$\boxed{8.7}$ Improper Integrals (Type I and Type II)

Def: Improper Integrals of Type I are integrals with infinite limits of integration:

$\boxed{1}$ $$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$ where f is continous on $[a,\infty)$

$\boxed{2}$ If f is continuous on $(-\infty, a)$, then

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{b\to-\infty} \int_b^a f(x)\,dx$$

$\boxed{3}$ If f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx \quad , c \in \mathbb{R}$$

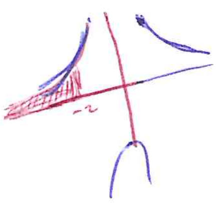
In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area"

otherwise the improper integral diverges "infinit Area"

Exp ① $$\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \tan^{-1}x \Big|_0^b = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}$$

② $$\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{2\,dx}{(x-1)(x+1)}$$

$$\frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$ $A=1$, $B=-1$



$$= \int_{-\infty}^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to-\infty} \int_b^{-2} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \lim_{b\to-\infty} \left(\ln|x-1| - \ln|x+1| \right)\Big|_b^{-2}$$

$$= \lim_{b\to-\infty} \left[\ln\left|\frac{x-1}{x+1}\right| \right]\Big|_b^{-2} = \lim_{b\to-\infty} \left[\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right| \right]$$

$$= \ln 3 - \ln \lim_{b\to-\infty} \frac{b-1}{b+1} = \ln 3 - \ln 1 = \ln 3$$

Note that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Exp $\displaystyle\int_0^{\infty} \frac{16\tan^{-1}x}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^{b} \frac{16\tan^{-1}x}{1+x^2}\,dx$

$u = \tan^{-1}x$

$du = \dfrac{1}{1+x^2}\,dx$

$$= \lim_{b\to\infty} \int_0^{\tan^{-1}b} 16u\,du = \lim_{b\to\infty} 8u^2\Big|_0^{\tan^{-1}b}$$

$$= \lim_{b\to\infty} 8\left[(\tan^{-1}b)^2 - (\tan^{-1}0)^2 \right] = 8\left(\frac{\pi}{2}\right)^2 = 2\pi^2$$

Def Improper Integrals of Type II are integrals of functions that become infinite at a point within the interval of integration (Vertical Asymptotes)

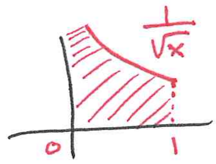
1. If f is discontinuous at a, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$

2. If f is discontinuous at b, then $\displaystyle\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$

3. If f is discontinuous at c, where $a<c<b$, then $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

In each case, if the limit is finite, then the improper integral converges and it is equal to this limit "Area".
Otherwise the improper integral diverges "Infinite area"

Exp $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c\to 0^+} 2\sqrt{x}\Big|_c^1$



$\displaystyle = \lim_{c\to 0^+} \left(2 - 2\sqrt{c}\right) = 2 - 0 = 2$

Exp $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} \int_0^b \frac{dx}{\sqrt{4-x}} = \lim_{b\to 4^-} -2\sqrt{4-x}\Big|_0^b$

$\displaystyle = \lim_{b\to 4^-} \left[-2\sqrt{4-b} + 4\right] = 4$

Exp* For what values of p does the integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converge?

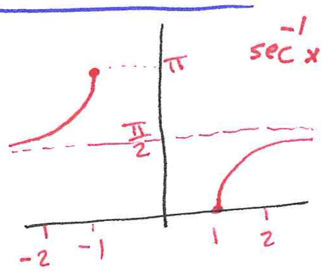
$\displaystyle p \neq 1 \Rightarrow \int_1^\infty \frac{dx}{x^p} = \lim_{c\to\infty} \int_1^c x^{-p}\,dx = \lim_{c\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^c$

$\displaystyle = \lim_{c\to\infty} \left[\frac{c^{1-p}}{1-p} - \frac{1}{1-p}\right] = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$

diverges

$\displaystyle \boxed{p=1} \Rightarrow \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty} \ln|b| = \infty$

Exp $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \int_1^2 \frac{dx}{x\sqrt{x^2-1}} + \int_2^\infty \frac{dx}{x\sqrt{x^2-1}}$



$\displaystyle = \lim_{b\to 1^+} \int_b^2 \frac{dx}{x\sqrt{x^2-1}} + \lim_{c\to\infty} \int_2^c \frac{dx}{x\sqrt{x^2-1}}$

$\displaystyle = \lim_{b\to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim_{c\to\infty} \left[\sec^{-1}|x|\right]_2^c$

$\displaystyle = \lim_{b\to 1^+} \left[\underbrace{\cancel{\sec^{-1} 2}}_{\frac{\pi}{3}} - \sec^{-1} b\right] + \lim_{c\to\infty} \left[\sec^{-1} c - \underbrace{\cancel{\sec^{-1} 2}}_{\frac{\pi}{3}}\right]$

$\displaystyle = \lim_{b\to 1^+} -\sec^{-1} b + \lim_{c\to\infty} \sec^{-1} c$

$\displaystyle = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

Tests for Convergence and Divergence

Th (Direct Comparasion Test)

Let f and g be continuous on $[a, \infty)$ with

$$0 \le f(x) \le g(x) \text{ for all } x \ge a.$$

* If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
* If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Exp $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because $0 \le \underbrace{\frac{\sin^2 x}{x^2}}_{f(x)} \le \underbrace{\frac{1}{x^2}}_{g(x)}$ on $[1, \infty)$

and $\int_1^\infty \frac{1}{x^2}$ converges "see exp *"

Thus, by the Direct Comparasion Test $\int_1^\infty \frac{\sin^2}{x^2}\,dx$ converges

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Exp $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because $\underbrace{\frac{1}{\sqrt{x^2 - 0.1}}}_{g(x)} \ge \underbrace{\frac{1}{x}}_{f(x)}$ on $[1, \infty)$

and $\int_1^\infty \frac{1}{x}$ diverges "see exp *"

Thus, by the Direct Comparision Test, $\int_1^\infty \frac{dx}{\sqrt{x^2 - 0.1}}$ diverges

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Exp $\int_0^\pi \frac{dt}{\sqrt{t} + \sin t}$ converges because $0 \le \underbrace{\frac{1}{\sqrt{t} + \sin t}}_{f(x)} \le \underbrace{\frac{1}{\sqrt{t}}}_{g(t)}$ on $[0, \pi]$

and $\int_0^\pi \frac{dt}{\sqrt{t}} = \lim_{b \to 0^+} \int_b^\pi t^{-\frac{1}{2}}\,dt = \lim_{b \to 0^+} \left. 2\sqrt{t}\, \right|_b^\pi = \lim_{b \to 0^+} 2\sqrt{\pi} - 2\sqrt{b}$

$= 2\sqrt{\pi}$

which converges, Thus, by the Direct Comparasion test, $\int_0^\pi \frac{dt}{\sqrt{t} + \sin t}$ converges.

2 Th (limit Comparision Test)

If • f and g are positive and continous functions on $[a,\infty)$

• $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$,

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or both diverge.

Exp $\int_1^\infty \frac{dx}{1+x^2}$

• let $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{1+x^2}$

• f and g are positive continuous on $[1,\infty)$

• $\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{x^2} = 1$ "finite +"

and $\int_1^\infty \frac{dx}{x^2}$ $= \frac{1}{2-1} = 1$ converges "see exp*" . Thus, by the Limit Comparasion Test

$\Rightarrow \int_1^\infty \frac{dx}{1+x^2}$ converges. $\frac{\pi}{4}$

Exp $\int_2^\infty \frac{dx}{\sqrt{x-1}}$

• Let $f(x) = \frac{1}{\sqrt{x-1}}$, $g(x) = \frac{1}{\sqrt{x}}$

• f and g are positive continuous on $[2,\infty)$

• $\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x-1}}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x}}} = 1$

and $\int_2^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_2^b x^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} 2\sqrt{x}\Big|_2^b = \lim_{b\to\infty} \left[2\sqrt{b} - 2\sqrt{2}\right] = +\infty$

which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x-1}}$ diverges by the limit Comparasion Test.