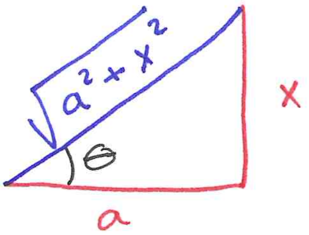
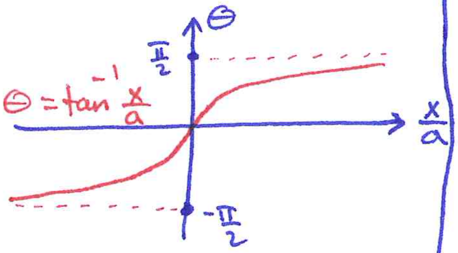
$\boxed{8.3}$ Trigonometric Substitution



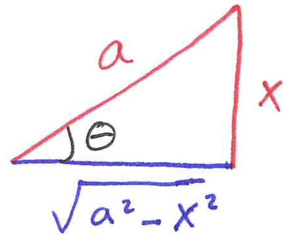
$\boxed{x = a\tan\theta}$ requires

$\theta = \tan^{-1}\frac{x}{a}$ with

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ♡[1]



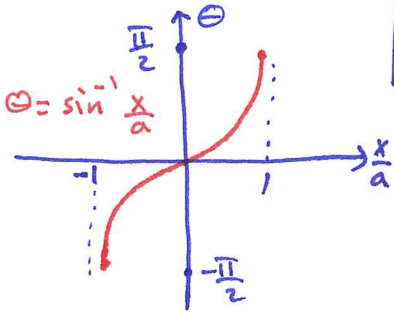
$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$$
$$= a\sqrt{1+\tan^2\theta}$$
$$= a\sqrt{\sec^2\theta}$$
$$= a\,|\sec\theta| \quad ♡^1$$
$$= a\sec\theta$$



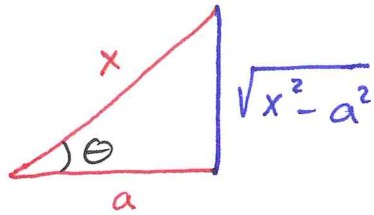
$\boxed{x = a\sin\theta}$ requires

$\theta = \sin^{-1}\frac{x}{a}$ with

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ ♡[2]



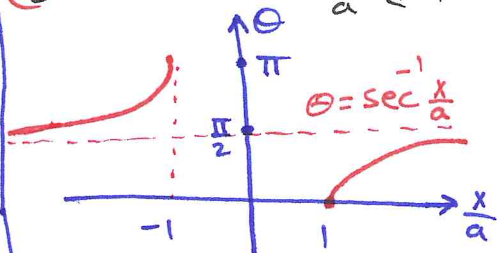
$$\bullet\ \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$$
$$= a\sqrt{1-\sin^2\theta}$$
$$= a\sqrt{\cos^2\theta}$$
$$= a\,|\cos\theta| \quad ♡^2$$
$$= a\cos\theta$$



$\boxed{x = a\sec\theta}$ requires

$\theta = \sec^{-1}\frac{x}{a}$ with

$$\begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1 \end{cases} \quad ♡^3$$

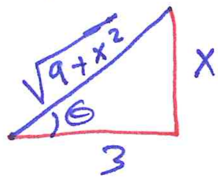


$$\bullet\ \sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$$
$$= a\sqrt{\sec^2\theta - 1}$$
$$= a\sqrt{\tan^2\theta}$$
$$= a\,|\tan\theta| \quad ♡^3$$
$$= a\tan\theta$$

Exp $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$ $\quad x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{|\sec\theta|}\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C = \ln\left|\sqrt{9+x^2} + x\right| + C.$$

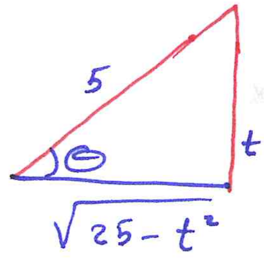
Exp $\int \sqrt{25-t^2}\,dt$ $\quad t = 5\sin\theta \Rightarrow dt = 5\cos\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \sqrt{25-t^2}\,dt = \int \sqrt{25-25\sin^2\theta}\ 5\cos\theta\,d\theta$$

$$= 25\int \sqrt{1-\sin^2\theta}\ \cos\theta\,d\theta = 25\int \sqrt{\cos^2\theta}\ \cos\theta\,d\theta$$

$$= 25\int \cos^2\theta\,d\theta = \frac{25}{2}\int (1+\cos 2\theta)\,d\theta = \frac{25}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{25}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$



$$= \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \frac{t}{5}\,\frac{\sqrt{25-t^2}}{5}\right] + C$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$$

Exp $\int \frac{\sqrt{y^2-25}}{y^3}\,dy$ $\quad y = 5\sec\theta \Rightarrow dy = 5\sec\theta\tan\theta\,d\theta$

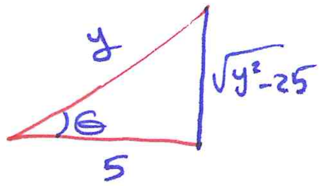
$\boxed{0 < \theta < \frac{\pi}{2}}$

$$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\ 5\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{5}\int \frac{\sqrt{\sec^2\theta - 1}}{\sec^2\theta}\tan\theta\,d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{10}\int (1-\cos 2\theta)\,d\theta = \frac{1}{10}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{10}\left[\theta - \sin\theta\cos\theta\right] + C$$



$$= \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y}\left(\frac{5}{y}\right)\right] + C$$

Exp $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$ $\quad x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta \, d\theta$ (38)

$$\boxed{-\frac{\pi}{4} < \theta < \frac{\pi}{4}}$$

when $x = -2 \Rightarrow \theta = \tan^{-1}(-1) = -\frac{\pi}{4}$

$x = 2 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\int_{-2}^{2} \frac{dx}{4+x^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta \, d\theta}{4+4\tan^2\theta} = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta} \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi}{4}$$

Note that $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_{-2}^{2}$

$$= \frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}-1\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - -\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$