$\boxed{A.7}$ Complex Numbers

* Complex numbers have the form $a + ib = (a, b)$
  where
  - a and b are real numbers
  - a is called the real part
  - b is called the Imaginary part
  - $i = \sqrt{-1}$

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*1 Natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ = positive integers

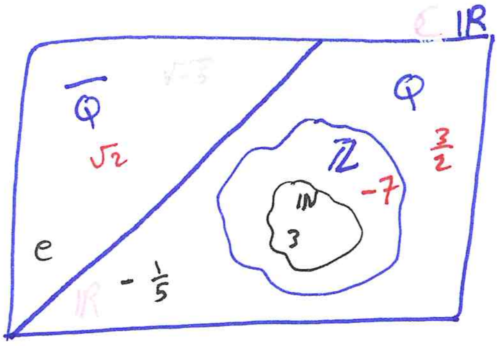
*2 Integer numbers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

*3 Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ with } n \neq 0\}$

*4 Irrational numbers $\overline{\mathbb{Q}}$

*5 Real numbers $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$

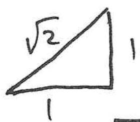
*6 Complex numbers $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$



$\mathbb{C}$ $\sqrt{-5}$

Note that:

- $\mathbb{N}$ is closed under $+$ and $\times$
- $\mathbb{Z}$ is closed under $+, -, \times$
- $\mathbb{Q}$ is closed under $+, -, \times, \div$ except division by zero
- There are some numbers that are not in $\mathbb{Q}$.
  $\overline{\mathbb{Q}}$ contains all such numbers like $\pm\sqrt{2}, \pm\sqrt{3}, \dots, e, \dots$



$x^2 = \sqrt{2}$
How to solve

$\Rightarrow$ We can have a sequence of rational numbers

$$\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \dots$$

whose squares form a sequence

$$\frac{1}{1}, \frac{49}{25}, \frac{1681}{841}, \frac{57121}{28561}, \dots \quad \text{converges to } 2$$

$L^2 = 2$ and $L \notin \mathbb{Q}$

- Hence Real numbers $\mathbb{R}$ includes rational numbers and the limits of an increasing bounded of rational numbers.

• The complex numbers contain the solution of equations like $x^2+1=0$

* Properities of $\mathbb{C}$: $i=\sqrt{-1}$, $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i, \dots$

[1] Equality : $a+ib = c+id \iff a=c$ and $b=d$

[2] Addition : $(a+ib)+(c+id) = (a+c)+i(b+d)$

[3] Multiplication: $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

• $c(a+ib) = ac + i(bc)$

[4] Division : If $a+ib \neq 0$, then

$$\frac{c+id}{a+ib} = \frac{c+id}{a+ib} \cdot \frac{a-ib}{\boxed{a-ib}} = \frac{(ac+bd)+i(ad-bc)}{a^2+b^2}$$

$$= \frac{ac+bd}{a^2+b^2} + i\,\frac{(ad-bc)}{(a^2+b^2)}$$

is called the complex conjugate of $a+ib$

Exp [1] $(5+2i)+(3-4i) = (5+3)+i(2-4) = 8-2i$

[2] $(5+2i)(3-4i) = 15-20i+6i-8i^2$
$= 15+8-14i = 23-14i$

[3] $\frac{5+2i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{15+20i+6i+8i^2}{9+16} = \frac{7+26i}{25}$
$= \frac{7}{25} + i\,\frac{26}{25}$

[4] $\overline{5+2i} = 5-2i$

[5] $(\overline{5+2i})(5+2i) = (5-2i)(5+2i) = 25-4i^2 = 29$

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* When the imaginary part in complex numbers is zero "$b=0$", then the complex numbers have all properities of real numbers.

* Recall that $a+ib = (a,b)$

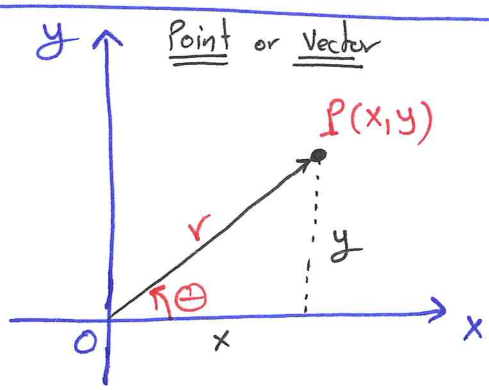
$\Rightarrow$ The complex number $(0,0)$: $(0,0)\cdot(a,b) = (0,0)$

$\Rightarrow$ The complex number $(1,0)$: $(1,0)\cdot(a,b) = (a,b)$

$\Rightarrow$ The complex number $(0,1) = i$: $(0,1)\cdot(0,1) = (-1,0) = -1$

so $(0,1)^2 + (1,0) = (0,0)$

Argand Diagrams: Using this diagram



* we can represent the complex number $z = x + iy$ in the complex plane where:

→ x-axis is the real axis
→ y-axis is the imaginary axis.
→ θ is called the polar angle

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

→ r is the length of the vector $\vec{OP}$ which is defined as the Absolute value of the complex number z:

$$r = |x+iy| = \sqrt{x^2+y^2}$$

→ θ is the argument of z and is written as $\theta = \arg z$

- $\theta + 2\pi m$ is also an argument of the complex number z
- Note that $\boxed{z\cdot\bar{z} = |z|^2}$
  - Exp Let $z = 1+2i \Rightarrow \bar{z} = 1-2i$
    $\Rightarrow z\cdot\bar{z} = (1+2i)(1-2i) = 1-4i^2 = 5$
    $\Rightarrow |z| = \sqrt{1+4} = \sqrt{5} \Rightarrow |z|^2 = 5$ ✓

→ The complex number:

$$\left.\begin{aligned} z = x+iy &= r\cos\theta + i(r\sin\theta) \\ &= r(\cos\theta + i\sin\theta) \end{aligned}\right\} *$$

Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

This comes from $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

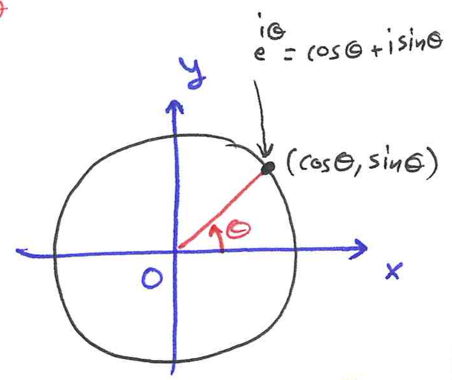
using $i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \cos\theta + i\sin\theta$$

- Now $*$ becomes $z = x + iy = re^{i\theta}$

→ Make $r = 1$ ⇒ means that the complex number z is a unit vector $e^{i\theta}$ that makes an angle θ with positive x-axis



Argand diagram for $e^{i\theta} = \cos\theta + i\sin\theta$ as a point.

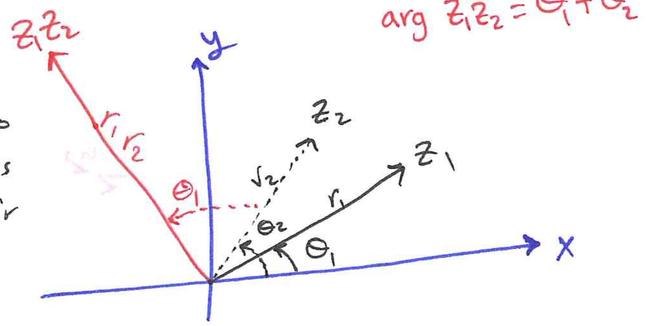
Products of complex numbers: $z_1 = r_1 e^{i\theta_1}$
$z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$, where $r_1 = |z_1|$, $r_2 = |z_2|$

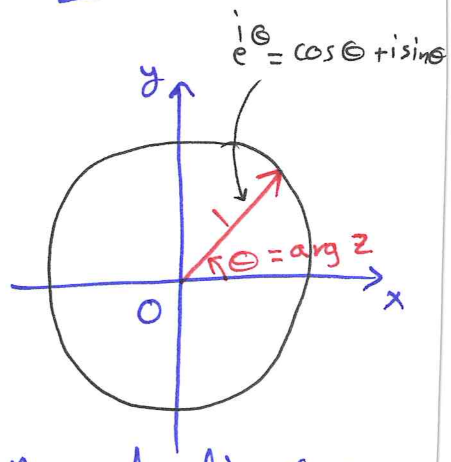
Hence,

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$$

$\arg z_1 z_2 = \theta_1 + \theta_2$



To multiply two complex numbers we multiply their absolute values and add their arguments.



Argand diagram for $e^{i\theta} = \cos\theta + i\sin\theta$ as a vector

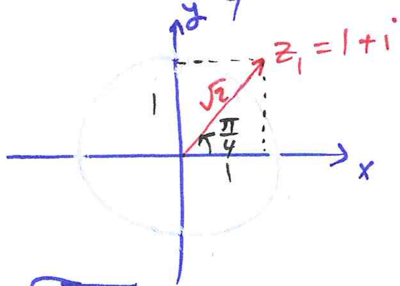
Exp Plot the following complex numbers in an Argand diagram:

1. $z_1 = 1 + i \Rightarrow r_1 = |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$ The polar angle $\theta = \frac{\pi}{4}$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$

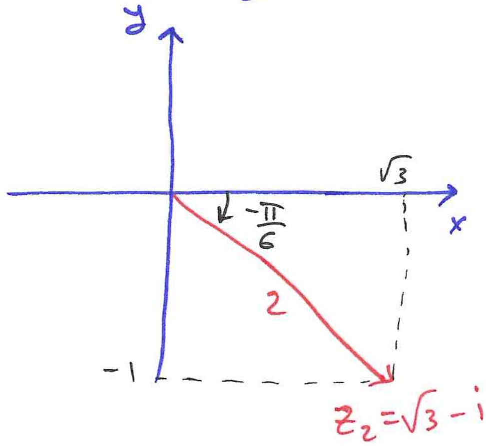


2. $z_2 = \sqrt{3} - i \Rightarrow r_2 = |\sqrt{3} - i| = \sqrt{3+1} = \sqrt{4} = 2$

$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$ The polar angle $\theta = -\frac{\pi}{6}$

$= 2\left(\cos\frac{-\pi}{6} + i \sin\left(-\frac{\pi}{6}\right)\right)$

$= 2\, e^{-i\frac{\pi}{6}}$



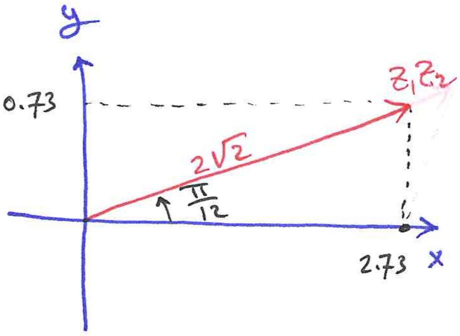
3. $z_1 z_2 = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)\left(2\, e^{-i\frac{\pi}{6}}\right)$

$= 2\sqrt{2}\, e^{i\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}$

$= 2\sqrt{2}\, e^{i\frac{\pi}{12}}$

$= 2\sqrt{2}\left[\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right]$

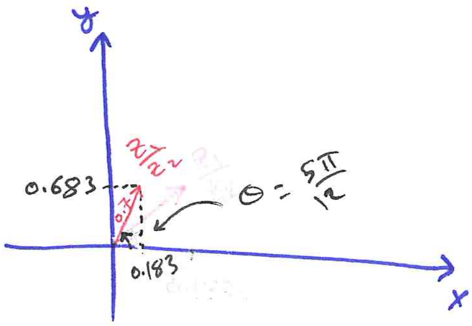
$\approx 2.73 + 0.73 i$



4. $\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2\, e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}}\, e^{i\frac{5\pi}{12}}$

$= 0.707\left[\cos\frac{5\pi}{12} + i \sin\frac{5\pi}{12}\right]$

$\approx 0.183 + 0.683 i$



Powers of complex numbers

Recall the complex number $z = r e^{i\theta}$

$$z^n = \left(r e^{i\theta}\right)^n = r^n e^{in\theta}$$

Note that $e^{in\theta} = \left(e^{i\theta}\right)^n$

$$\boxed{\cos n\theta + i \sin n\theta = \left(\cos\theta + i\sin\theta\right)^n}$$

De Moivre's Theorem

Exp $z = 1 + \sqrt{3}\, i \Rightarrow r = |1+\sqrt{3}\, i| = \sqrt{1+3} = \sqrt{4} = 2$

$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) \Rightarrow \theta = \frac{\pi}{3}$

$= 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

$= 2 e^{i\frac{\pi}{3}}$

Thus, $z^6 = \left(2 e^{i\frac{\pi}{3}}\right)^6$

$= 2^6 e^{i2\pi}$

$= 64\left(\cos 2\pi + i \sin 2\pi\right)$

$= 64 (1 + 0)$

$= 64$

Roots of complex numbers:

If $z = r e^{i\theta}$, then $\sqrt[n]{z} = \sqrt[n]{r e^{i\theta}} = \sqrt[n]{r}\, e^{i\frac{\theta}{n}}$

$= \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right)$

$= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$

$k = 0, \pm 1, \pm 2, \ldots$

Th "Fundemental Theorem of Algebra"

Every polynomial of degree n has exactly n roots.

Exp Find the four fourth roots of -16

$$-16+0i \Rightarrow r = |-16+0i| = \sqrt{(-16)^2+0^2} = 16$$

$$= 16(-1+0i) \Rightarrow \text{The polar angle } \theta = \pi$$

$$= 16(\cos\pi + i\sin\pi) \qquad z^4 = -16 \Leftrightarrow z = (-16)^{\frac{1}{4}}$$

Thus, $z = (-16+0i)^{\frac{1}{4}}$

$$= (16)^{\frac{1}{4}}\left[\cos(\pi+2\pi m) + i\sin(\pi+2\pi m)\right]^{\frac{1}{4}}$$

$$= 2\left(\cos\left(\frac{\pi}{4}+\frac{\pi m}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi m}{2}\right)\right)$$

The four roots are when $m = 0, 1, 2, 3$

when $m=0 \Rightarrow w_0 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = 2\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = \sqrt{2}+\sqrt{2}i$

$m=1 \Rightarrow w_1 = 2\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right] = 2\left[\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = -\sqrt{2}+\sqrt{2}i$

$m=2 \Rightarrow w_2 = 2\left[\cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right)\right] = 2\left[\frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = -\sqrt{2}-\sqrt{2}i$

$m=3 \Rightarrow w_3 = 2\left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)\right] = 2\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = \sqrt{2}-\sqrt{2}i$