8.4 Integration of Rational Functions By Partial Fractions

Exp ① $\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x+6)(x-1)}\,dx$

Heaviside "cover up" method

$$\frac{x+4}{\underbrace{(x+6)}_{A}\underbrace{(x-1)}_{B}} = \frac{A}{x+6} + \frac{B}{x-1}$$

$$x+4 = A(x-1) + B(x+6)$$
$$= (A+B)x + 6B - A$$

$A+B=1$, $-A+6B=4$ $\Rightarrow$ $A=\frac{2}{7}$, $B=\frac{5}{7}$

$$\int \frac{x+4}{(x+6)(x-1)}\,dx = \int \frac{\frac{2}{7}}{x+6}\,dx + \int \frac{\frac{5}{7}}{x-1}\,dx$$
$$= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$$
$$= \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

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* The partial fraction Method: is a method for writing $\frac{f(x)}{g(x)}$ "rational functions" as a sum of simpler fractions.

* The Heaviside "cover up" method can be used when $g(x)$ can be written as product of distinct linear factors.
  (distinct → non repeated)

* The degree of f must be less than the degree of g.
  If not we use long division:

Exp $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

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$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

cover up method

$$A = \frac{1+4+1}{(2)(4)} = \frac{6}{(2)(4)} = \frac{3}{4}$$

$$B = \frac{1-4+1}{(-2)(2)} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = \frac{9-12+1}{(-4)(-2)} = \frac{-2}{8} = \frac{-1}{4}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{4}}{x+3}\right) dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Exp $\displaystyle\int \frac{dx}{x^3+x^2-2x} = \int \frac{dx}{x(x^2+x-2)} = \int \frac{dx}{x(x+2)(x-1)}$

"cover up"

$$\frac{1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$A = \frac{-1}{2}$

$B = \frac{1}{6}$

$C = \frac{1}{3}$

$$\int \frac{dx}{x(x+2)(x-1)} = \int \left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{6}}{x+2} + \frac{\frac{1}{3}}{x-1}\right) dx$$

$$= -\frac{1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

Exp $\displaystyle\int \frac{x^3}{x^2+2x+1}\,dx = \int\left(x-2+\frac{3x+2}{x^2+2x+1}\right)dx$

$$\begin{array}{r} x-2 \\ x^2+2x+1 \,\overline{)\, x^3 \qquad\qquad} \\ \underline{x^3+2x^2+x} \\ -2x^2-x \\ \underline{-2x^2-4x-2} \\ 3x+2 \end{array}$$

$$= \int (x-2)\,dx + \int \frac{3x+2}{x^2+2x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + \int \frac{3x+2}{(x+1)^2}$$

Repeated linear factor

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$3x+2 = A(x+1) + B$$
$$= Ax + A + B$$

$\boxed{A=3}$ $\boxed{B=-1}$

$$= \frac{x^2}{2} - 2x + \int\left(\frac{3}{x+1} - \frac{1}{(x+1)^2}\right)dx$$

$$= \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + c$$

Exp (irreducible Quadratic Factors) + repeated linear Factor) $\displaystyle\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx$

$$\frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{Ax+B}{x^2+1} + \frac{C(x-1)+D}{(x-1)^2}$$

$$4-2x = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$
$$= (Ax+B)(x^2-2x+1) + (x-1)(x^2+1) + D(x^2+1)$$
$$4-2x = A(x^3-2x^2+x) + B(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$A+C=0$, $-2A+B-C+D=0$, $-2=A-2B+C$, $4=B-C+D$

$\boxed{A=2}$, $\boxed{B=1}$, $\boxed{C=-2}$, $\boxed{D=1}$

$$\int \frac{4-2x\;dx}{(x^2+1)(x-1)^2} = \int \frac{2x+1}{x^2+1}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1} - 2\ln|x-1| - \frac{1}{x-1} + C$$

$\ln(x^2+1)$ $\qquad \tan^{-1}x$

Exp $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

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$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A}{x} + \frac{(Bx+C)(x^2+1)+(Dx+E)}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x$$

$$* \quad 1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$

$A+B=0$, $\boxed{C=0}$, $2A+B+D=0$, $C+E=0$, $\boxed{A=1}$

$\boxed{B=-1}$

$\boxed{D=-1}$

$\boxed{E=0}$

$$\int \frac{dx}{x(x^2+1)^2} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\right) dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C$$

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* can be differentiated to find the coefficients

- $A=1$ by cover up method
- $0 = A(4x^3+4x) + B(4x^3+2x) + C(3x^2+1) + 2Dx + E$ $\qquad \boxed{E+C=0}$

$0 = A(12x^2+4) + B(12x^2+2) + C(6x) + 2D$ $\qquad 4A+2B+2D=0$, $\boxed{2A+B+D=0}$

$0 = A(24x) + B(24x) + 6C$ $\qquad \boxed{C=0}$

$A+B=0$