8.2 Trigonometric Integrals

$$\int \sin^m x \ \cos^n x \ dx$$

Case 1 m is odd $\Rightarrow m = 2k+1$ we use $\sin^2 x = 1 - \cos^2 x$

$$\sin^m x = \sin^{2k+1} x = \left[\sin^2 x\right]^k \sin x = \left[1-\cos^2 x\right]^k \sin x$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

Cas 2 m is even and n is odd $\Rightarrow n = 2k+1$ we use $\cos^2 x = 1 - \sin^2 x$

$$\cos^n x = \cos^{2k+1} x = \left[\cos^2 x\right]^k \cos x = \left[1-\sin^2 x\right]^k \cos x$$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

Case 3 m and n are both even: we use

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Exp $\int \sin^3 x \cos^3 x \, dx = \int \sin^2 x \cos^3 x \sin x \, dx$ $\quad u = \cos x$, $du = -\sin x \, dx$

$= \int (1-\cos^2 x) \cos^3 x \sin x \, dx$

$= -\int (1-u^2)\, u^3 \, du = -\int (u^3 - u^5)\, du = \frac{u^6}{6} - \frac{u^4}{4} + c$

$= \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + c$

Exp $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$ $\quad u = \sin x$, $du = \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \int u^2 (1-u^2)\, du$

$= \int (u^2 - u^4)\, du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + c$

3 Exp $\int 16\sin^2 x\cos^2 x\,dx = 16\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx$

$= 4\int(1-\cos^2 2x)\,dx = 4x - 4\int\cos^2 2x\,dx$

$= 4x - 4\int\left(\frac{1+\cos 4x}{2}\right)dx = 4x - \frac{4}{2}x - 2\int\cos 4x\,dx$

$= 4x - 2x - \frac{1}{2}\sin 4x + c = 2x - \frac{\sin 4x}{2} + c$

Product of sin and cos

$\int \sin mx \sin nx\,dx$

$\int \sin mx \cos nx\,dx$

$\int \cos mx \cos nx\,dx$

$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Exp $\int \cos 3x\cos 4x\,dx = \int \frac{1}{2}\left[\cos(-x) + \cos 7x\right]dx$

$= \frac{1}{2}\int(\cos x + \cos 7x)\,dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + c$

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* Eliminating Square Roots:

Exp $\int_0^{\pi}\sqrt{1-\sin^2 x}\,dx = \int_0^{\pi}\sqrt{\cos^2 x}\,dx = \int_0^{\pi}|\cos x|\,dx$

$= \int_0^{\frac{\pi}{2}}\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos x\,dx$

$= \sin x\Big|_0^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\pi}$

$= \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 1 + 1 = 2$

Power of tanx and secx

Exp* $\int 4\tan^3 x\,dx = 4\int \tan^2 x\,\tan x\,dx$

$= 4\int (\sec^2 x - 1)\tan x\,dx = 4\int \sec^2 x\tan x\,dx - 4\int \tan x\,dx$

$= 2\tan^2 x - 4\int \frac{\sin x}{\cos x}\,dx$

$= 2\tan^2 x + 4\ln|\cos x| + c$

$= 2\tan^2 x - 2\ln|\sec^2 x| + c$

$= 2\tan^2 x - 2\ln(1+\tan^2 x) + c$

Exp $\int \sec^4 x\,dx = \int \sec^2 x\,\sec^2 x\,dx$

$= \int (1+\tan^2 x)\sec^2 x\,dx$

$u = \tan x$
$du = \sec^2 x\,dx$

$= \int (1+u^2)\,du$

$= u + \frac{u^3}{3} + c$

$= \tan x + \frac{1}{3}\tan^3 x + c$

$= \tan x + \frac{1}{3}[\sec^2 x - 1]\tan x + c$

$= \tan x - \frac{1}{3}\tan x + \frac{1}{3}\sec^2 x\tan x + c$

$= \frac{2}{3}\tan x + \frac{1}{3}\sec^2 x\tan x + c.$
