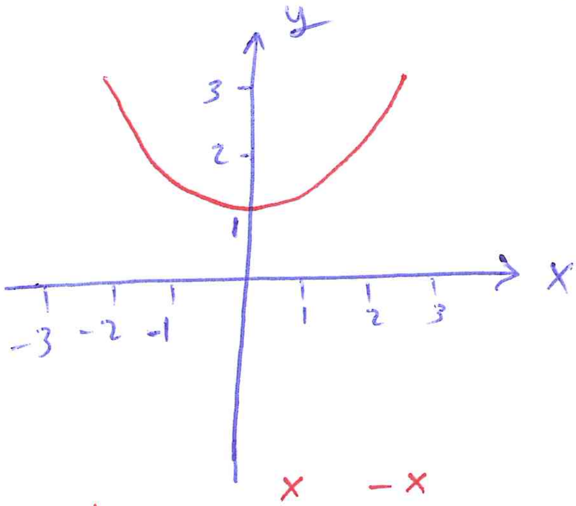
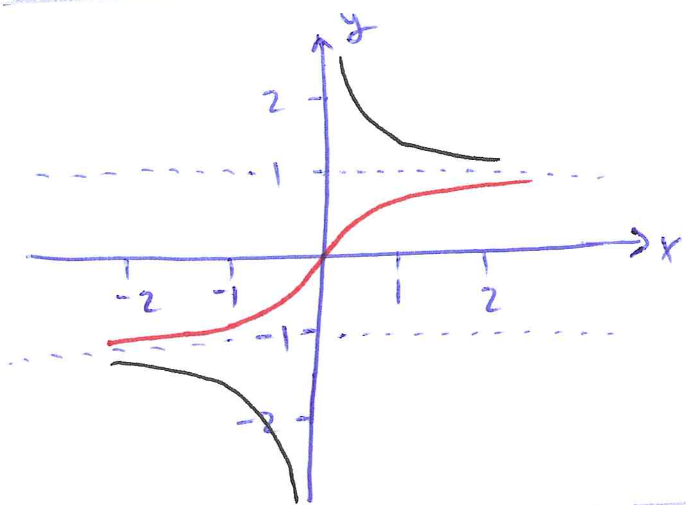


$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

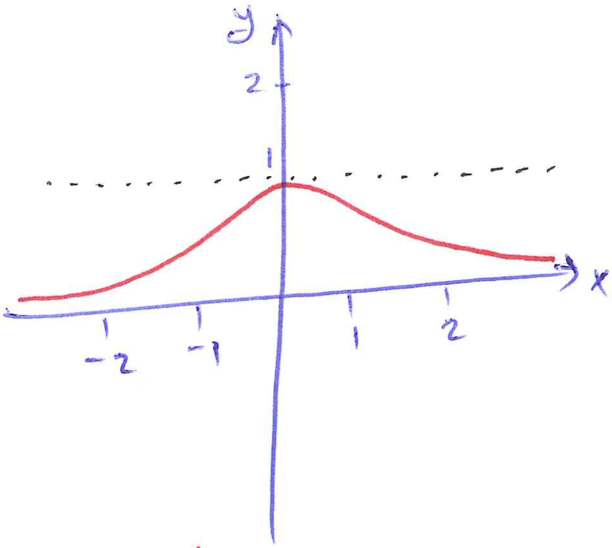


$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



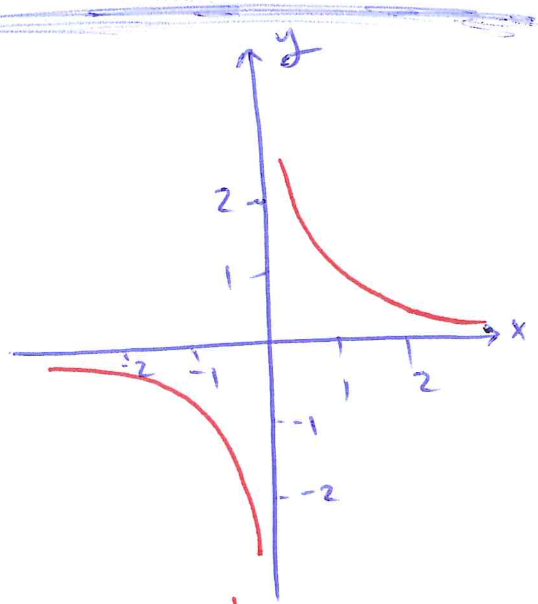
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$



$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$= \frac{2}{e^{x} + e^{-x}}$$



$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$= \frac{2}{e^{x} - e^{-x}}$$

* Identities For Hyperbolic Functions

- $\cosh^2 x - \sinh^2 x = 1$ $\Rightarrow$ Proof $\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2 =$
$\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4} =$
$\frac{4}{4} = 1$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 2\sinh^2 x + 1$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$
- $\coth^2 x - \operatorname{csch}^2 x = 1$

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* Derivatives of Hyperbolic Functions

✓ • $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

• $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$ $\Rightarrow$ Proof

$\cosh u = \frac{e^{u}+e^{-u}}{2} \Rightarrow$

$\frac{d}{dx}(\cosh u) = \left(\frac{e^{u}-e^{u}}{2}\right)\frac{du}{dx}$

$= \sinh u \frac{du}{dx}$

✓ • $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$

• $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$ $(\cot u)' = -\csc^2 u$

• $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$

✓ • $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} u \coth u \frac{du}{dx}$

Exp. Find y' for ① $y = \ln(\sinh x)$ $\Rightarrow y' = \frac{\cosh x}{\sinh x} = \coth x$

② $y = 4\cosh\frac{x}{2}$ $\Rightarrow y' = 2\sinh\frac{x}{2}$

* Integrals of Hyperbolic Functions

- $\int \sinh u \, du = \cosh u + c$
- $\int \cosh u \, du = \sinh u + c$
- $\int \operatorname{sech}^2 u \, du = \tanh u + c$
- $\int \operatorname{csch}^2 u \, du = -\coth u + c$
- $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + c$
- $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + c$

Exp: ① $\int \sinh 2x \, dx = \frac{1}{2}\int \sinh u \, du = \frac{1}{2}\cosh 2x + c$

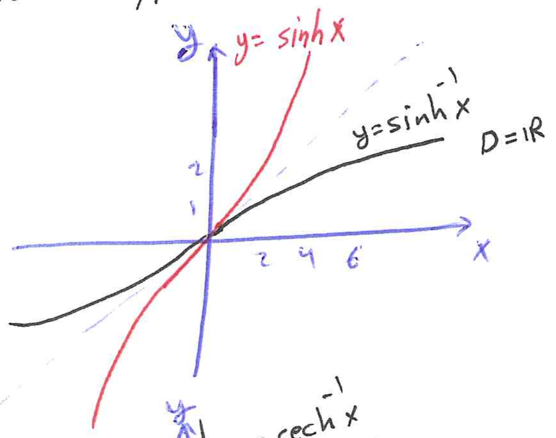
$u = 2x$, $du = 2\,dx$

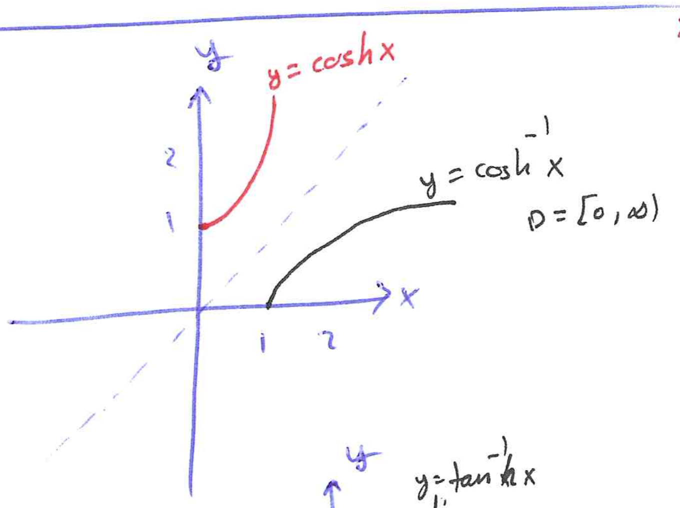
② $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} dx$

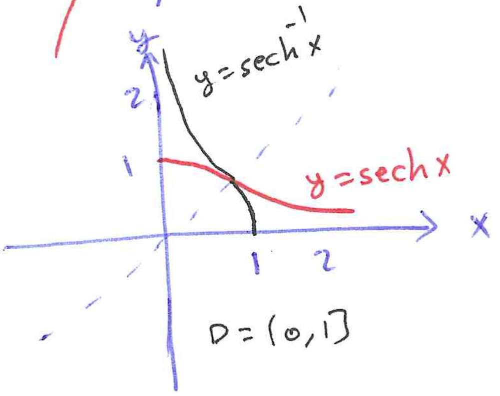
$u = \sinh x$, $du = \cosh x \, dx$

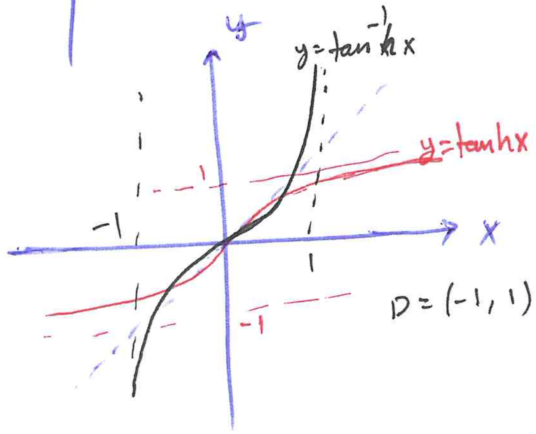
$= \int \frac{du}{u} = \ln|u| + c = \ln|\sinh x| + c$

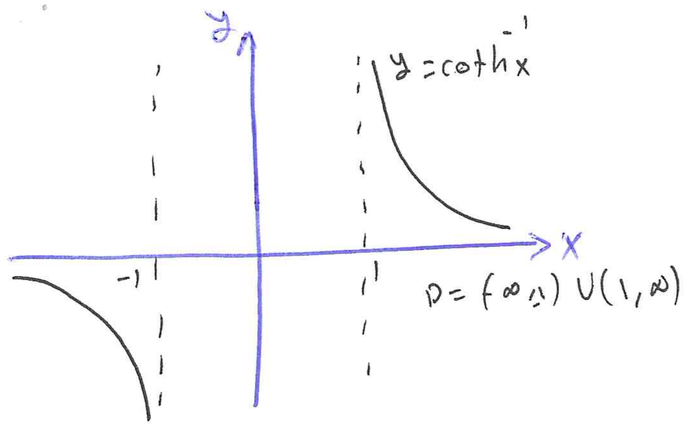
* Inverse Hyperbolic Functions:

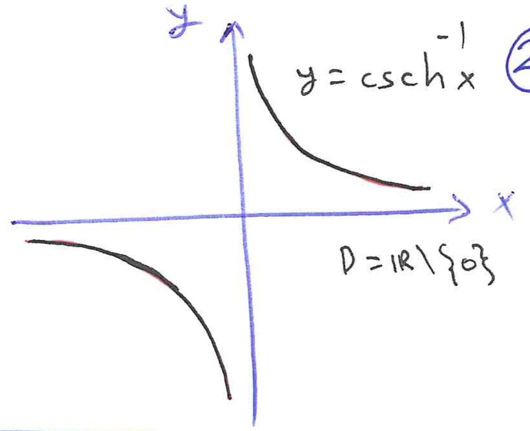












* Identities for inverse hyperbolic functions:

- $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$
- $\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$
- $\coth^{-1} x = \tanh^{-1} \frac{1}{x}$

$\Rightarrow$ Proof $\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$

$= \operatorname{sech}\left(\operatorname{sech}^{-1} x\right)$

* Derivatives of inverse hyperbolic functions

- $\frac{d}{dx}\left(\sinh^{-1} u\right) = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
- $\frac{d}{dx}\left(\cosh^{-1} u\right) = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$, $u > 1$
- $\frac{d}{dx}\left(\tanh^{-1} u\right) = \frac{1}{1-u^2} \frac{du}{dx}$, $|u| < 1$
- $\frac{d}{dx}\left(\coth^{-1} u\right) = \frac{1}{1-u^2} \frac{du}{dx}$, $|u| > 1$
- $\frac{d}{dx}\left(\operatorname{sech}^{-1} u\right) = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}$, $0 < u < 1$
- $\frac{d}{dx}\left(\operatorname{csch}^{-1} u\right) = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}$, $u \neq 0$

$f(x) = \cosh x$
$f^{-1}(x) = \cosh^{-1} x$

Proof $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$

$= \frac{1}{\sinh(\cosh^{-1} x)}$

$= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}}$

$= \frac{1}{\sqrt{x^2-1}}$

* Integrals leading to inverse hyperbolic functions

- $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad a>0$

- $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u>a>0$

- $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u^2<a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u^2>a^2 \end{cases}$

- $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C \quad , \quad 0<u<a$

- $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad , \quad u\neq 0$ and $a>0$

Exp. $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{du}{\sqrt{1+u^2}} = 0$ $\qquad u=\sin x$, $du = \cos x\,dx$

$\displaystyle\int_{\frac{1}{5}}^{\frac{3}{13}} \frac{dx}{x\sqrt{1-16x^2}} = \int_{\frac{4}{5}}^{\frac{12}{13}} \frac{du}{u\sqrt{1-u^2}}$ $\qquad u = 4x$, $du = 4\,dx$

$\displaystyle = -\operatorname{sech}^{-1}(u)\Big|$ $\qquad a=1$, $0<\frac{12}{13}<1$, $0<\frac{4}{5}<1$

$\displaystyle = -\operatorname{sech}^{-1}\left(\frac{12}{13}\right) + \operatorname{sech}^{-1}\left(\frac{4}{5}\right)$
