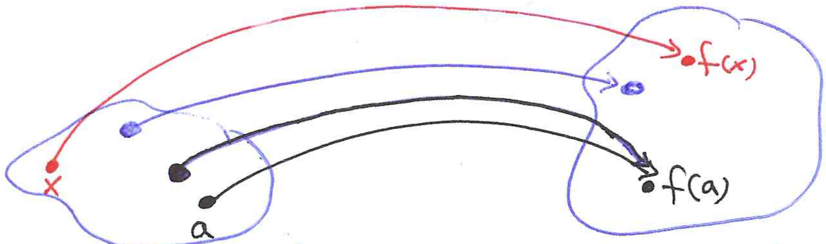
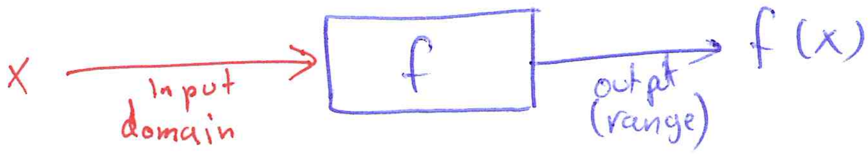
1.1 Functions and Their Graphs

Def: A function f from a set D to a set Y is a rule that assignes a unique (single) element $f(x) \in Y$ to each element $x \in D$.



D = domain set : the largest set of real x-values that gives real y-values.

Y = set contains the range



$y = f(x)$

x is independent variable
y is dependent variable

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* A function whose range is a set of real numbers is called real-valued function.

<u>Example</u>: Find the natural domain and range of the following functions

1. $y = x^2$ $\quad D = (-\infty, \infty)$ $\quad R = [0, \infty)$
2. $y = \frac{1}{x}$ $\quad D = (-\infty, 0) \cup (0, \infty) = R$
3. $y = \sqrt{x}$ $\quad D = [0, \infty)$ i.e $x \geq 0$ $\quad R = [0, \infty)$
4. $y = \sqrt{4-x}$ $\quad D = (-\infty, 4]$ i.e $4 - x \geq 0$ $\quad R = [0, \infty)$
5. $y = \sqrt{1-x^2}$ $\quad D = [-1, 1]$ i.e $1 - x^2 \geq 0$ $\quad R = [0, 1]$

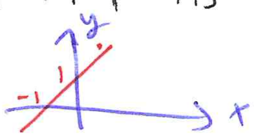
$1 \geq x^2$

$1 \geq |x|$

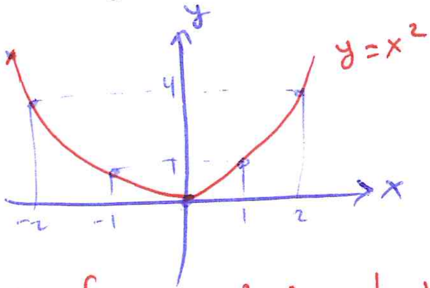
Graphs of functions

The graph of a function f whose domain is D, consistes of the points in the Cartesian plane whose coordinates are the (input, output) pairs for f i.e: $\{(x, f(x)) \mid x \in D\}$

Example: The graph of $f(x) = x+1$ is the set of points with coordinates $(x, x+1)$

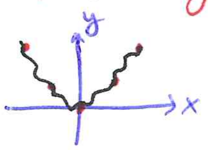


Example: Graph the function $y = x^2$ over the interval $[-2, 2]$

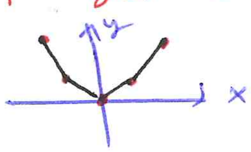


x	$y = x^2$
-2	4
-1	1
0	0
1	1
2	4

Question: why the graph of $y = x^2$ is not like

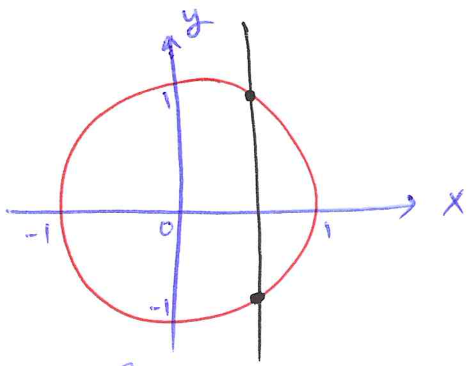


or



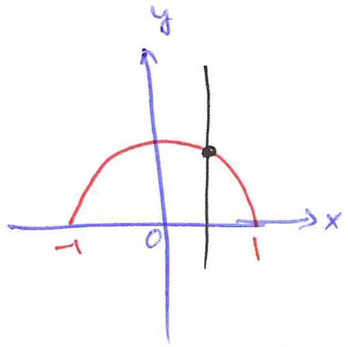
We will learn derivatives in Ch 3

Vertical line Test for a function



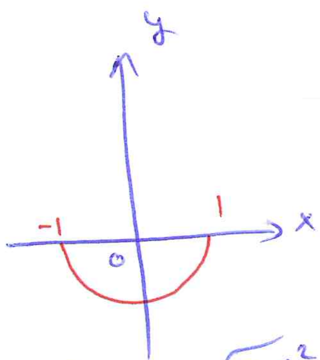
ⓐ $x^2 + y^2 = 1$

The circle is not a graph of a function. It fails the vertical line test.



ⓑ $y = \sqrt{1-x^2}$

The upper semicircle is the graph of a function

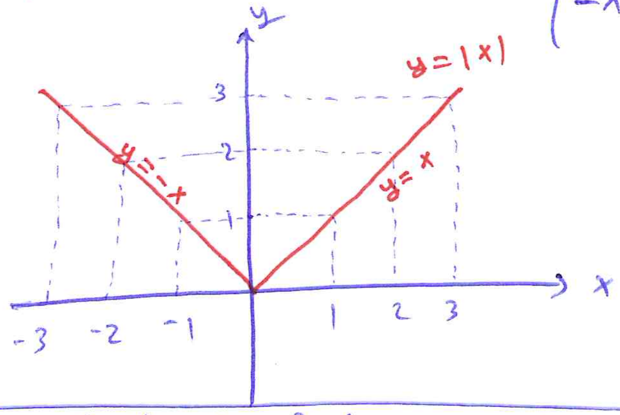


ⓒ $y = -\sqrt{1-x^2}$

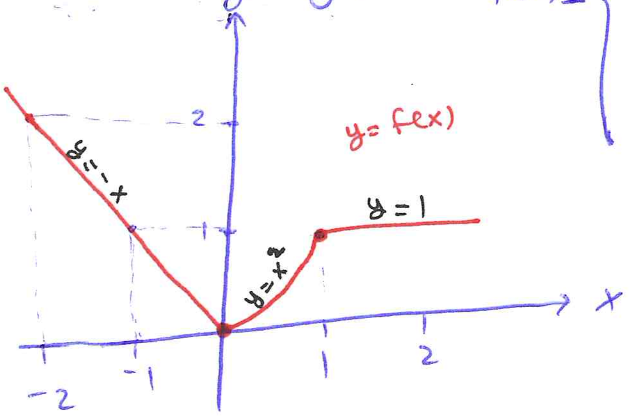
The lower semicircle is the graph of a function

Piecewise Defined Functions

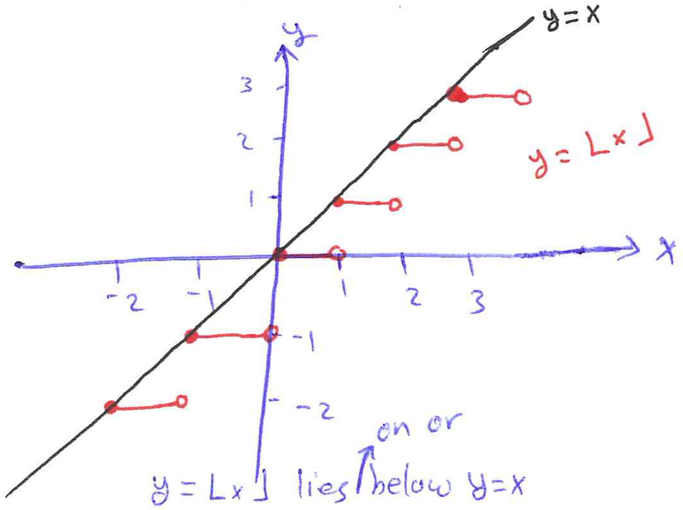
Examples 1 Absolute value function $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$



2 Consider the following function $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$



3 The greatest integer function: is a function whose value at any number x is the greatest integer less than or equal to x. It is also called the integer floor function. It is denoted by $\lfloor x \rfloor$.



$\lfloor 1.5 \rfloor = 1$

$\lfloor 1.9 \rfloor = 1$

$\lfloor 1.3 \rfloor = 1$

$\lfloor 0.2 \rfloor = 0$

$\lfloor 0 \rfloor = 0$

$\lfloor -1.2 \rfloor = -2$

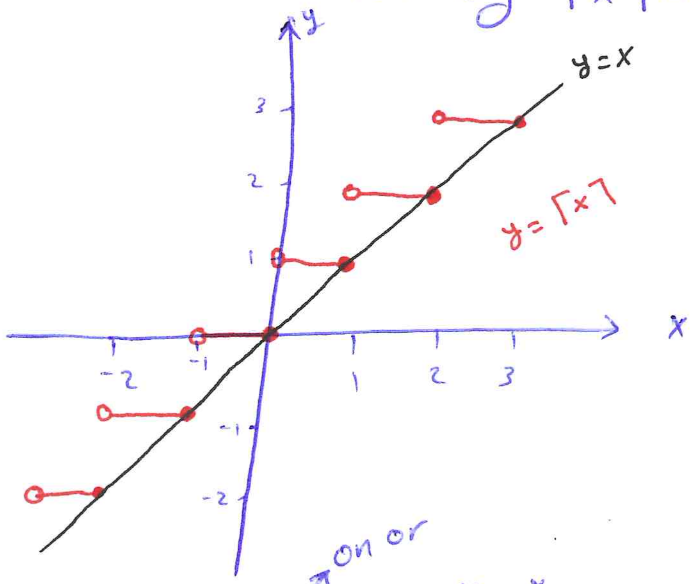
$\lfloor -0.3 \rfloor = -1$

$\lfloor 5 \rfloor = 5$

4 The least integer function: is a function whose value at any number x is the smallest integer greater than or equal to x.

$\Rightarrow$ It is also called the **integer ceiling function**.

$\Rightarrow$ It is denoted by $\lceil x \rceil$.



$\lceil 1.5 \rceil = 2$

$\lceil 1.9 \rceil = 2$

$\lceil 1.3 \rceil = 2$

$\lceil 0.2 \rceil = 1$

$\lceil 0 \rceil = 0$

$\lceil -1.2 \rceil = -1$

$\lceil -0.3 \rceil = 0$

$\lceil 5 \rceil = 5$

$y = \lceil x \rceil$ lies on or above $y = x$

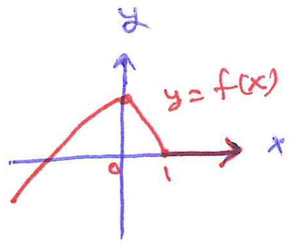
Increasing and decreasing functions

Def: Let f be a function defined on an interval I.
Let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is an increasing on I

2. If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is a decreasing on I.

Example: The function y is an increasing on $(-\infty, 0]$ and decreasing on $[0,1]$. The function is neither increasing nor decreasing on $[1, \infty)$.



Even and Odd Functions

* A function $y=f(x)$ is even if $f(-x)=f(x)$ for every x in the domain of f

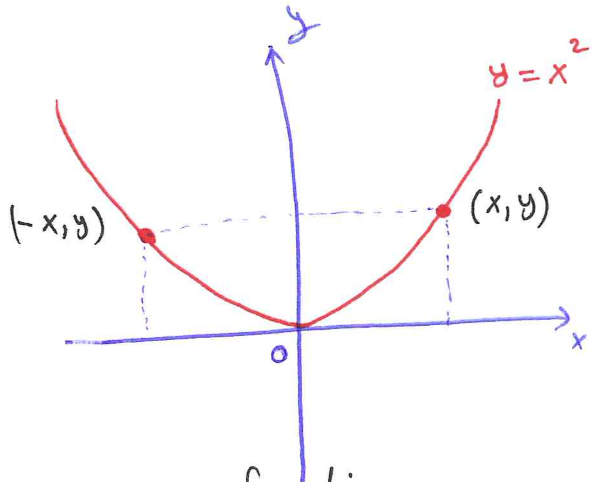
* A function $y=f(x)$ is odd if $f(-x)=-f(x)$ for every x in the domain of f.

Example : $f(x)=x^2$ is even because $f(-x)=(-x)^2=x^2=f(x)$

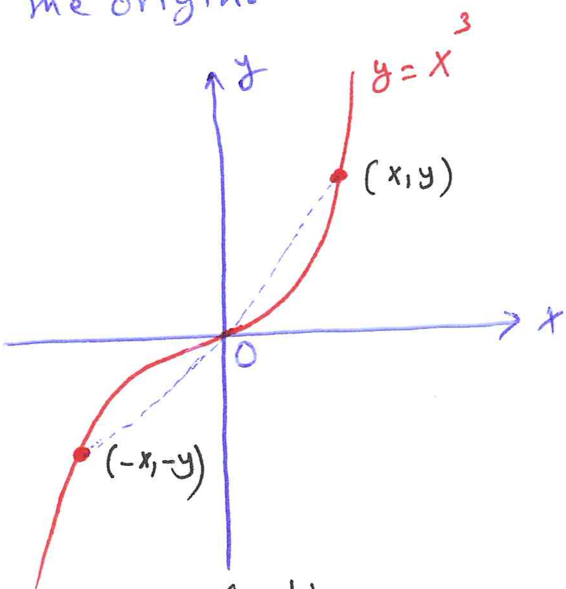
$f(x)=x^3$ is odd because $f(-x)=(-x)^3=-x^3=-f(x)$

Note that * the graph of an even function is symmetric about the y-axis

* the graph of an odd function is symmetric about the origin.



Even function symmetric about y-axis



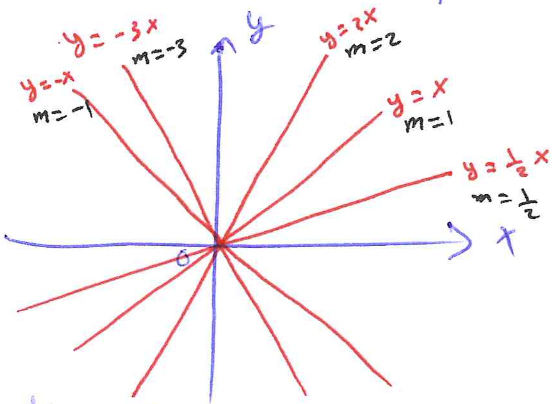
Odd function symmetric about the origin

Common Functions

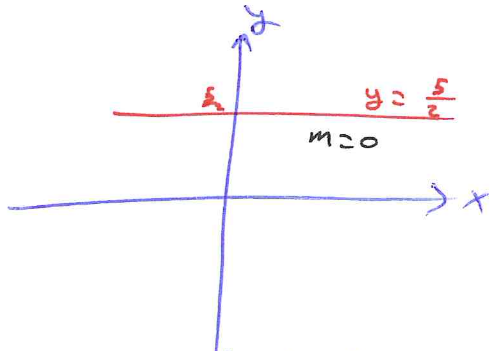
1. linear function: $f(x) = mx + b$ m, b are constants

m → slop, b → y-intercept

- when $m=1$ and $b=0$ $\Rightarrow$ $f(x)=x$ is called the idenity function
- when $m=0$ $\Rightarrow$ $f(x)=b$ is called constant function



lines through the origin



Constant fnction

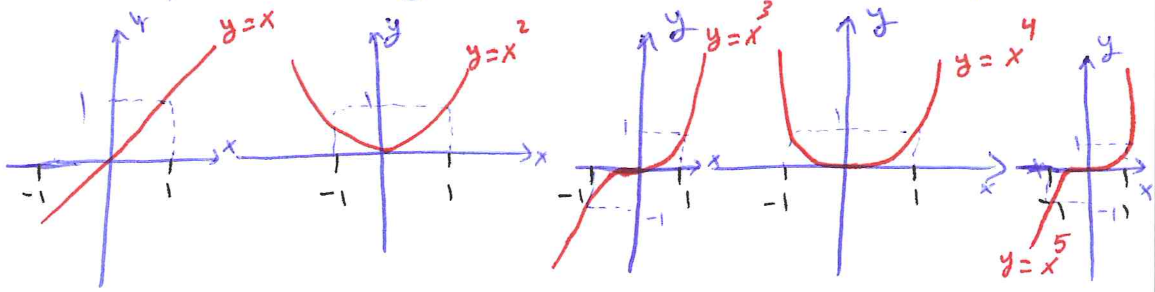
- A linear function with positive slope and passes through origin is called proportionality relationship.

<u>Def</u>: Two variables y and x are proportional (one to another) if one is always a constant multiple of the other, i.e: $y = kx$ for some nonzero constant k

If the variable is proportional to the reciprocal $\frac{1}{x}$, then y is inversely proportional to x.

2 Power functions $f(x) = x^a$, a is constant

a a is positive integer $f(x) = x^n$, $n = 1, 2, 3, 4, 5$



- as the power $n \uparrow$, the curves get more flat toward x-axis on the interval $(-1, 1)$ and more steeply for $|x| > 1$
- all curves passes through $(1,1)$ and origin.
- functions with even power are symmetric about y-axis
- functions with odd power are symmetric about the origin
- Even functions are $\downarrow$ on the interval $(-\infty, 0]$ and $\uparrow$ on $[0, \infty)$
- Odd functions are $\uparrow$ over the entire real line $(-\infty, \infty)$

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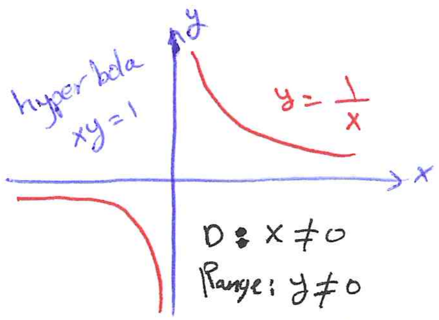
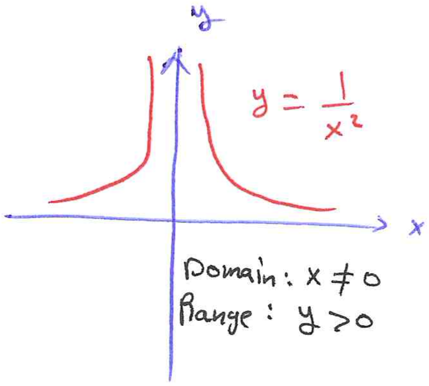
b $a = -1$ or $a = -2$





(a) $f(x) = x^{-1} = \frac{1}{x}$

$f \downarrow$ on $(-\infty, 0)$ and $(0, \infty)$

f is symmetric about origin

(b) $g(x) = x^{-2} = \frac{1}{x^2}$

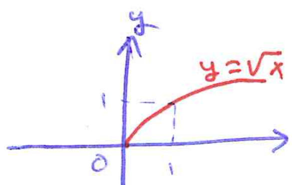
$g \uparrow$ on $(-\infty, 0)$

$g \downarrow$ on $(0, \infty)$

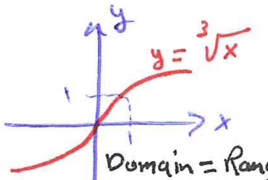
g is symmetric about y-axes

(8)

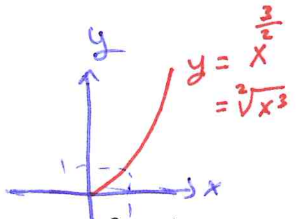
(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



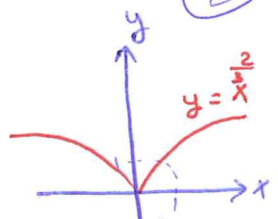
Domain : $0 \le x < \infty$
Range : $0 \le y < \infty$



Domain = Range
$-\infty < x < \infty$
$-\infty < y < \infty$



Domain : $0 \le x < \infty$
Range $0 \le y < \infty$



D : $-\infty < x < \infty$
R : $0 \le y < \infty$

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3) **Polynomials** : A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real constants called the coefficients of p.

* All polynomials have Domain $(-\infty, \infty)$
* If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the degree of the polynomial p.

- Linear functions $p(x) = mx + b$ with $m \neq 0$ are polynomials of degree 1.
- Quadratic functions $p(x) = ax^2 + bx + c$ with $a \neq 0$ are polynomials of degree 2.
- Cubic functions $p(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ are polynomials of degree 3. ...

⋮

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4) **Rational functions:** are a quotient or ratio

$$f(x) = \frac{p(x)}{g(x)}$$, where p, g are polynomials

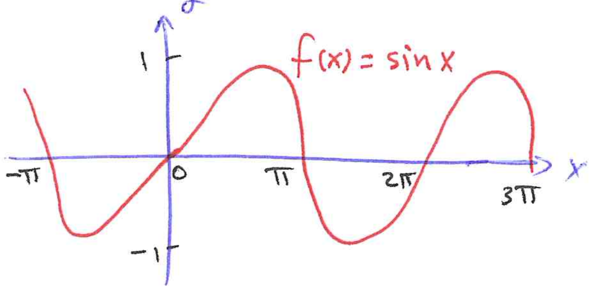
Example: $f(x) = \frac{x^2 - 3}{2x + 1}$

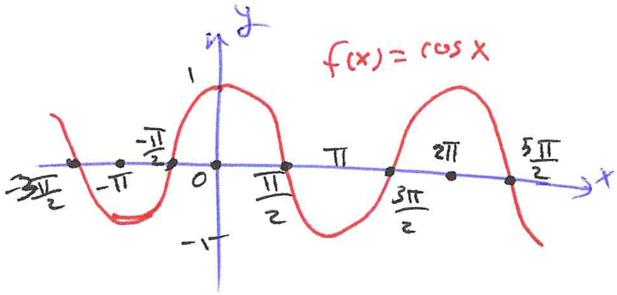
5) **Algebraic functions:** are any function constructed from polynomials using algebraic operation ($+, -, \times, \div$ and taking roots)

Example (1) All rational functions are algebraic.

(2) $y = x^2 (1 - x)^{\frac{3}{4}}$

6 Trigonometric functions: sin, cos, tan, csc, sec, cot (section 1.3)





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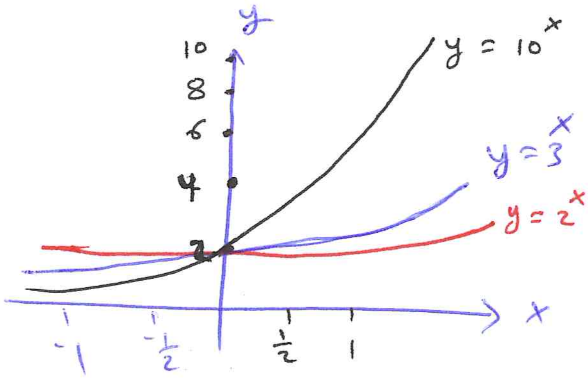
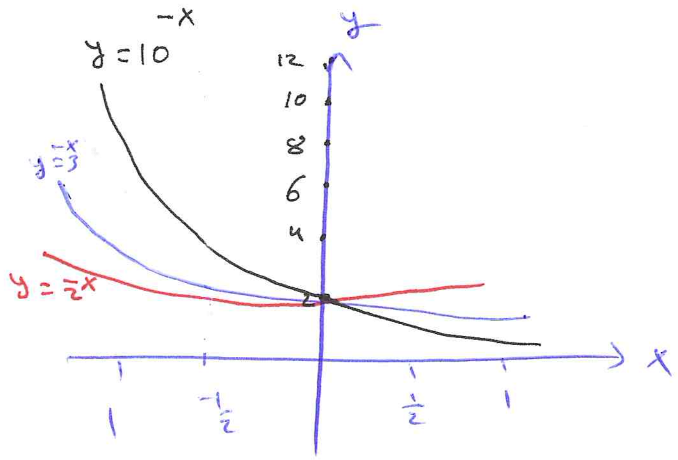
7 Exponential functions

$f(x) = a^x$, $a > 0$ and $a \neq 1$

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$ $\Big\rangle$ always





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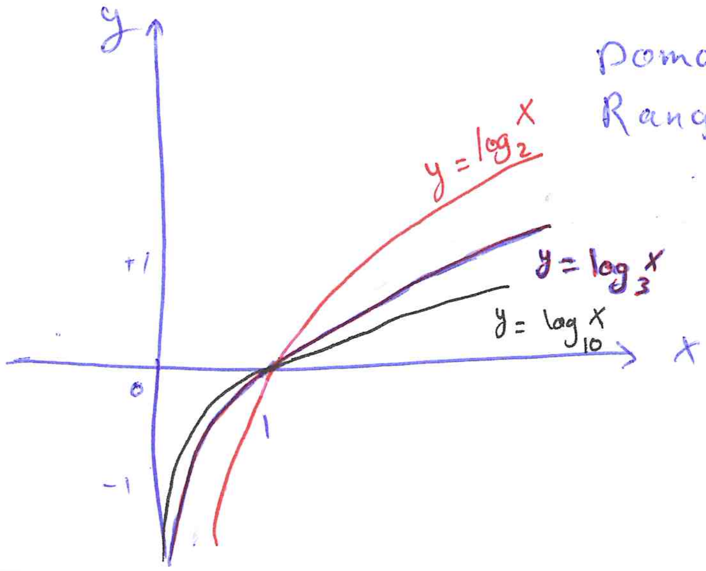
8 logarithmic functions

$f(x) = \log_a x$ the base $a \neq 1$ and $a > 0$

They are the inverse functions of the exponential functions.

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$ $\Big\rangle$ always



1.2 Combining functions; shifting and scaling Graphs

* If f and g are functions, then for every $x \in D(f) \cap D(g)$, we define:

operation of addition of functions → addition of real numbers

$$(f+g)(x) = f(x)+g(x)$$ "sum"

$$(f-g)(x) = f(x)-g(x)$$ "Difference"

$$(fg)(x) = f(x)\,g(x)$$ "product"

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$ "Quotient"

$$(cf)(x) = c f(x), \quad c \in \mathbb{R}$$ "Multiply by constant"

Example: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$, then

$(2f)(x) = 2f(x) = 2\sqrt{x}$ with domain $[0,\infty)$

$(f+g)(x) = f(x)+g(x) = \sqrt{x}+\sqrt{1-x}$ with domain $[0,\infty) \cap (-\infty,1] = [0,1]$

$(f-g)(x) = \sqrt{x}-\sqrt{1-x}$ with domain $[0,1]$

$(f\cdot g)(x) = \sqrt{x}\,\sqrt{1-x} = \sqrt{x(1-x)}$ with domain $[0,1]$

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{1-x}} = \sqrt{\frac{x}{1-x}}$ with domain $[0,1)$

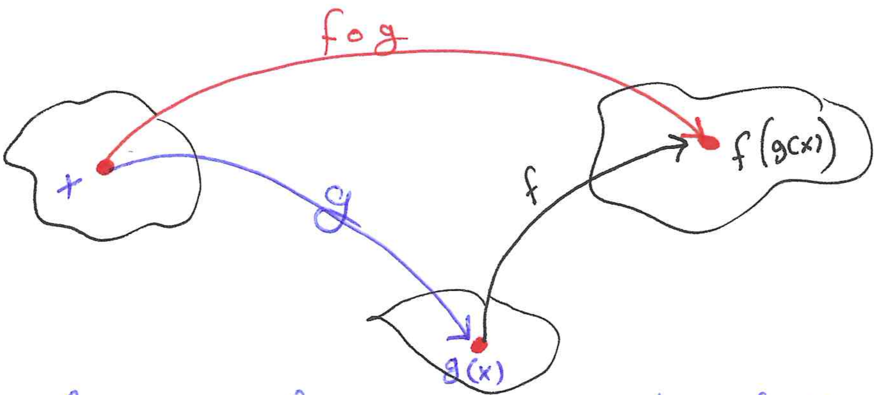
$\left(\frac{g}{f}\right)(x) = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}}$ with domain $(0,1]$

Composite functions

Def: Let f and g be two functions. The composite function $f \circ g$ "f composed with g" is defined by

$$(f \circ g)(x) = f(g(x))$$

$$R(g) \subset D(f)$$



The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

Example If $f(x) = \sqrt{x}$ and $g(x) = x + 1$ find

(a) $(f \circ g)(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$, $D = [-1, \infty)$

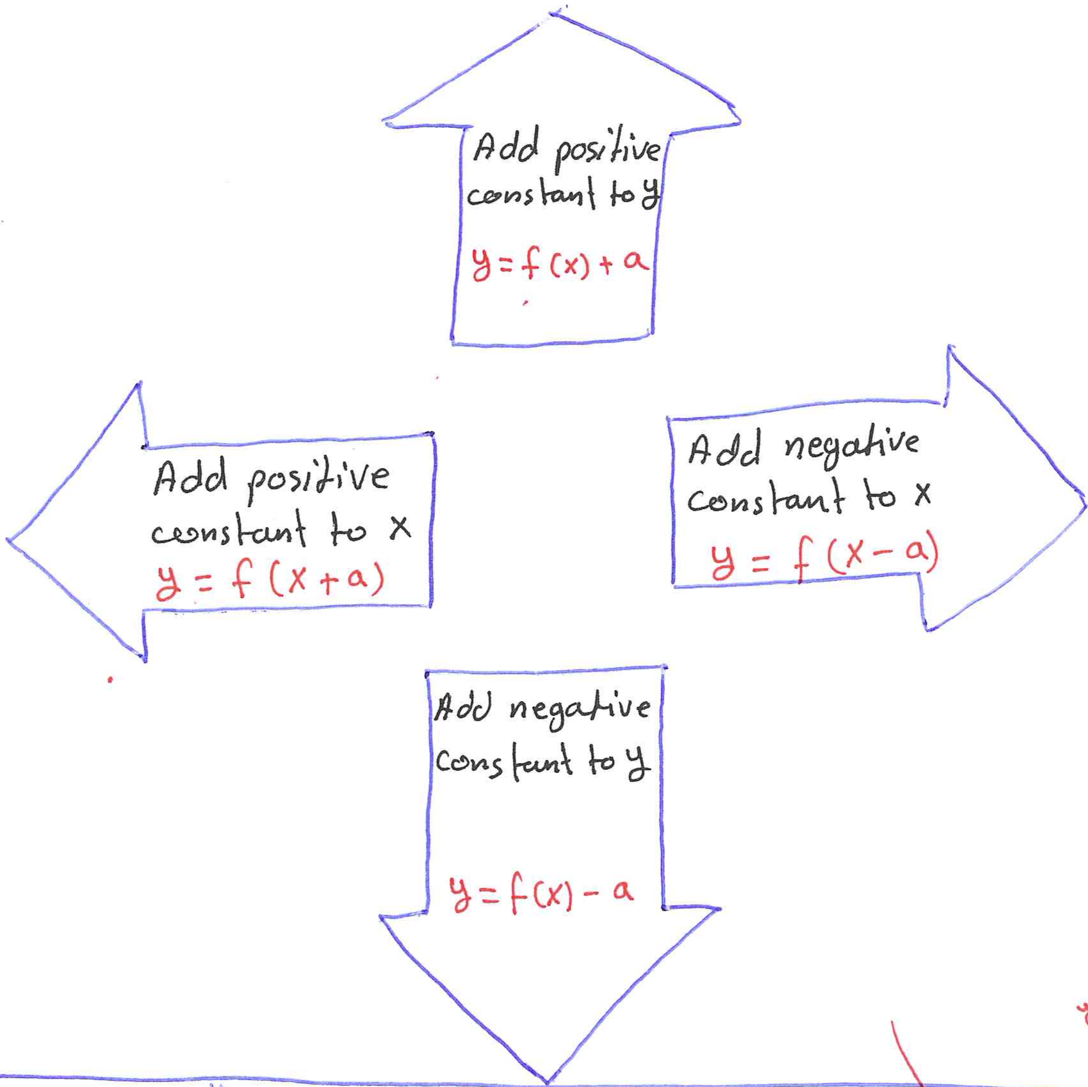
(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$, $D = [0, \infty)$

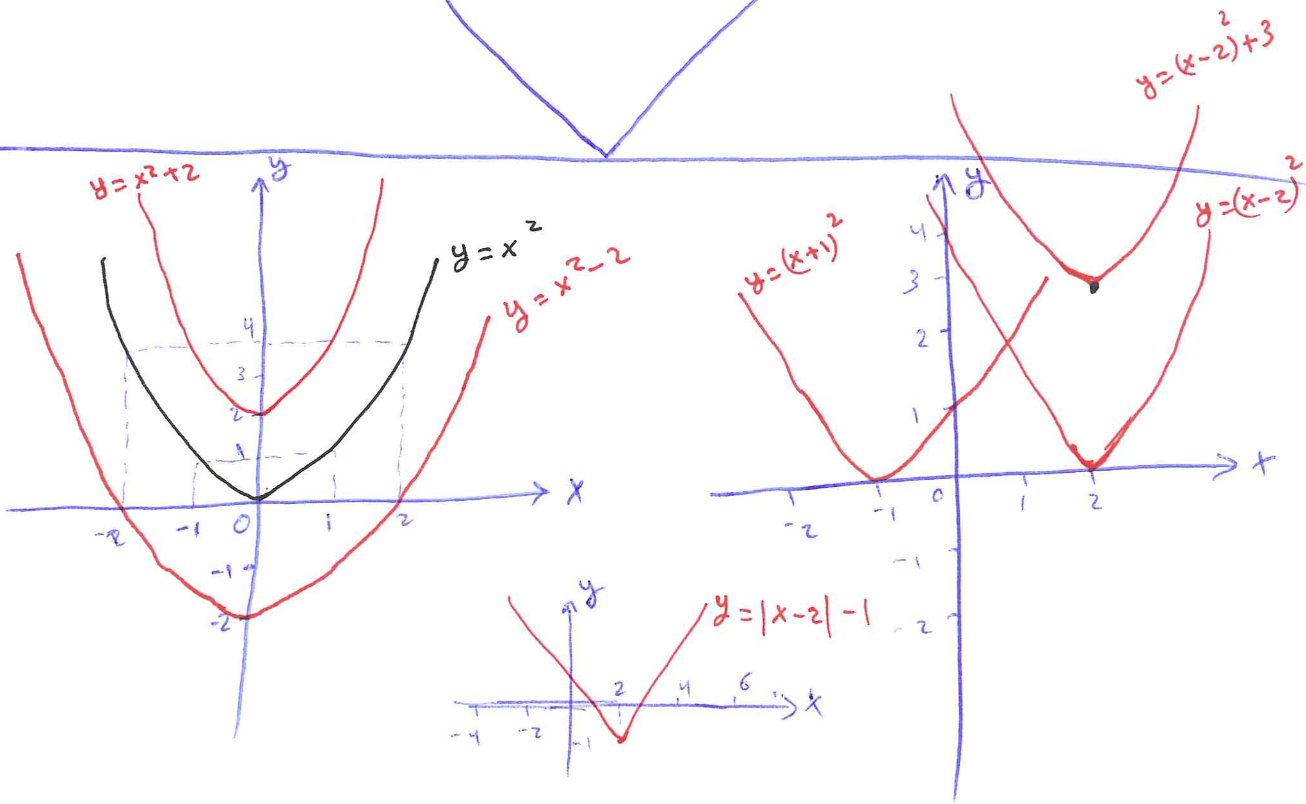
(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$, $D = [0, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x+1) = x + 1 + 1 = x + 2$, $D = \mathbb{R}$

Shifting Graphs

* Horizontal and Vertical shift: Let $a > 0$ and $y = f(x)$

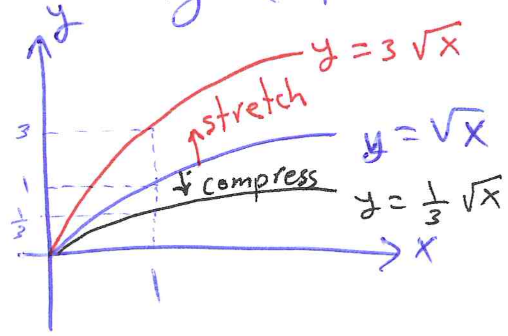




Vertical scaling:

If $c > 1$, then

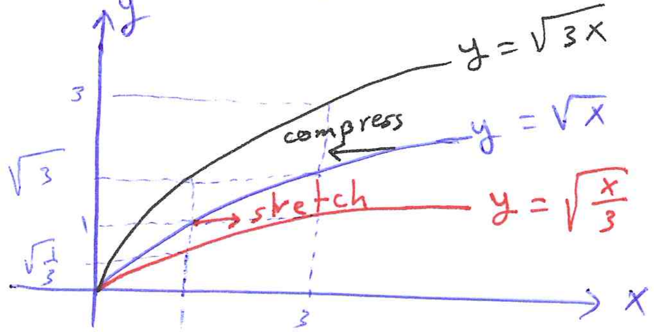
- $y = c\,f(x)$ stretches تمدد the graph of f vertically by a factor of c.
- $y = \frac{1}{c} f(x)$ Compresses ضغط the graph of f vertically by a factor of c.



Horizontal scaling

If $c > 1$, then

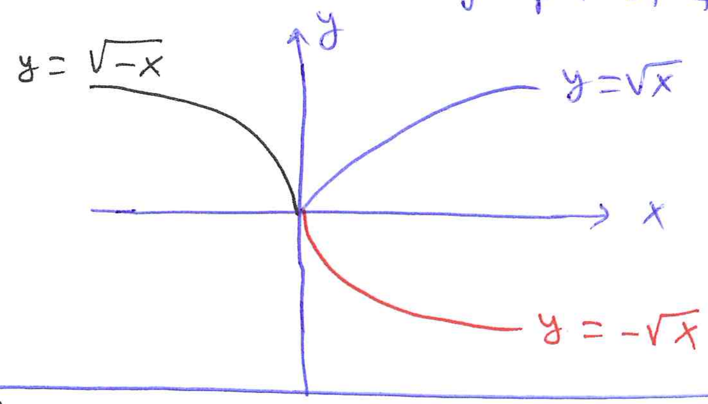
- $y = f(cx)$ compresses the graph of f horizontally by a factor of c
- $y = f\left(\frac{x}{c}\right)$ stretches the graph of f horizontally by a factor of c.



Reflections

* If $c = -1$, then

- $y = -f(x)$ reflects the graph of f across the x-axis
- $y = f(-x)$ reflects the graph of f across the y-axis



Example: Let $f(x) = x^4 - 2x^3 + 1$. Find formulas to

(a) compress the graph horizontally by a factor of 2 followed by a reflection across the y-axis.

compress horizontally by 2 $\Rightarrow f(2x)$
reflection across y-axis $\Rightarrow f(-2x)$

$$f(-2x) = (-2x)^4 - 2(-2x)^3 + 1$$
$$= 16x^4 + 16x^3 + 1$$

(b) compress the graph vertically by a factor 2 followed by a reflection across the x-axis.

compress the graph vertically by 2 $\Rightarrow \frac{1}{2} f(x)$
reflection across x-axis $\Rightarrow -\frac{1}{2} f(x)$

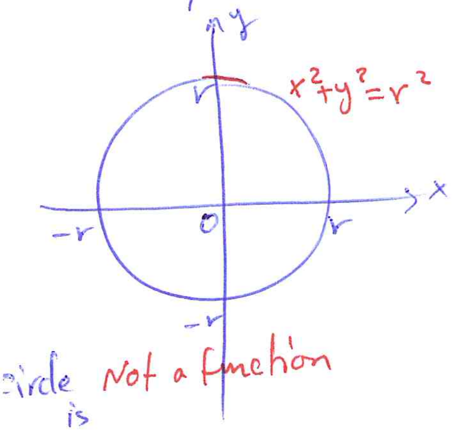
$$-\frac{1}{2} f(x) = -\frac{1}{2}\left[x^4 - 2x^3 + 1\right]$$
$$= -\frac{1}{2}x^4 + x^3 - \frac{1}{2}$$

Ellipses القطع الناقص

* A standard equation for the circle of radius r and centered at point (h, k) is

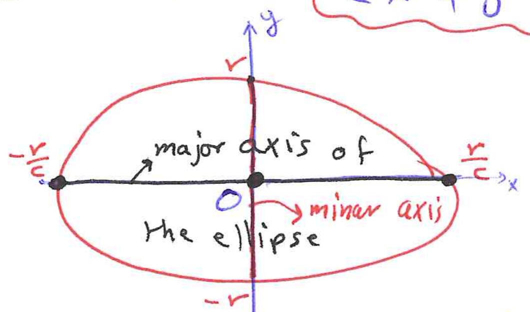
$$(x-h)^2+(y-k)^2=r^2$$

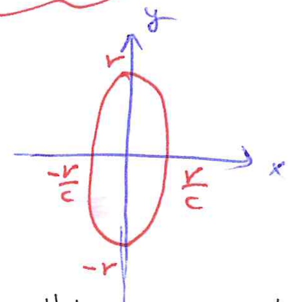
* Circle centered at origin has the following equation $x^2+y^2=r^2$ ---- ①



ircle is Not a function

Substitute cx for x in equation ① we get $c^2x^2+y^2=r^2$





y-intercept is the same $r, -r$ in the three figures

• Major axis is the longer line segment.

①• ellipse: $0<c<1$

②• not a function

③• stretches the circle

①• ellipses $c>1$

②• not a function

③• compressed horizontally

④• Major axis is the line segment joining the points $(-\frac{r}{c}, 0)$ and $(\frac{r}{c}, 0)$

⑤• Minor axis is the line segment joining the points $(0, -r)$ and $(0, r)$

④• Major axis is the line segment joining the points $(0, -r)$ and $(0, r)$

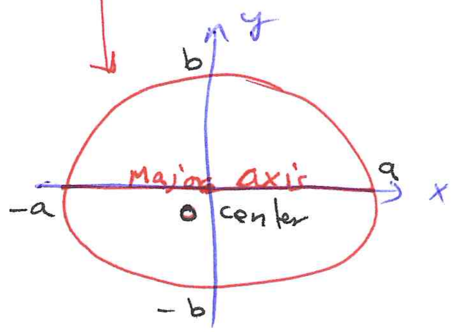
⑤• Minor axis is the line segment joining the points $(-\frac{r}{c}, 0)$ and $(\frac{r}{c}, 0)$

Divid * by r^2 $\Rightarrow$ $\boxed{\dfrac{c^2x^2}{r^2} + \dfrac{y^2}{r^2} = 1} \rightarrow (2)$

Take $a = \frac{r}{c}$ and $b = r$

Equation (2) becomes: $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1} \rightarrow$ Ellipse centered at origin

- If $a > b$, then the major axis is horizontal
- If $a < b$, then " " " " vertical



$\boxed{a > b}$

The standard equation of an ellipse centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

1.3 Trigonometric Functions

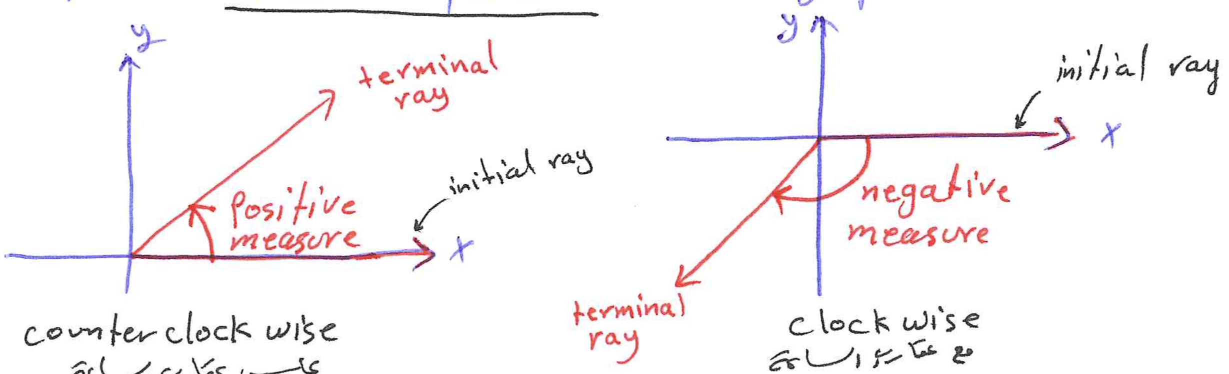
* Angles are measured either by degrees or radians.

$$\boxed{\pi \text{ radians} = 180^\circ}$$

1 radian $= \frac{180}{\pi} \approx 57^\circ$

$1^\circ = \frac{\pi}{180} \approx 0.02$ rad

* Angles in <u>standarad position</u> in the xy-plane



counterclockwise
عكس عقارب الساعة

clockwise
مع عقارب الساعة

⇒ An angle in the xy-plane is in standard position if its vertex lies at orgin an its initial ray lies along the positive x-axis.

* Conversion formulas:

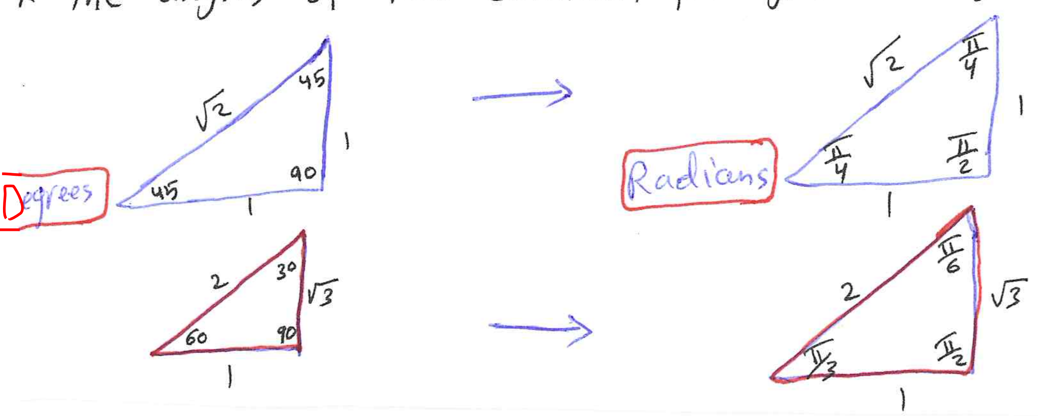
Degrees to radian : multiply by $\frac{\pi}{180}$

Radians to Degrees : multiply by $\frac{180}{\pi}$

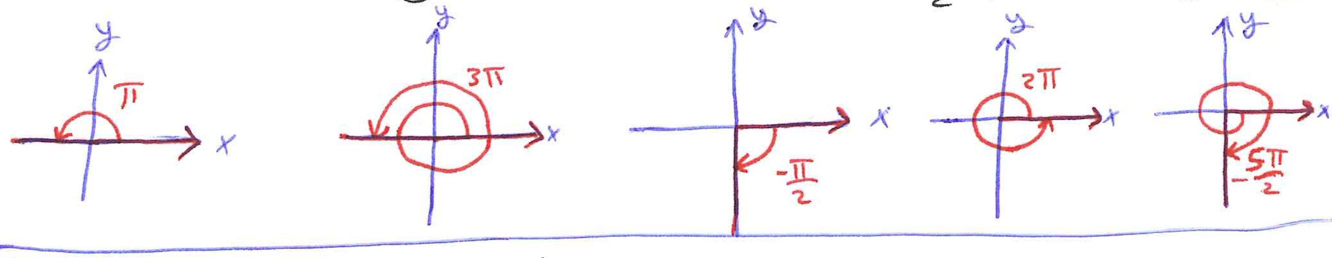
<u>Examples</u> : Convert 45° to radians: $45^\circ \times \frac{\pi}{180} = \frac{\pi}{4}$ rad

Convert $\frac{\pi}{6}$ rad to degrees: $\frac{\pi}{6} \times \frac{180}{\pi} = 30^\circ$

* The angles of two common triangles in degrees and radians:



* Draw the following Angles: π, 3π, $-\frac{\pi}{2}$, 2π, $-\frac{5\pi}{2}$ 18

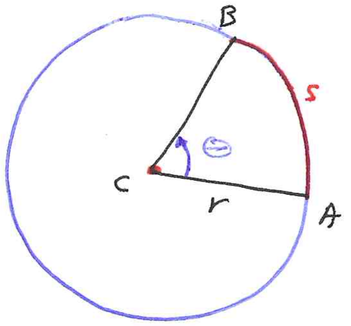


* Radian Measure and Arc Length:

Let s be the arc length AB of a circle of radius r.

The angle ACB is θ measured in radians.

$$\boxed{s = r\theta}$$



* The unit cicle has arc length $s = \theta$

Example: Consider a circle of radius 8

(a) Find the central angle subtended (مقصوصة/ يسوي) by an arc of length 2π

(b) Find the length of an arc subtending a central angle of $\frac{3\pi}{4}$

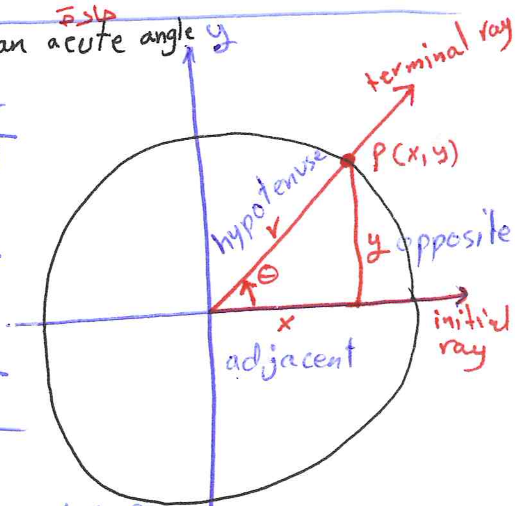
(a) $\theta = \frac{s}{r} = \frac{2\pi}{8} = \frac{\pi}{4}$

(b) $s = r\theta = 8\left(\frac{3\pi}{4}\right) = 6\pi$

obtuse: منفرجة

* The Six Basic Trigonometric Functions of an acute (حادة) angle

Sine: $\sin\theta = \frac{y}{r}$	Cosecant: $\csc\theta = \frac{r}{y}$
Cosine: $\cos\theta = \frac{x}{r}$	Secant: $\sec\theta = \frac{r}{x}$
Tangent: $\tan\theta = \frac{y}{x}$	Cotangent: $\cot\theta = \frac{x}{y}$



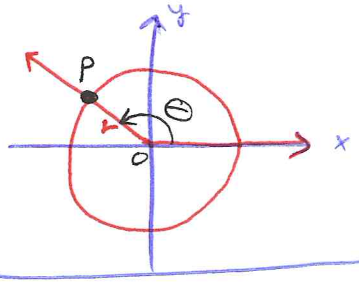
* Note that when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ we have $x = 0$ and so $\tan\theta$ and $\sec\theta$ are not defined.

* Note that when $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ we have $y = 0$ and so $\cot\theta$ and $\csc\theta$ are not defined.

Note also that $x = r\cos\theta$ and $y = r\sin\theta$

$$P(x,y) = (r\cos\theta,\ r\sin\theta)$$

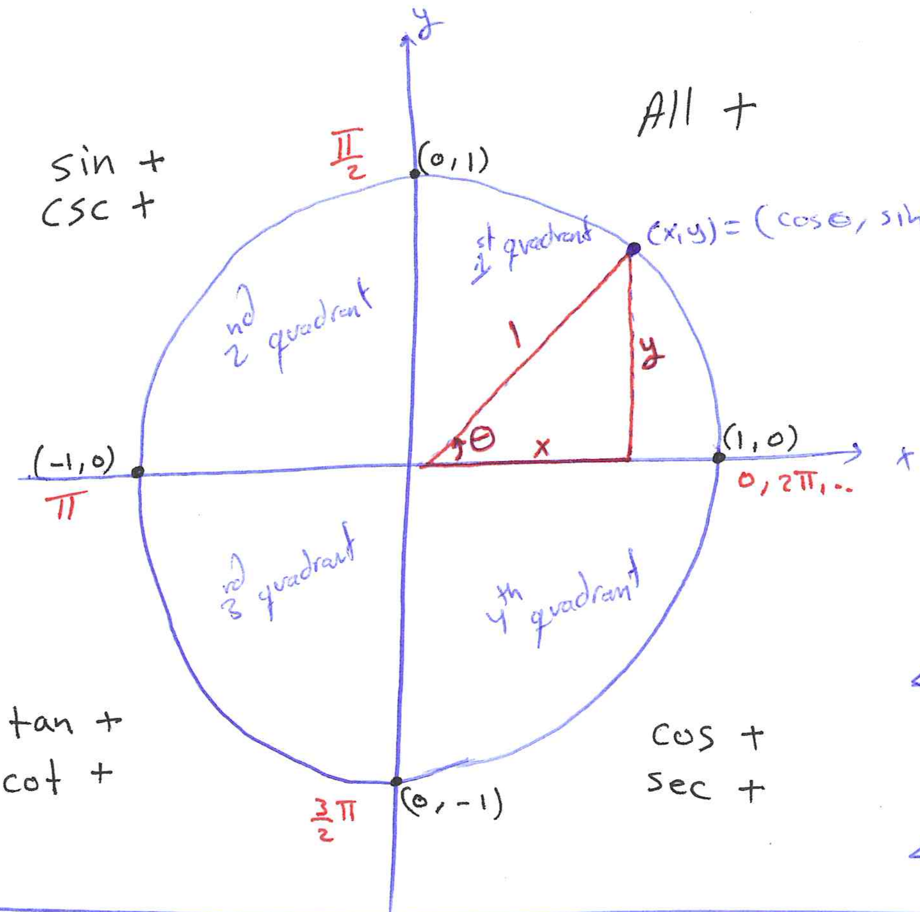
Note that in Unit circle $(x,y) = (\cos\theta, \sin\theta)$

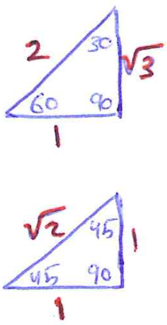


$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}$$

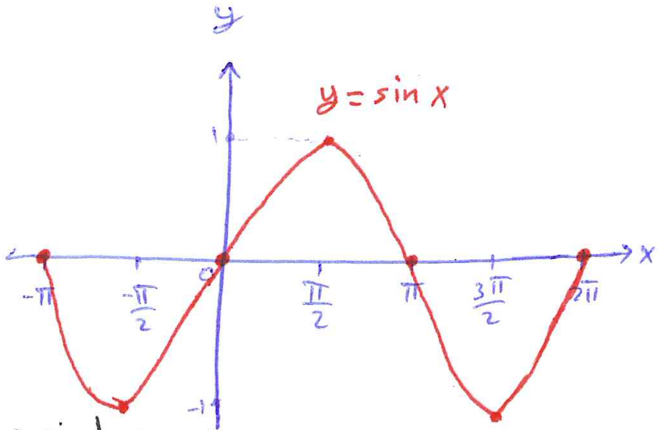
Unit Circle





Degrees	30	60	45	0	90	180	270	360	-45	-135	135	-180
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$-\frac{\pi}{4}$	$-\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$-\pi$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	0	1	0	-1	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	0	-1	0	1	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\tan\theta$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	0		0		0	-1	1	-1	0

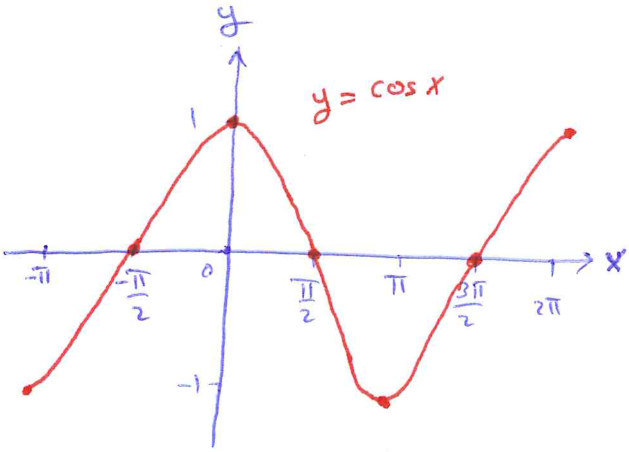
Graphs of Trigonometric Functions



period : 2π

Domain = (−∞, ∞)

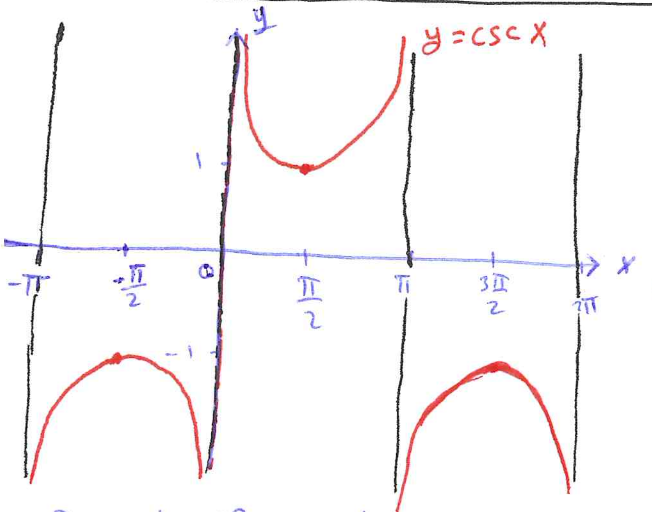
Range = [−1, 1]



Domain = (−∞, ∞)

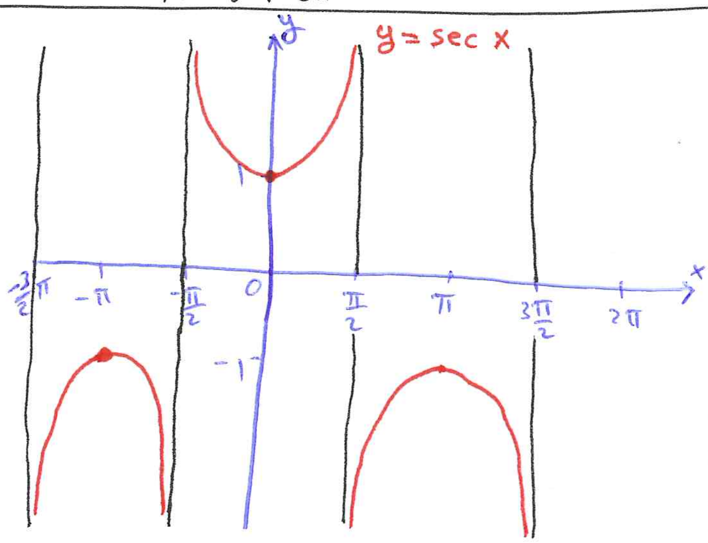
Range = [−1, 1]

period : 2π



Domain = ℝ except 0, ±π, ±2π, ...

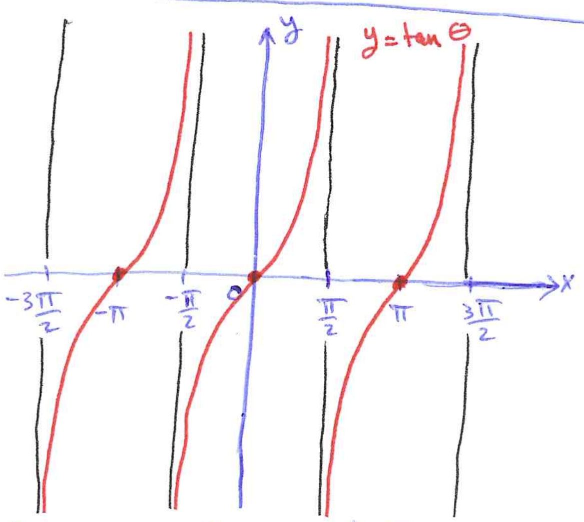
Range = (−∞, −1] ∪ [1, ∞) period : 2π



Domain = ℝ except ±π/2, ±3π/2, ...

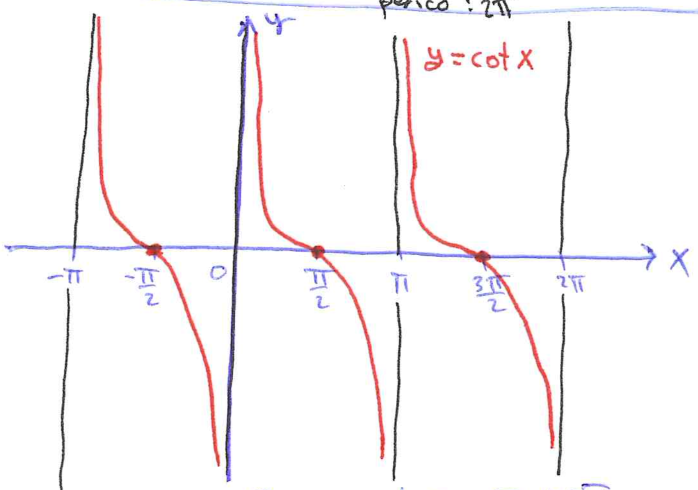
Range = (−∞, −1] ∪ [1, ∞)

period : 2π



Domain = ℝ except ±π/2, ±3π/2, ...

Range = (−∞, ∞) period : π



Domain = ℝ except 0, ±π, ±2π, ...

Range = (−∞, ∞) period : π

Periodicity

*** Definition:** A function $f(x)$ is periodic if there is a positive number p s.t $f(x+p) = f(x)$.
The smallest such value of p is the period of f.

* Periods of trigonometric functions:

Period π examples: $\tan x = \tan(x+\pi)$

$\cot x = \cot(x+\pi)$

See page 29

Period 2π examples: $\sin x = \sin(x+2\pi)$

$\cos x = \cos(x+2\pi)$

$\sec x = \sec(x+2\pi)$

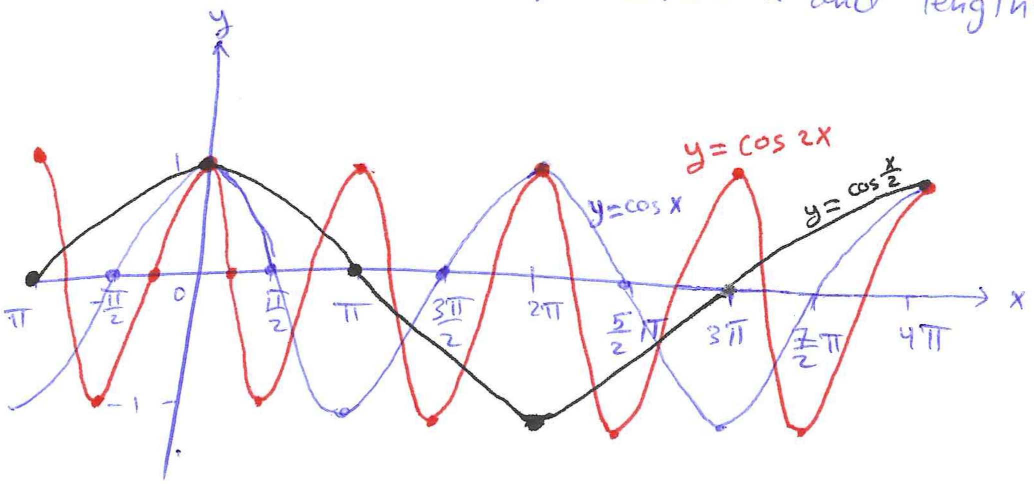
$\csc x = \csc(x+2\pi)$

Example: Draw ① $y = \cos 2x$

② $y = \cos \frac{x}{2}$

Note that ① Multiplying x by a number greater than 1 speeds up the trigonometric function (increase the frequency). ($p\downarrow$)

② Multiplying x by a positive number less than 1 slows the trigonometric function down and lengthens its period ($p\uparrow$).



$\cos x$ has $p = 2\pi$

$\cos \frac{x}{2}$ has $p = 4\pi$

$\cos 2x$ has $p = \pi$

* Even trigonometric functions:

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

* Odd trigonometric functions:

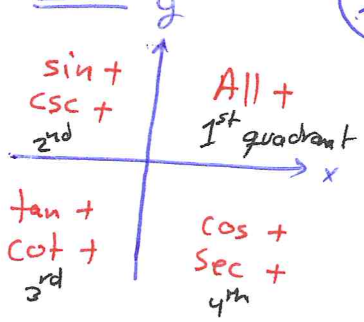
$$\sin(-x) = -\sin x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\cot(-x) = -\cot x$$

Remember



see also page 29

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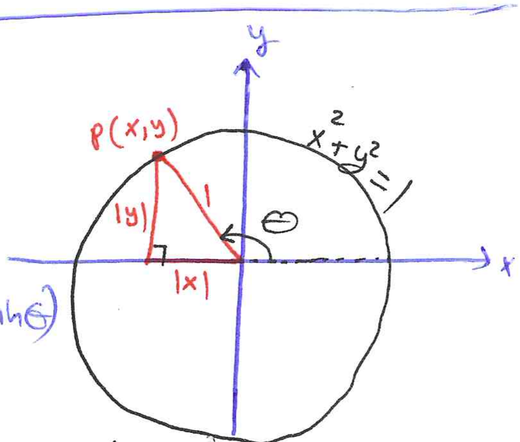
* Identities

Recall the unit cycle $x^2+y^2=1$

Remembe that the point $P(x,y) = P(\cos\theta, \sin\theta)$

Apply Pythagorean theorem $\Rightarrow$

$$|x|^2 + |y|^2 = 1^2 \Rightarrow \boxed{\cos^2\theta + \sin^2\theta = 1} \quad (1)$$



The triangle for a general angle θ

---

* Divid equation (1) by $\cos^2\theta$, we get

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

Divid equation (1) by $\sin^2\theta$, we get

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

Angle Sum Formulas:

$$\boxed{\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \end{aligned}} \quad (2)$$

* Double-angle Formulas:

Make $A = B = \theta$ in equation (2), we get

$$\boxed{\begin{array}{l} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \sin 2\theta = 2\sin\theta\cos\theta \end{array}} \quad (3)$$

* Additional double-angle Formulas:

→ Add equation (1) to equation (3), we get

$$2\cos^2\theta = 1 + \cos 2\theta \Rightarrow \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}} \quad (4)$$

→ substract (3) from (1), we get

$$(1) - (3) \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \Rightarrow \boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}} \quad (5)$$

* Half-angles formulas:

→ Apply $\frac{\theta}{2}$ in equation (4), we get

$$\boxed{\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}}$$

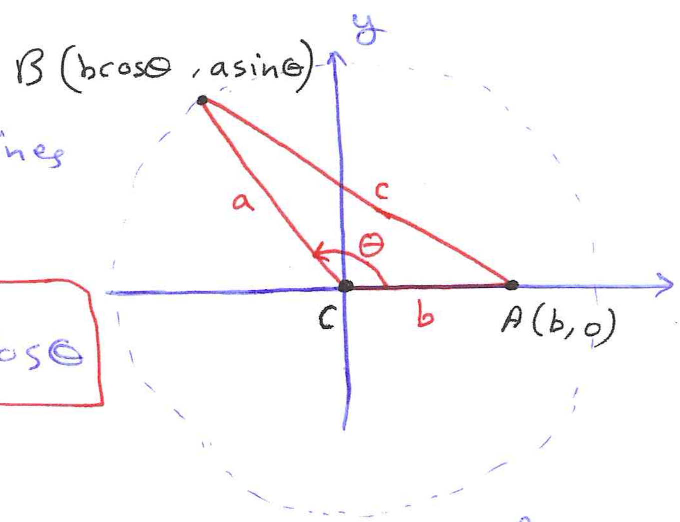
→ Apply $\frac{\theta}{2}$ in equation (5), we get

$$\boxed{\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}}$$

The law of Cosines

Consider the triangle ABC. The law of Cosines is given by

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$



Proof: The distance between A and B is given by

$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$
$$= a^2\cos\theta^2 - 2ab\cos\theta + b^2 + a^2\sin\theta^2$$
$$= a^2(\cos\theta^2 + \sin\theta^2) + b^2 - 2ab\cos\theta$$
$$= a^2 + b^2 - 2ab\cos\theta$$

---

* Note that the law of cosines generalizes the Pythagorean theorem. If $\theta = \frac{\pi}{2}$, then $\cos\frac{\pi}{2} = 0$ and $c^2 = a^2 + b^2$.

---

* Transformation of Trigonometric functions:

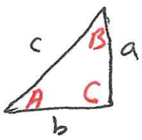
$$y = a\, f(b(x+c)) + d$$

- a: vertical stretch or compression; reflection about x-axis if $a<0$
- d: vertical shift
- c: Horizontal shift
- b: • Horizontal stretch or compression • reflection about y-axis if $b<0$

* For any angle θ measured in radians:

$$-|\theta| \le \sin\theta \le |\theta| \quad \text{and} \quad -|\theta| \le 1-\cos\theta \le |\theta|$$

Note that the law of sines says that if a, b, c are sides opposites the angles A, B, C in a triangle, then



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

2·1 2.2 Limits and Continuity

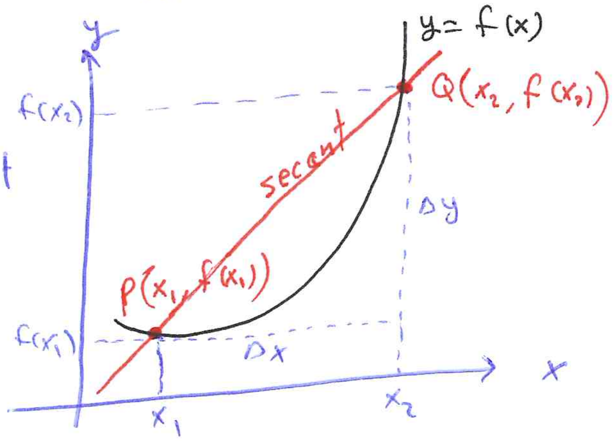
Rates of change and limits

Def: The average rate of change of the function $y = f(x)$ w.r.t x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

h is the length of the interval

* Note that the average rate of change = slope of the secant



Example 1: Find the average rate of change of the function $f(x) = \sqrt{x}$ over $[4, 9]$

The average rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(9) - f(4)}{9 - 4} = \frac{\sqrt{9} - \sqrt{4}}{9 - 4} = \frac{3-2}{5} = \frac{1}{5}$

Example 2 : see the end

2.2 * limits of function values

Example: How does the function $f(x) = \frac{x^2 - 1}{x - 1}$ behave near $x = 1$?

$\Rightarrow$ The problem is when $x = 1$ (we can't divid over zero)

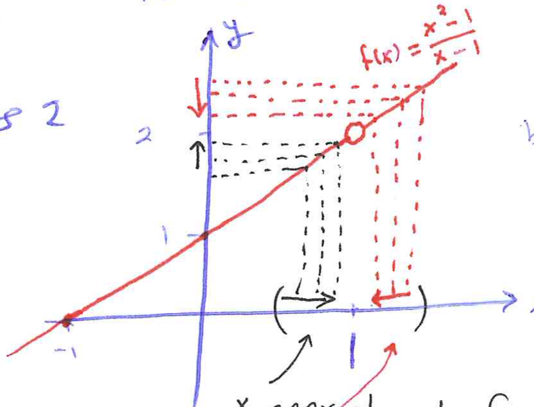
$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1$ for all values of x except $x = 1$

$\Rightarrow$ We say $f(x)$ approches 2 as x approches 1.

We write this as

$\lim_{x \to 1} f(x) = 2$ or

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$



x	f(x)
0.9	1.9
0.99	1.99
0.999	1.999
1	
1.001	2.001
1.01	2.01
1.1	2.1

below 1

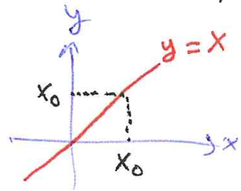
x approches 1 from left $x \to 1^-$

x approches 1 from right $x \to 1^+$

Example: [a] If f is the identity function $f(x)=x$,

then for any value x_0

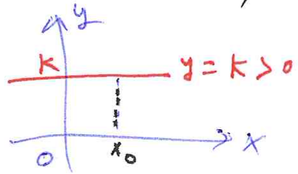
$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0$$



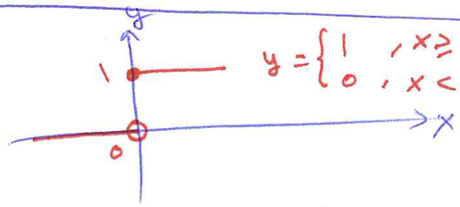
[b] If f is the constant function $f(x)=k$,

then for any value x_0

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} k = k$$



Example:

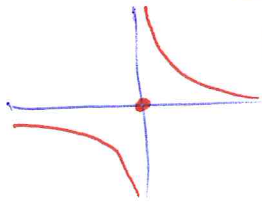


$y = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}$

$\lim_{x\to 0} f(x)$ DNE because at $x=0$, y jumbs

As $x \to 0^-$, $y \to 0$

As $x \to 0^+$, $y \to 1$



$y = \begin{cases} \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

$\lim_{x\to 0} f(x)$ DNE

As $x \to 0^-$, $y \to -\infty$

As $x \to 0^+$, $y \to \infty$

Theorem 1 (limit laws) If L, M, c, k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

then:

1. Sum Rule: $\lim_{x\to c} (f(x)+g(x)) = L + M$
2. Difference Rule: $\lim_{x\to c} (f(x)-g(x)) = L - M$
3. Constant Multiply Rule: $\lim_{x\to c} k f(x) = kL$
4. Product Rule: $\lim_{x\to c} f(x) g(x) = LM$
5. Quotient Rule: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. Power Rule: $\lim_{x\to c} (f(x))^n = L^n$, n is positive integer
7. Root Rule: $\lim \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, n = =

If n is even, we assume that $L > 0$.

Example: Find

(a) $\lim_{x\to 1} (x^3 - 4x^2 + 1) = (1)^3 - 4(1)^2 + 1 = 1 - 4 + 1 = -2$

(b) $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 2+2 = 4$

(c) $\lim_{x\to -2} \sqrt{4x^2-7} = \lim \sqrt{4(-2)^2-7} = \sqrt{16-7} = \sqrt{9} = 3$

Theorem 2 (limits of Polynomials)

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then

$$\lim_{x\to c} p(x) = p(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$$

Theorem 3 (limits of Rational functions)

If $p(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$,

then $\lim_{x\to c} \dfrac{p(x)}{Q(x)} = \dfrac{p(c)}{Q(c)}$

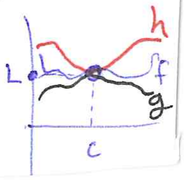
Example: Find $\lim_{x\to -1} \dfrac{x^3+2x^2-1}{x^2+3} = \dfrac{(-1)^3 + 2(-1)^2 - 1}{(-1)^2+3} = \dfrac{-1+2-1}{4} = \dfrac{0}{4} = 0$

Example: Find $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \dfrac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} = \lim_{x\to 1} \dfrac{x+2}{x} = \dfrac{3}{1} = 3$

Example: Find $\lim_{x\to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} = \lim_{x\to 0} \dfrac{\sqrt{x^2+9}-3}{x^2} \cdot \dfrac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$ multiply by the conjugate of numerator

$$= \lim_{x\to 0} \frac{\cancel{x^2}+\cancel{9}-\cancel{9}}{\cancel{x^2}(\sqrt{x^2+9}+3)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Theorem 4 (Sandwich Theorem) Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then $\lim_{x\to c} f(x) = L$



<u>Example</u>: Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \neq 0$

find $\lim_{x \to 0} u(x)$.

$$\lim_{x \to 0} \left(1 - \frac{x^2}{4}\right) = 1 = \lim_{x \to 0} \left(1 + \frac{x^2}{2}\right)$$

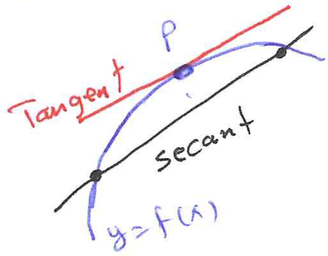
Thus, by sandwich theorem $\lim_{x \to 0} u(x) = 1$

<u>Theorem 5</u>: If $f(x) \le g(x)$ $\forall x$ in some open interval containing c, except possibly at $x = c$, and the limits of f and g both exist as x approaches c, then

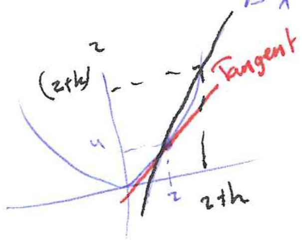
$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$$

<u>Example</u>: Find the slope of $= x^2$ at point $(2,4)$.

Write an equation for the tangent at this point.



$$\text{Secant Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x_1+h) - f(x_1)}{h} = \frac{f(2+h) - f(2)}{h} = \frac{(2+h)^2 - 2^2}{h}$$



$$= \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h} = \frac{h(4+h)}{h} = 4 + h$$

$$\text{Tangent slope} = \lim_{h \to 0} \text{Secant slop} = 4$$

$$y - y_0 = m(x - x_0)$$

$$y - 4 = 4(x - 2) = 2x - 8$$

$$y = 4x - 4$$

The Precise Definitions of Limits

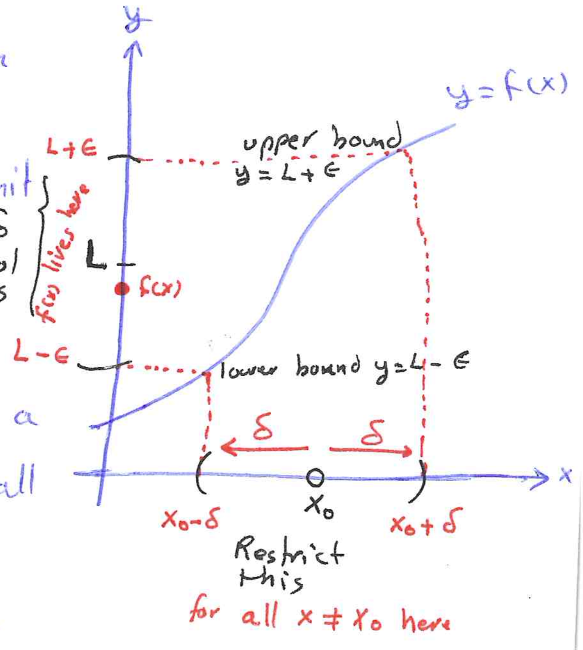
Def: Example $f(x) = 2x+1 \quad \epsilon = 2 \quad \Rightarrow \delta = 1$

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say $f(x)$ approaches the limit L as x approaches x_0, and we write:

$$\lim_{x \to x_0} f(x) = L,$$

if for every number $\epsilon > 0$, there exists a corrsponding number $\delta > 0$ such that for all x

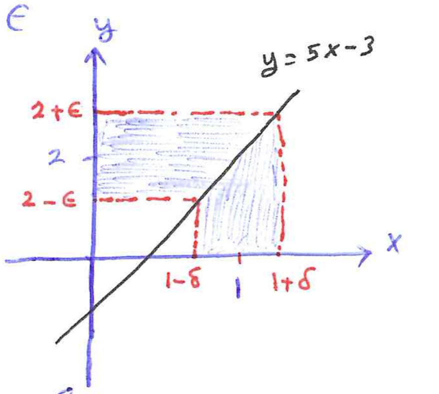
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



Example: show that $\lim_{x \to 1} (5x-3) = 2$

→ Let $\epsilon > 0$, we must find $\delta > 0$ s.t for all x
if $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

→ $x_0 = 1$, $L = 2$, $f(x) = 5x - 3$
That is we need to find $\delta > 0$ s.t for all x if $|x-1| < \delta$ then
$|(5x-3) - 2| < \epsilon$



$\Rightarrow |5x-5| < \epsilon \Rightarrow 5|x-1| < \epsilon \Rightarrow |x-1| < \frac{\epsilon}{5}$

Thus, we can take $0 < \delta \le \frac{\epsilon}{5}$ because if

$|x-1| < \delta = \frac{\epsilon}{5}$, then $|(5x-3)-2| = |5x-5| = 5|x-1| < \cancel{5}\frac{\epsilon}{\cancel{5}} = \epsilon$

To find the maximum δ^* in which $0 < \delta \le \delta^*$, we may use $f(x_0 - \delta^*) = L - \epsilon$ or $f(x_0 + \delta^*) = L + \epsilon$

In the previous example:

$f(x_0 - \delta^*) = L - \epsilon \Rightarrow f(1-\delta^*) = 2 - \epsilon \Rightarrow$

$5(1-\delta^*) - 3 = 2 - \epsilon \Rightarrow 5 - 5\delta^* - 3 = 2 - \epsilon \Rightarrow \cancel{2} - 5\delta^* = \cancel{2} - \epsilon$

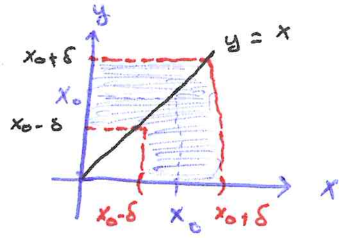
$\Rightarrow \boxed{\delta^* = \frac{\epsilon}{5}}$

Example: show that $\lim_{x \to x_0} x = x_0$ $\quad f(x) = x$ and $L = x_0$

let $\epsilon > 0$, we need to find $\delta > 0$ such that for all x

if $|x - x_0| < \delta$ then $|x - x_0| < \epsilon$

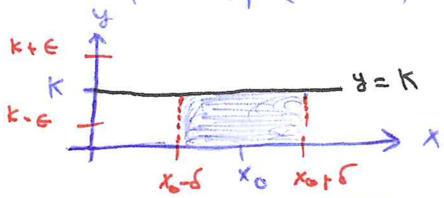
Take $0 < \delta \leq \epsilon$. This proves that $\lim_{x \to x_0} x = x_0$



Example: Prove that $\lim_{x \to x_0} k = k$

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - x_0| < \delta$ then $|k - k| < \epsilon$.



$0 < \epsilon$ This is true for any $\delta > 0$.

This proves that $\lim_{x \to x_0} k = k$

How to find δ for a given f, L, x_0 and ϵ:

Two steps to find $\delta > 0$ s.t for all x

if $|x - x_0| < \delta$ then $|f(x) - L| < \epsilon$

$\boxed{1}$ Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (a, b) about x_0 on which the inequality holds for all $x \neq x_0$.

$\boxed{2}$ Find $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

Example: Prove that $\lim_{x \to 2} f(x) = 4$ where $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$

let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - 2| < \delta$ then $|f(x) - 4| < \epsilon$

step $\boxed{1}$: solve $|f(x) - 4| < \epsilon$ to find an open interval about $x_0 = 2$ in which the inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq x_0$

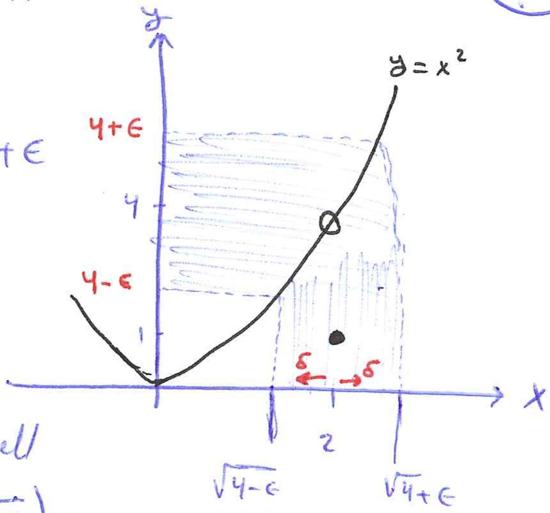
$\Rightarrow$ For $x \neq x_0 = 2$ we have $f(x) = x^2$

$\Rightarrow |x^2 - 4| < \epsilon \Rightarrow$

$-\epsilon < x^2 - 4 < \epsilon \Rightarrow 4 - \epsilon < x^2 < 4 + \epsilon$

$\Rightarrow \sqrt{4-\epsilon} < |x| < \sqrt{4+\epsilon}$ (Assume $\epsilon < 4$)

$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}$



The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.

step $\boxed{2}$ Find $\delta > 0$ that places the centered interval $(2-\delta, 2+\delta)$ inside the interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$

Take $\delta = \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$.

Thus, the inequality $|x-2| < \delta$ will automatically place x between $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ to make $|f(x) - 4| < \epsilon$.

Example: Given that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. We now can prove Theorems

Prove that $\lim_{x \to c} (f(x) + g(x)) = L + M$.

Let $\epsilon > 0$, we need to find $\delta > 0$ s.t for all x

if $|x - c| < \delta$ then $|f(x) + g(x) - (L+M)| < \epsilon$.

> since $\lim_{x \to c} f(x) = L$, $\exists\ \delta_1 > 0$ s.t for all x if $|x-c| < \delta_1$ then $|f(x) - L| < \frac{\epsilon}{2}$

> Since $\lim_{x \to c} g(x) = M$, $\exists\ \delta_2 > 0$ s.t for all x if $|x-c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$

step $\boxed{1}$ $|f(x) + g(x) - (L+M)| = |(f(x) - L) + (g(x) - M)|$

$\leq |f(x) - L| + |g(x) - M|$ Triangle Inequality

$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

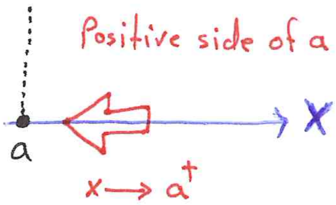
step $\boxed{2}$ Take $\delta = \min\{\delta_1, \delta_2\}$, so that

If $|x-c| < \delta_1$ then $|x-c| < \delta$, thus $|f(x) - L| < \frac{\epsilon}{2}$.

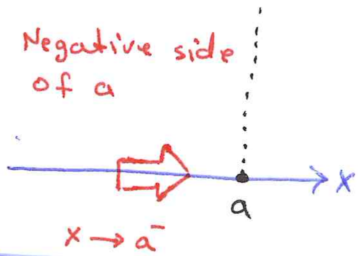
If $|x-c| < \delta_2$ then $|x-c| < \delta$, thus $|g(x) - M| < \frac{\epsilon}{2}$. Thus

A: One-sided limits: is two parts:

1 Right-hand limit ($x \to a^+$) means x approaches a from the positive side of a through values greater than a.

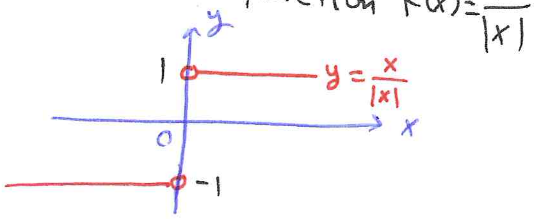


2 Left-hand limit ($x \to a^-$) means x approaches a from the negative side of a through values less than a.



Example 1 Find $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to 0^-} f(x)$ for the function $f(x) = \dfrac{x}{|x|}$

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$\lim\limits_{x \to 0^-} f(x) = -1$ (left-hand limit) and $\lim\limits_{x \to 0^+} f(x) = 1$ (Right-hand limit)

Let $f(x)$ be defined on an interval (a, b) where $a < b$.

If $f(x)$ approaches L as x approaches a within the interval (a, b), then we say f has right-hand limit L at a, and we write

$$\lim_{x \to a^+} f(x) = L$$

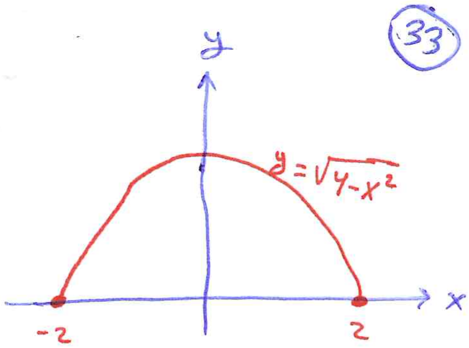
Let $f(x)$ be defined on an interval (c, a) where $c < a$.

If $f(x)$ approaches M as x approaches a within the interval (c, a), then we say f has left-hand limit M at a, and we write

$$\lim_{x \to a^-} f(x) = M$$

Example [2] Let $f(x)=\sqrt{4-x^2}$. Find

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to -2^+} f(x)$



Note that the domain of $f(x)$ i $[-2,2]$ and the graph of f is semicircle

(a) $\lim\limits_{x\to 2^-} f(x) = 0$ (b) $\lim\limits_{x\to -2^+} f(x) = 0$

(c) $\lim\limits_{x\to -2^-} f(x) = DNE$ (d) $\lim\limits_{x\to 2^+} f(x) = DNE$

Theorem (~~One-sided vs. Two-sided Limits~~)

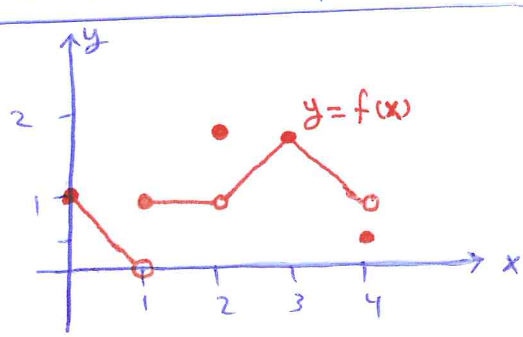
A function $f(x)$ has a limit L as x approaches c iff the one-sided limits of f are equal:

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Note that in Example 1: $\lim\limits_{x\to 0} f(x)$ does not exist because the right and left hand limits are not equal.

Example 2: $\lim\limits_{x\to 2} f(x)$ and $\lim\limits_{x\to -2} f(x)$ do not exist because the right and left hand limits are not equal.

Example [3] Consider the graph $y=f(x)$. Find



a) $\lim\limits_{x\to 0^+} f(x) = 1$

$\lim\limits_{x\to 0^-} f(x) = DNE$

$\lim\limits_{x\to 0} f(x) = DNE$

) $\lim\limits_{x\to 1^-} f(x) = 0$ $f(1)=1$

$\lim\limits_{x\to 1^+} f(x) = 1$

$\lim\limits_{x\to 1} f(x) = DNE$

(c) $\lim\limits_{x\to 2^-} f(x) = 1$ $f(2)=2$

$\lim\limits_{x\to 2^+} f(x) = 1$

$\lim\limits_{x\to 2} f(x) = 1$

(d) $\lim\limits_{x\to 3^-} f(x) = 2$ $f(3)=2$

$\lim\limits_{x\to 3^+} f(x) = 2$

$\lim\limits_{x\to 3} f(x) = 2$

(e) $\lim\limits_{x\to 4^-} f(x) = 1$ $f(4)=\frac{1}{2}$

$\lim\limits_{x\to 4^+} f(x) = DNE$

$\lim\limits_{x\to 4} f(x) = DNE$

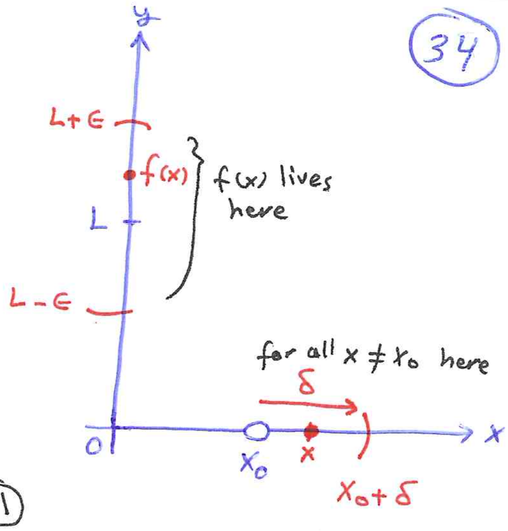
Definition (Right-hand limit)

$f(x)$ has right-hand limit L at x_0

$$\lim_{x \to x_0^+} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$\boxed{\text{if } 0 < x - x_0 < \delta \text{ then } |f(x) - L| < \epsilon} \to \text{(D1)}$$



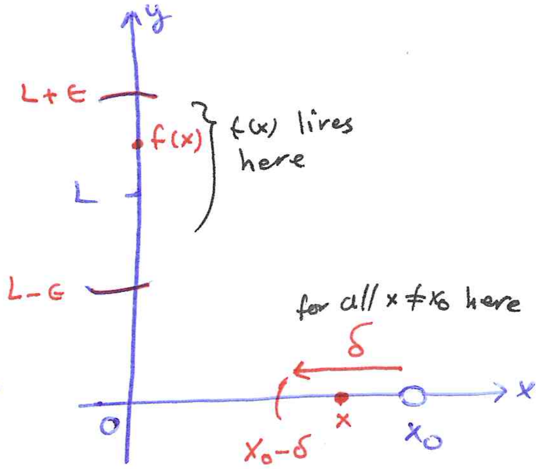
Definition (Left-hand limit)

$f(x)$ has left-hand limit L at x_0

$$\lim_{x \to x_0^-} f(x) = L$$

if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$\boxed{\text{if } -\delta < x - x_0 < 0 \text{ then } |f(x) - L| < \epsilon} \to \text{(D2)}$$



* Note that the relation between One and Two sided Limits is clear from (D1) and (D2):

→ For the right-hand limit we have (D1)

→ For the left-hand limit we have (D2)

$\boxed{\text{D1}}$ together with $\boxed{\text{D2}}$ provid the two sided limit

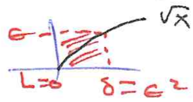
if $\quad |x - x_0| < \delta \quad$ then $\quad |f(x) - L| < \epsilon$

Thus, $\quad \lim_{x \to x_0} f(x) = L$

Example: Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$ $\qquad L = 0, \; x_0 = 0, \; f(x) = \sqrt{x}$

let $\epsilon > 0$, we must find $\delta > 0$ s.t for all x if $0 < x - 0 < \delta$ then $|\sqrt{x} - 0| < \epsilon$

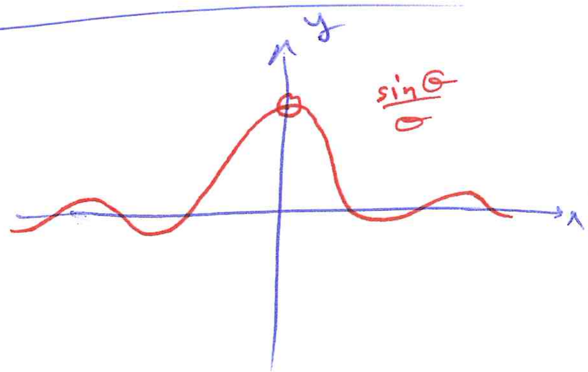
$0 < x < \delta$ then $\sqrt{x} < \epsilon \quad x < \epsilon^2 \quad \Rightarrow \quad 0 < \delta < \epsilon^2$



Theorem 7: $\displaystyle\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (θ in radians)

Example: Show that $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{4x} = \frac{3}{4}$

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \lim_{x \to 0} \frac{\sin 3x}{\frac{4}{3}\, 3x} = \frac{3}{4} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{4}$$



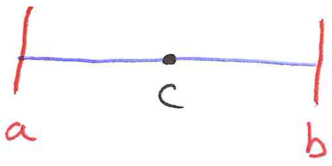
Example: Find $\displaystyle\lim_{t \to 0} \frac{\tan t \sec 2t}{3t}$

$$\lim_{t \to 0} \frac{\sin t}{\cos t} \frac{1}{3t} \frac{1}{\cos 2t} = \lim_{t \to 0} \frac{1}{3} \frac{\sin t}{t} \frac{1}{\cos t} \frac{1}{\cos 2t}$$

$$= \frac{1}{3} (1) (1) (1) = \frac{1}{3}$$

2.5 Continuity

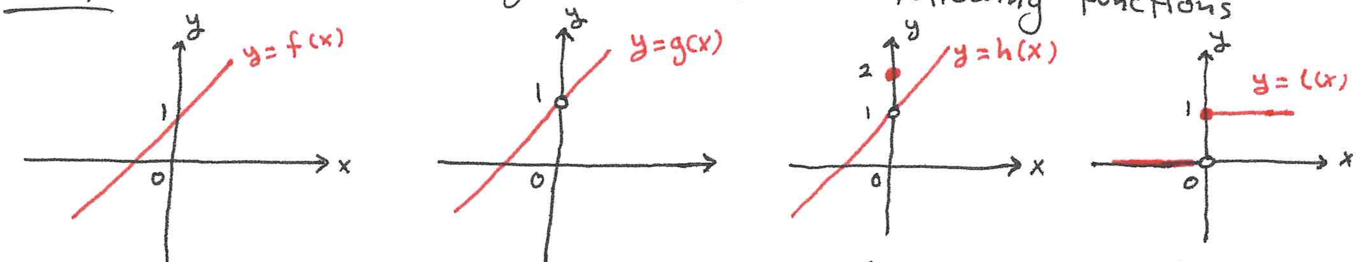
* Points are 3 kinds: ① interior points (c)
② left endpoints (a)
③ right endpoints (b)



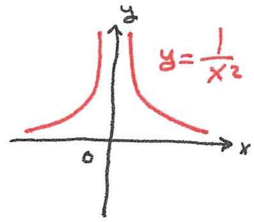
Definition: (Continuity at point)

A function f is continuous at an interior point $x=c$ of its domain if $\lim_{x\to c} f(x) = f(c)$

Example: Discuss the continuity at $x=0$ to the following functions



(a) (b) (c) (d)



(e)

) f is continuous at $x=0$ because $\lim_{x\to 0} f(x) = f(0) = 1$

) g is not continuous at $x=0$ because $\lim_{x\to 0} g(x) = 1 \neq g(0) \to$ DNE

h is not continuous at $x=0$ because $\lim_{n\to 0} h(x) = 1 \neq h(0) = 2$

l is not continuous at $x=0$ because $1 = f(0) \neq \lim_{x\to 0} l(x) \to$ DNE

$y=\frac{1}{x^2}$ is not continuous at $x=0$ because $\lim_{x\to 0} \frac{1}{x^2} = \infty$ and $f(0)$ is not defined.

) and (c) are called removable discontinuity because in
(b) g would be continuous if $g(0)=1$
(c) h would be continous if $h(0)=1$ instead of 2

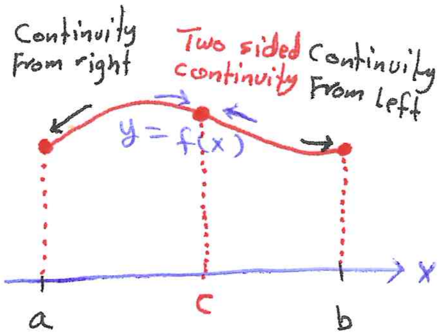
d) is called jumb discontinuity.

) is called infinite discontinuity.

Definition: A function f is

- continous at left endpoint $x=a$ of its domain if $\lim_{x \to a^+} f(x) = f(a)$

- continous at right endpoint $x=b$ of its domain if $\lim_{x \to b^-} f(x) = f(b)$

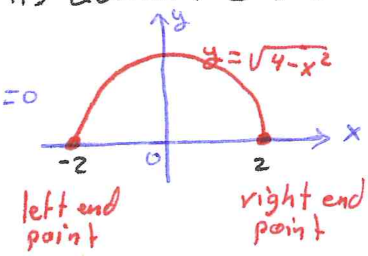


Example: $f(x)=\sqrt{4-x^2}$ is continuous at every point of its domain $[-2,2]$

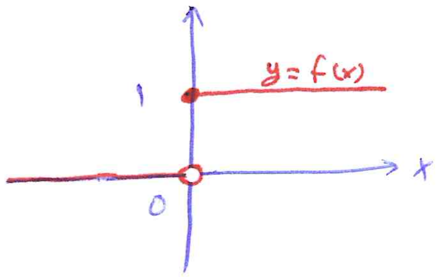
→ f is continous at -2 because $\lim_{x \to -2^+} \sqrt{4-x^2} = f(-2) = 0$

→ f is continous at 2 because $\lim_{x \to 2^-} \sqrt{4-x^2} = f(2) = 0$

→ f is continous on an interval $[-2,2]$.



Example: $f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$



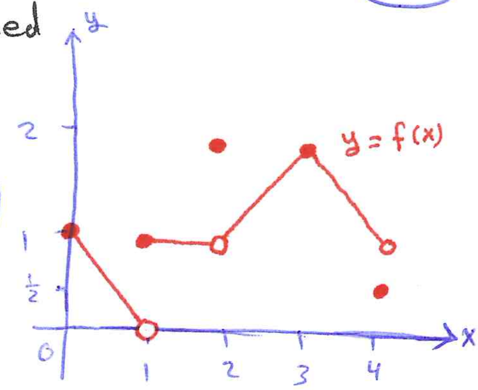
- f is right continuous at $x=0$ because $\lim_{x \to 0^+} f(x) = f(0) = 1$
- f is not left continuous at $x=0$ because $\lim_{x \to 0^-} f(x) = 0 \neq f(0) = 1$
- f is not continuous at $x=0$ because $\underbrace{\lim_{x \to 0} f(x)}_{\text{DNE}} \neq f(0) = 1$

Test of Continuity:

$f(x)$ is continuous at $\underbrace{x=c}$ (an interior point) iff the following three conditions hold:

1) $f(c)$ exists where $c \in D(f)$ domain of f
2) $\lim_{x \to c} f(x)$ exists
3) $\lim_{x \to c} f(x) = f(c)$

Example: Discuss the continuity of f at $x = 0, 1, 2, 3, 4$, where f is as given in the graph defined over the domain $[0, 4]$.



(a) f is continuous at $x = 0$ because
$\lim_{x \to 0^+} f(x) = f(0) = 1$ (right-hand limit exists at the left endpoint)

(b) f is discontinuous at $x = 1$ because
$\lim_{x \to 1} f(x)$ DNE

(c) f is discontinuous at $x = 2$ because $\lim_{x \to 2} f(x) = 1 \neq f(2) = 2$

(d) f is continuous at $x = 3$ because $\lim_{x \to 3} f(x) = f(3) = 2$

e) f is discontinuous at $x = 4$ because $\lim_{x \to 4^-} f(x) = 1 \neq f(4) = \frac{1}{2}$ (left-hand limit exists at the right end point).

Theorem: If the functions f and g are continuous at $x = c$, then the following functions are continuous at $x = c$:

1) $f \pm g$ 2) fg 3) Kf, where $K \in \mathbb{R}$

4) $\frac{f}{g}$ where $g(c) \neq 0$

5) $[f(x)]^{\frac{m}{n}}$ where $(f(x))^{\frac{m}{n}}$ is defined on an interval containing c, and $m, n \in \mathbb{Z}$

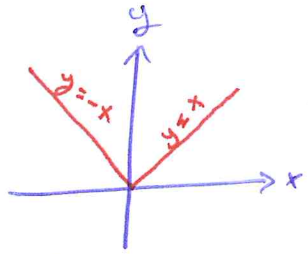
Example:
- Every polynomial is continuous at every point of the real line.
- Every rational function is continuous at every point where it's defined (denominator is different from zero).

Example: $f(x) = x^3 - 2x^2 + 1$ is continuous at every point x.

$$g(x) = \frac{f(x)}{x^2 - 4} = \frac{x^3 - 2x^2 + 1}{(x-2)(x+2)}$$ is continuous at every value of x except $x = 2$ and $x = -2$ where the denominator is zero.

Example $f(x) = |x|$ is continuos

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



if $x > 0$ then $f(x) = x$ polynomial which is continuous

if $x < 0$ then $f(x) = -x$ polynomial which is continuous

if $x = 0$ then $\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = f(0) = 0$

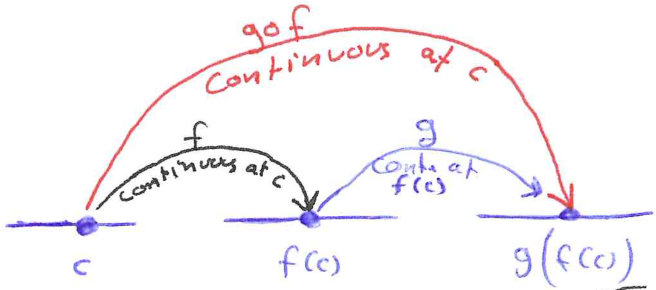
Continuity of trigonometric functions:

* The functions $\sin x$ and $\cos x$ are continuous at every value x.
* The functions $\tan x = \frac{\sin x}{\cos x}$, $\cot = \frac{\cos x}{\sin x} = \frac{1}{\tan}$, $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$

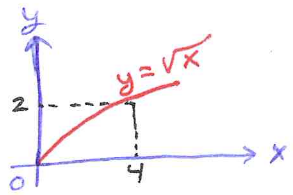
are continuous at every point except where they are not defined.

Theorem (Continuity of Composition)

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

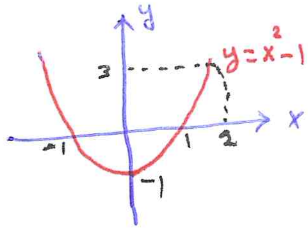


Example: Let $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$ show that $g \circ f$ is continuous at $x = 4$



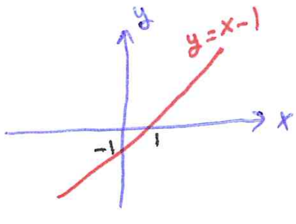
1. $f(x)$ is continuous at $x = 4$ because

$$\lim_{x \to 4} \sqrt{x} = f(4) = 2 \quad \text{and}$$



$g(x)$ is continuous at $x = f(4) = 2$ because

$$\lim_{x \to 2} (x^2 - 1) = g(2) = 3.$$



Thus by Theorem above $g \circ f$ is continuous at $x = 4$.

2. Note that $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 - 1 = x - 1$.

This is polynomial and continuous every where. Thus continuous at $x = 4$.

Continuous Extension to point

* A rational function f may have a limit L at point $x=c$ even if $f(c)$ is not defined (the denominator is zero).

Example: $f(x) = \dfrac{x^2-4}{x-2}$

- If $x=2 \Rightarrow f(2)$ is not defined but
- If $x \neq 2 \Rightarrow$ $f(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{(x-2)} = x+2$

The function $F(x) = x+2$ is th same as $f(x) = \dfrac{x^2-4}{x-2}$ for all $x \neq 2$
The only difference is that $F(x)$ is continuous at $x=2$ because

$$\lim_{x\to 2} F(x) = \lim_{x\to 2}(x+2) = 4 = F(2)$$

but $f(x)$ is not continuous at $x=2$ because

$$\lim_{x\to 2} f(x) = 4 \neq f(2)$$

Thus, $F(x)$ is called the continuous extension of $f(x)$ at $x=c$, and we write

$$F(x) = \begin{cases} f(x) & \text{if } x\neq c \text{ and } x\in D(f) \\ L & \text{if } x=c \end{cases}$$

where $\lim_{x\to c} f(x) = L$.

Example: show that $f(x) = \dfrac{x^2+x-6}{x^2-4}$ has a continuous extension at $x=2$, and find that extension.

- If $x=2 \Rightarrow f(2)$ is not defined.
- If $x\neq 2 \Rightarrow f(x) = \dfrac{x^2+x-6}{x^2-4} = \dfrac{(x-2)(x+3)}{(x-2)(x+2)} = \dfrac{x+3}{x+2}$

The function $F(x) = \dfrac{x+3}{x+2}$ is the same as $f(x)$ for all $x\neq 2$

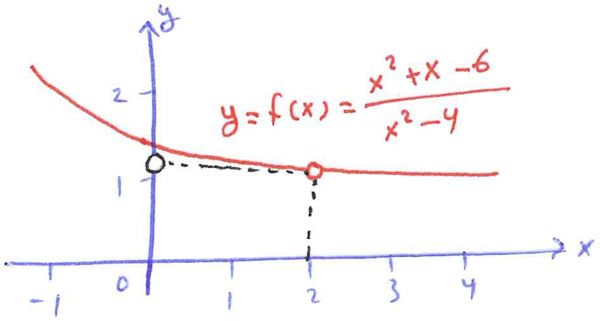
$\Rightarrow$ But $F(x)$ is continuous at $x=2$ because

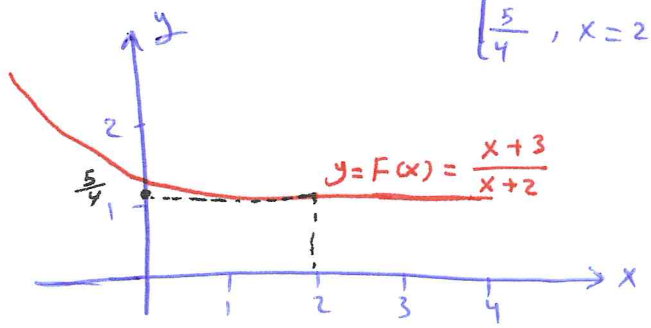
$$\lim_{x\to 2} F(x) = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4} = F(2)$$

and $f(x)$ is not continuous at $x=2$ because

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{x+3}{x+2} = \frac{5}{4} \neq f(2)$$

Thus, F is the continuous extension of f to $x=2$. $F(x)=\begin{cases} \dfrac{x^2+x-6}{x^2-4}, & x\neq 2 \\ \dfrac{5}{4}, & x=2 \end{cases}$





Continuity on Intervals

- Let $D(f)$ be the domain of the function f:

  → A function f is continuous if it is continuous at every where in $D(f)$

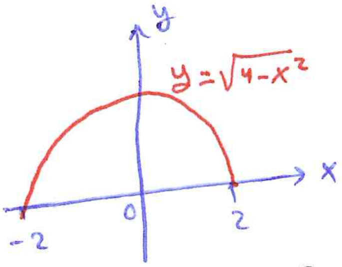
  → A function f is continuous on an interval $I \subset D(f)$ if f is continuous at every point in I.

  → If the function f is continuous on an interval I, then f is continuous on any interval $J \subset I$.

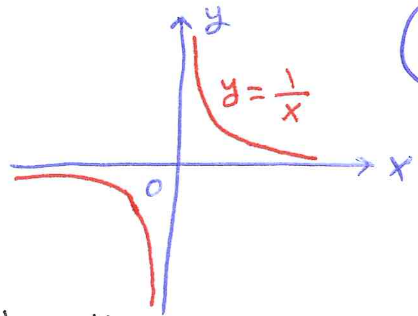
Example: → Polynomials functions are continuous on every interval.

→ Rational functions are continuous on every interval on which they are defined.

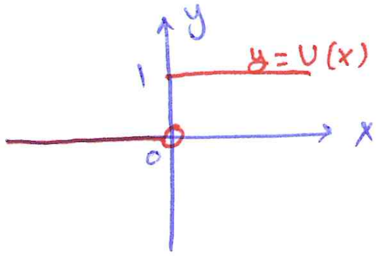
Example*:



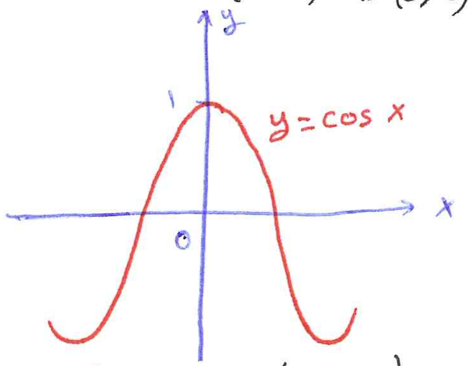
(a) continuous on $[-2,2]$



(b) continuous on $(-\infty,0)$ and $(0,\infty)$



(c) Continuous on $(-\infty,0)$ and $[0,\infty)$

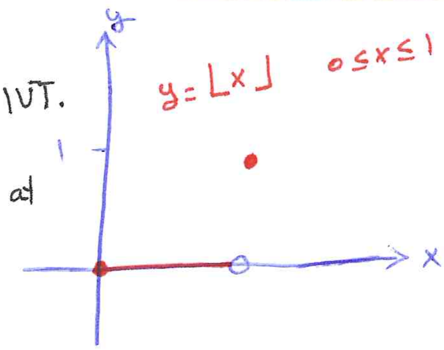


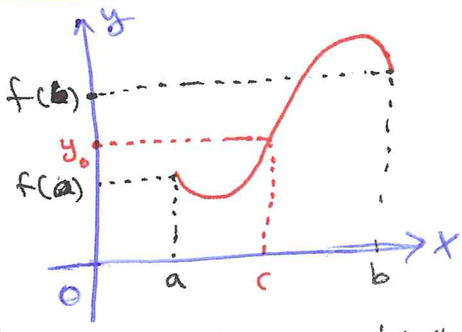
(d) Continuous on $(-\infty,\infty)$

Theorem; The Intermediate Value Theorem (IVT)

Suppose $f(x)$ is continuous on an interval $I=[a,b]$. If y_0 is any number between $f(a)$ and $f(b)$, then there exists a number c between a and b such that $f(c)=y_0$

The continuity of f on I is essential to IVT. If f is discontinuous at even one point of I, then IVT may fail





IVT geometrically

Consequences of the IVT

- IVT is the reason that the graph of a function continuous on an interval I can not have any breaks. It will be **connected** and **single unbroken curve**. For instance (a) and (d) in Example*. It will not have jumps like (c) in Example* or separate branches like (b) in Example*.

Theorem: (limits of continuous functions)

If g is continuous at the point : b, and $\lim_{x\to c} f(x) = b$, then $\lim_{x\to c} g(f(x)) = g(b)$

$$= g\left(\lim_{x\to c} f(x)\right)$$

Example: $\lim_{x\to\frac{\pi}{2}} \cos\left(2x + \sin\left(\frac{3\pi}{2}+x\right)\right) = \cos\left(\lim_{x\to\frac{\pi}{2}} 2x + \lim_{x\to\frac{\pi}{2}} \sin\left(\frac{3\pi}{2}+x\right)\right)$

$$= \cos\left(\pi + \sin 2\pi\right)$$

$$= \cos\pi = -1$$

Example: Show that $\exists$ a root of the equation $x^3 - x - 1 = \boxed{0}$ between 1 and 2.

let $f(x) = x^3 - x - 1$

$f(1) = 1 - 1 - 1 = -1 < 0$

$f(2) = 8 - 2 - 1 = 5 > 0$

Since $0 = y_0$ is between $f(1)$ and $f(2)$

Since f is continuous (polynomial). Thus, by IVT, $\exists$ a (zero / root) of f between 1 and 2. $\boxed{x = 1.32}$

2.6 Limits involving infinity; Asymptotes of Graph

Def [1] $f(x)$ has the limit L as x approaches infinity and we write $\lim\limits_{x\to\infty} f(x) = L$ if for every number $\epsilon > 0$, there exist a corresponding number M such that for all x

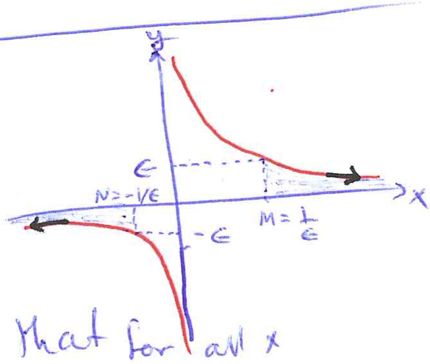
$$x > M \Rightarrow |f(x) - L| < \epsilon.$$

[2] $f(x)$ has the limit L as x approaches minus infinity and we write $\lim\limits_{x\to-\infty} f(x) = L$ if for every number $\epsilon > 0$, there exist a corresponding number N such that for all x

$$x < N \Rightarrow |f(x) - L| < \epsilon.$$

Example: $y = \frac{1}{x}$ show that

a) $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ (b) $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$



a) Let $\epsilon > 0$. We need to find a number M such that for all x

if $\boxed{x > M} \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$ $\boxed{|x| > \frac{1}{\epsilon}}$ This is true if $M \geq \frac{1}{\epsilon}$

This proves that $\lim\limits_{x\to\infty} \frac{1}{x} = 0$

b) let $\epsilon > 0$. We need to find a number N such that for all x

if $\boxed{x < N} \Rightarrow \left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \epsilon \Rightarrow |x| > \frac{1}{\epsilon}$ $\boxed{x < -\frac{1}{\epsilon}}$ This is true if $N \leq \frac{-1}{\epsilon}$

This proves that $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

Note that $\lim\limits_{x\to\pm\infty} \frac{1}{x} = 0$ and $\lim\limits_{x\to\pm\infty} k = k$

limit at infinity of Rational functions

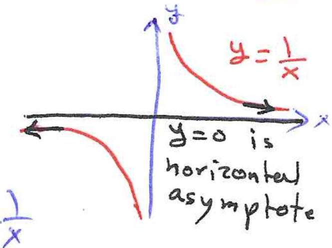
Example [1] $\lim\limits_{x\to\infty} \dfrac{2x^2 - x + 4}{3x^2 + 5} = \lim\limits_{x\to\infty} \dfrac{2 - \frac{1}{x} + \frac{4}{x^2}}{3 + \frac{5}{x^2}}$

$= \dfrac{2}{3}$

numerator بسط
denominator مقام

[2] $\lim\limits_{x\to-\infty} \dfrac{8x - 6}{4x^3 + 1} = \lim\limits_{x\to-\infty} \dfrac{\frac{8}{x^2} - \frac{6}{x^3}}{4 + \frac{1}{x^3}} = \dfrac{0+0}{4+0} = \dfrac{0}{4} = 0$

Horizontal Asymptotes

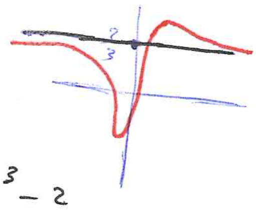


Example: $y = \frac{1}{x}$ the horizontal line $y=0$ "x-axis" is the horizontal asymptote of the graph $f(x) = \frac{1}{x}$

because $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

Def A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either $\lim\limits_{x\to\infty} f(x) = b$ or $\lim\limits_{x\to-\infty} f(x) = b$

Example [1] $\lim\limits_{x\to\pm\infty} \dfrac{2x^2 - x + 4}{3x^2 + 5} = \dfrac{2}{3}$

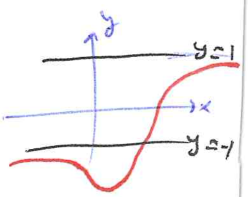


$\Rightarrow y = \frac{2}{3}$ is a horizontal asymptote

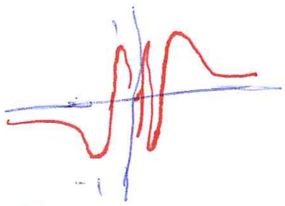
[2] Find the horizontal asymptote of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$

- For $x \geq 0 \Rightarrow \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^3 - 2}{x^3 + 1} = \lim\limits_{x\to\infty} \dfrac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} = 1$
- For $x \leq 0 \Rightarrow \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x^3 - 2}{-x^3 + 1} = \lim\limits_{x\to-\infty} \dfrac{1 - \frac{2}{x^3}}{-1 + \frac{1}{x^3}} = -1$

The horizontal asymptotes are $y = -1$ and $y = 1$



Example: Find (a) $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$

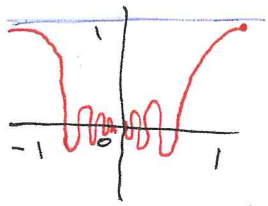


$\sin\frac{1}{x}$

take $t = \frac{1}{x}$ as $x \to \infty$, $t \to 0^+$

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \sin t = 0$$

$\boxed{b}$ $\lim_{x\to\pm\infty} x \sin\left(\frac{1}{x}\right)$



$x \sin\frac{1}{x}$

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \frac{\sin t}{t} = 1$$

$$\lim_{x\to -\infty} x\sin\left(\frac{1}{x}\right) = \lim_{t\to 0^-} \frac{\sin t}{t} = 1$$

Example: Find $\lim_{x\to\infty} \left(x - \sqrt{x^2+4}\right) = \lim_{x\to\infty} \left(x - \sqrt{x^2+4}\right) \cdot \frac{x+\sqrt{x^2+4}}{x+\sqrt{x^2+4}}$

$$= \lim_{x\to\infty} \frac{x^2 - (x^2+4)}{x+\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{-4}{x+\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{\frac{-4}{x}}{1+\sqrt{1+\frac{4}{x^2}}} = 0$$

Oblique Asymptotes

* If the degree of the numerator (البسط) of a rational function is 1 greater than the degree of the denominator, then the graph has an oblique (منحرف) or sant (مائل) line asymptote.

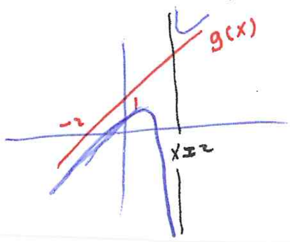
Example: Find the oblique asymptote of $f(x) = \frac{x^2-3}{2x-4}$

$$f(x) = \frac{x^2-3}{2x-4} = \underbrace{\frac{x}{2}+1}_{g(x)} + \underbrace{\frac{1}{2x-4}}_{\text{remainder } r(x)}$$

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x-4 \overline{\big) \; x^2 - 3} \\ \underline{-x^2 + 2x} \\ 2x - 3 \\ \underline{-2x + 4} \\ 1 \end{array}$$

oblique Asymptotes because

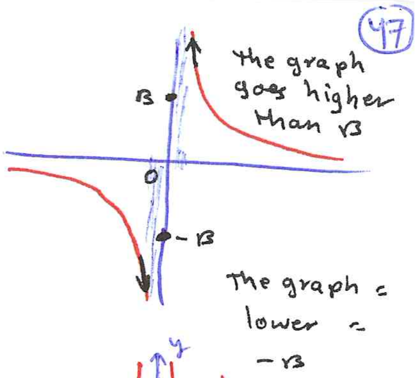
$$\lim_{x\to\infty} \frac{1}{2x-4} = 0$$



g(x) is dominante when x is Large
r(x) is dominante when x is near 2

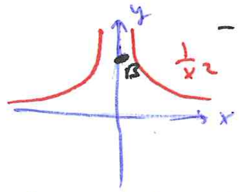
Infinite limits

Example: Find



a) $\lim_{x \to 0^+} \frac{1}{x} = \infty$ b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

c) $\lim_{x \to 1^+} \frac{1}{x-1} = \infty$ d) $\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$



e) $\lim_{x \to 0^+} \frac{1}{x^2} = \infty$ f) $\lim_{x \to 0^-} \frac{1}{x^2} = \infty$

Note that in a) and c) (b) and d)) we are not saying the limit exists, nor that there is a real number ∞. We are saying $\lim_{x \to 0^+} \frac{1}{x}$ DNE because $\frac{1}{x}$ becomes arbitrary large and positive as $x \to 0^+$

Def 1) $f(x)$ approaches infinity as x approaches x_0 and we write $\lim_{x \to x_0} f(x) = \infty$ if for every positive real number B, there exist a corresponding $\delta > 0$ such that for all x $\quad 0 < |x - x_0| < \delta \Rightarrow f(x) > B$.

2) $f(x)$ approaches minus infinity as x approaches x_0, and we write $\lim_{x \to x_0} f(x) = -\infty$ if for every negative number $-B$, there exists a corresponding $\delta > 0$ such that for all x $\quad 0 < |x - x_0| < \delta \Rightarrow f(x) < -B$

Example Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

let $B > 0$, we need to find $\delta > 0$ such that

$0 < |x - 0| < \delta \Rightarrow \frac{1}{x^2} > B$

$x^2 < \frac{1}{B}$

$|x| < \frac{1}{\sqrt{B}}$

$\Rightarrow 0 < \delta \le \frac{1}{\sqrt{B}}$ Now

If $|x| < \delta$ then $\frac{1}{x^2} > \frac{1}{\delta^2} \ge B$

Therefore, by definition

$\lim_{x \to 0} \frac{1}{x^2} = \infty$

Vertical Asymptotes

Def A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

Example: Find vertical asymptote of $y = \frac{1}{x}$

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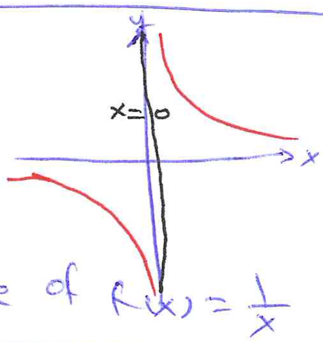
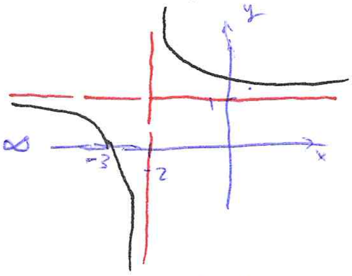


$\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\Rightarrow$ $(x=0)$ or (y-axis) is a vertical asymptote of $f(x) = \frac{1}{x}$

Example ① Find the horizontal and vertical asymptotes of

$$y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$$ قسمة طويلة



For Horizontal asymptotes: we check as $x \longrightarrow \pm\infty$

For Vertical asymptotes: we check as $x \longrightarrow -2$

$\lim_{x \to \pm\infty} \frac{x+3}{x+2} = \lim_{x \to \pm\infty} \frac{1 + \frac{3}{x}}{1 + \frac{2}{x}} = 1$ " horizontal asymptote " $y = 1$

$\lim_{x \to -2^+} \frac{x+3}{x+2} = \frac{1}{\text{small positive}} = \infty$

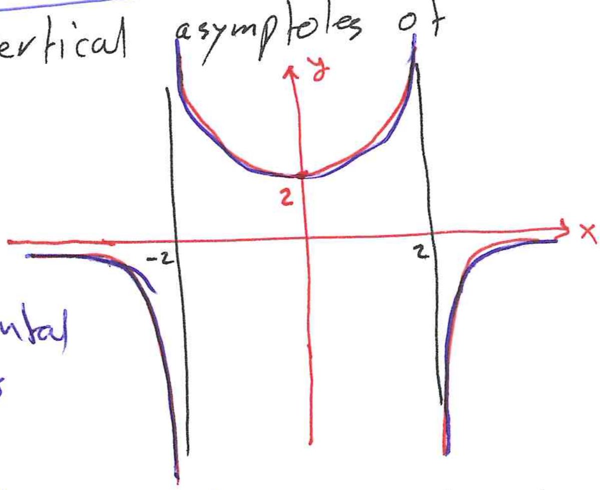
$\lim_{x \to -2^-} \frac{x+3}{x+2} = \frac{1}{\text{small negative}} = -\infty$

$\Rightarrow$ $x = -2$ is vertical asymptote

---

② Find the horizontal and vertical asymptotes of

$$y = -\frac{8}{x^2 - 4}$$



- $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to +\infty} f(x) = 0$ $\Big\rangle$ $y = 0$ is horizontal asymptotes

- $\lim_{x \to 2^+} f(x) = -\infty$, $\lim_{x \to 2^-} f(x) = \infty$ $\Big\rangle$ $x = 2$ is vertical asymptotes

also $x = -2$ " " " .

③ $y = \tan x$

odd multiple of $\frac{\pi}{2}$

Tangent and Derivative at a point

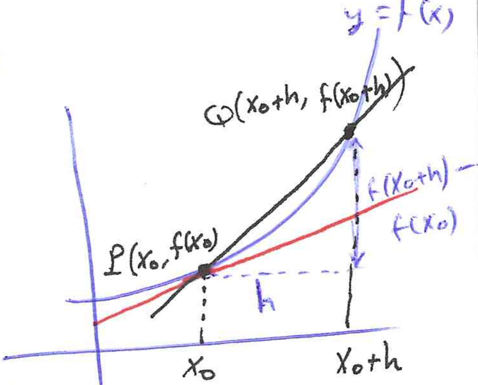
Def: The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is

$$m = \lim_{h \to 0} \boxed{\frac{f(x_0+h) - f(x_0)}{h}}$$ (provided the limit exists)

→ average rate of change

The tangent line to the curve at P with slope m is

$$y = m(x - x_0) + f(x_0)$$



Def: The derivative of a function f at a point x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$ « limit exists »

Example: find the slope of the function $f(x) = x^2 + 1$ at the point $(\overset{x_0}{2}, \overset{f(x_0)}{5})$ and find an equation for the tangent line

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + 1 - (2^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h + h^2 + 1 - 5}{h} = \lim_{h \to 0} \frac{h(4+h)}{h} = 4 + 0 = 4$$

$$y = m(x - x_0) + f(x_0)$$

$$= 4(x-2) + 5$$

$$y = 4x - 3$$ (tangent line)

3.2 The Derivative as a function

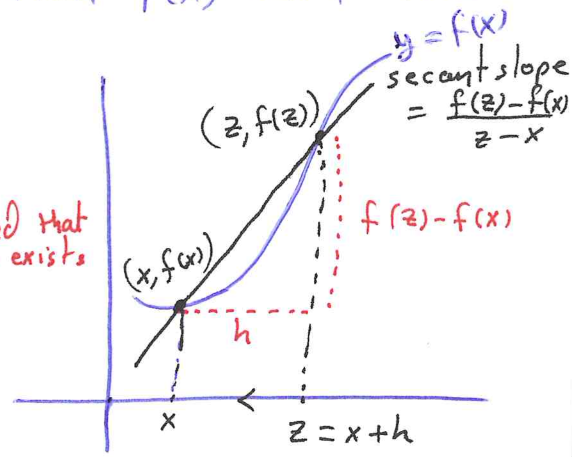
Def: The derivative of the function $f(x)$ w.r.t the variable x is

prime

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that limit exists

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$



- If f' exists at x, we say that f is differentiable (has derivative) at x.
- If f' exists at every point in the domain of f, we call f is differentiable.

- There are many ways to denote the derivative of $y = f(x)$

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x)$$

- The derivative at specified number $x = a$ is

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}$$

Example: Using the definition (a) find the derivative $f(x) = \sqrt{x}$ for $x > 0$

(a)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(b) $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

(c) tangent line at $x = 4$

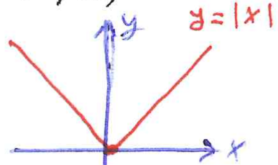
$m = f'(4) = \frac{1}{4}$, point $(4, 2)$

$y - 2 = \frac{1}{4}(x - 4) = \frac{1}{4}x - 1$

$$\boxed{y = \frac{1}{4}x + 1}$$

- A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has derivative at each point of the interval.
- A function $y = f(x)$ is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and
  - (a) $\lim\limits_{h \to 0^+} \dfrac{f(a+h) - f(a)}{h}$ exists and Right-hand derivative at a
  - (b) $\lim\limits_{h \to 0^-} \dfrac{f(b+h) - f(b)}{h}$ exists left-hand derivative at b

Example: show that $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$



y' is not defined at $x = 0$ (corner) because (a) $\neq$ (b)

For $x > 0 \Rightarrow |x| = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - x}{h} = 1$$

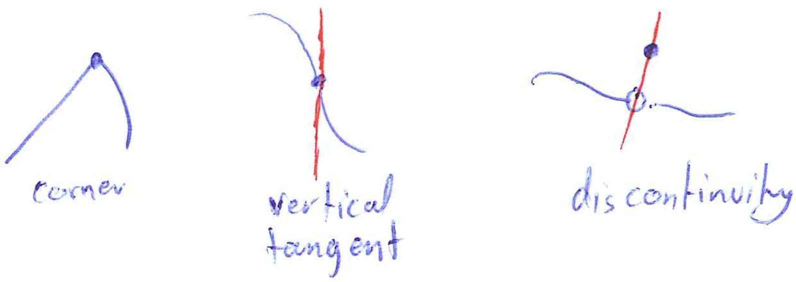
For $x < 0 \Rightarrow |x| = -x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-x - h + x}{h} = -1$$

At $x = 0$, there is no derivative because (a) $\neq$ (b)

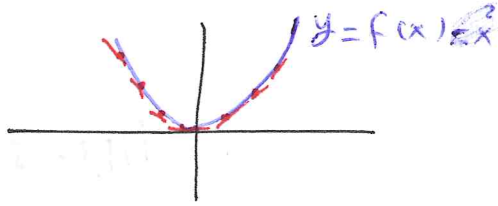
- Right-hand derivative at zero $= \lim\limits_{h \to 0^+} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^+} \dfrac{|h|}{h} = 1$
- left-hand derivative at zero $= \lim\limits_{h \to 0^-} \dfrac{|0+h| - |0|}{h} = \lim\limits_{h \to 0^-} \dfrac{|h|}{h} = -1$

The function has no derivative at corners, vertical tangent and discontinuity.

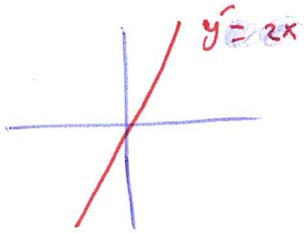


Theorem 1: If f has a derivative at $x=c$, then f is continuous at $x=c$

Example: Given the graph of $y=f(x)$. Graph the derivative



$\Rightarrow$



Proof: Assume that $f'(c)$ exists.

we need to show $\lim_{x\to c} f(x) = f(c) \Leftrightarrow$ $\quad x = c+h$

$$\lim_{h\to 0} f(c+h) = f(c)$$

If $h \neq 0$, then

$$f(c+h) = \cancel{f(c)+f(h)+f(c)-f(c)}$$

$$= f(c) + f(c+h) - f(c)$$

$$= f(c) + \frac{f(c+h)-f(c)}{h}\cdot h$$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}\cdot h$$

$$= f(c) + f'(c)\lim_{h\to 0} h$$

$$= f(c) + f'(c)\,(0)$$

$$\lim_{h\to 0} f(c+h) = f(c)$$

3.3 Differentiation Rules

① Derivative of a constant function

If $f(x) = c$, then $\frac{df}{dx} = 0$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$

---

② Power Rule

If $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$ $\quad n \in \mathbb{R}$

for all x, where x^n and x^{n-1} are defined.

Example ① Let $f(x) = \sqrt{3}$. $\Rightarrow f'(x) = 0$

② $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

③ $f(x) = \frac{1}{x^4} \Rightarrow f'(x) = -4x^{-5} = \frac{-4}{x^5}$

④ $g(x) = \sqrt{x^{2+\pi}} \Rightarrow g(x) = \left(x^{2+\pi}\right)^{\frac{1}{2}} = x^{1+\frac{\pi}{2}}$

$\Rightarrow g'(x) = \left(1 + \frac{\pi}{2}\right) x^{\frac{\pi}{2}} = \left(1 + \frac{\pi}{2}\right)\sqrt{x^{\pi}}$

⑤ $h(x) = x^{\sqrt{3}} \Rightarrow h'(x) = \sqrt{3}\, x^{\sqrt{3}-1}$

Let $f(x) = c\,u(x)$

③ where $u(x)$ is a differentiable function of x and c is a constant, then

$$f'(x) = \frac{d}{dx}(cu) = c\frac{du}{dx} = c\,u'(x)$$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c\,u(x+h) - c\,u(x)}{h}$

$= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} = c\,u'(x)$

Example: ① If $f(x) = 5x^2$, then $f'(x) = (5)(2)x = 10x$

② If $f(x) = -\sqrt{3}\,x^3$, then $f'(x) = -3\sqrt{3}\,x^2$

④ Derivative sum Rule:

If $f(x) = u(x) + v(x)$ where u and v are differentiable functions of x, then f is differentiable at every point where u and v are both differentiable

$$f'(x) = u'(x) + v'(x)$$

Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h}$

$= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right]$

$= u'(x) + v'(x)$

Example: ① Let $f(x) = x^3 - \frac{3}{2}x^2 - 7x + 3$.

$f'(x) = 3x^2 - 3x - 7$

② Does the curve $g(x) = x^2 + 1$ have any horizontal tangents?

Horizontal tangents $\Rightarrow g'(x) = 0 = 2x \Rightarrow \boxed{x = 0}$

$(0, g(0)) = (0, 1)$

5 Derivative Product Rule :

Let $f(x) = g(x)\, p(x)$, where g and p are differentiable at x, then f is differentiable at x :

$$f'(x) = g(x)\, p'(x) + p(x)\, g'(x)$$

Example: Find the derivative of $y = (x^2-1)(x^3+3)$

$$y' = (x^2-1)(3x^2) + (x^3+3)(2x)$$

$$= 3x^4 - 3x^2 + 2x^4 + 6x$$

$$= 6x^4 - 3x^2 + 6x$$

Proof: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{g(x+h)\,p(x+h) - g(x)\,p(x)}{h}$

$$= \lim_{h\to 0} \frac{g(x+h)\,p(x+h) - g(x+h)\,p(x) + g(x+h)\,p(x) - g(x)\,p(x)}{h}$$

$$= \lim_{h\to 0} \left[g(x+h)\, \frac{p(x+h)-p(x)}{h} + p(x)\, \frac{g(x+h)-g(x)}{h} \right]$$

$$= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} + p(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= g(x)\, p'(x) + p(x)\, g'(x)$$

6 Derivative Quotient Rule :

Let $f(x) = \dfrac{g(x)}{p(x)}$, $p(x) \neq 0$, p and g are differentiable

then f is differentiable at x:

$$f'(x) = \frac{p(x)\, g'(x) - g(x)\, p'(x)}{p^2(x)}$$

Example: Find the derivative of $y = \dfrac{3x-4}{x^2+1}$

$$y' = \frac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3-6x^2+8x}{(x^2+1)^2} = \frac{3+8x-3x^2}{(x^2+1)^2}$$

Proof: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{g(x+h)}{p(x+h)} - \frac{g(x)}{p(x)}}{h}$$

$$= \lim_{h\to 0} \frac{p(x)\,g(x+h) - g(x)\,p(x+h)}{h\,p(x)\,p(x+h)}$$

$$= \lim_{h\to 0} \frac{p(x)g(x+h) - p(x)g(x) + p(x)g(x) - g(x)p(x+h)}{h\,p(x)\,p(x+h)}$$

$$= \lim_{h\to 0} \frac{p(x)\frac{g(x+h)-g(x)}{h} - g(x)\frac{p(x+h)-p(x)}{h}}{p(x)\,p(x+h)}$$

$$= \frac{p(x)\,g'(x) - g(x)\,p'(x)}{p^2(x)}$$

---

6 Second and Higher-Order Derivatives:

Let $y = f(x)$

$y' = f'(x) = \dfrac{dy}{dx}$ 1^{st} derivative "y prime"

$y'' = f''(x) = \dfrac{d^2y}{dx^2}$ 2^{nd} derivative "y double prime"

$y''' = f'''(x) = \dfrac{d^3y}{dx^3}$ 3^{rd} derivative "y trible prime"

$\vdots$

$y^{(n)} = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$ n^{th} derivative "y super n"

Assuming f has all derivative

n is positive integer

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2\, f(x).$$

Example: Find the derivatives of all orders of the function $y = x^3 - 3x^2 + 2$

First derivative $y' = 3x^2 - 6x$

Second derivative $y'' = 6x$

Third derivative $y''' = 6$

Fourth derivative $y^{(4)} = 0$

The fifth and later derivatives are zero.

Proof of 2

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \lim_{z \to x} \frac{z^n - x^n}{z - x}$$

اذا مش لهدف وزع

$$= \lim_{z \to x} \frac{(z-x)\left(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}\right)}{(z-x)}$$

$$= \lim_{z \to x} \left(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}\right)$$

$$= n\,x^{n-1}$$

3.4 The Derivateves as a Rate of Change

Definition: The instantaneous rate of change of f w.r.t X at X_0 is

$$f'(X_0) = \lim_{h \to 0} \frac{f(X_0+h) - f(X_0)}{h}$$ "limit exists"

Example: Let $A = \frac{\pi}{4} D^2$

A : area of a circle
D : diameter of circle

Find the instantaneous rate of change of the area w.r.t the diameter when D = 10 m

$$\left.\frac{dA}{dD}\right|_{D=10} = \left.\frac{\pi}{4} 2D\right|_{D=10} = \left.\frac{\pi}{2} D\right|_{D=10} = \frac{\pi}{2}(10) = 5\pi \ m^2/m$$

Suppose an object is moving accordingly to

$$s = f(t)$$

t : time
s : distance

- The displacement of the object over the time interval $[t_1, t_2]$ is $\Delta t = t_2 - t_1$

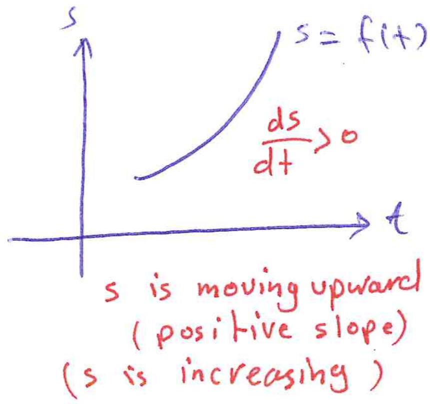
$$\Delta s = f(t_2) - f(t_1) = f(t_1 + \Delta t) - f(t_1)$$

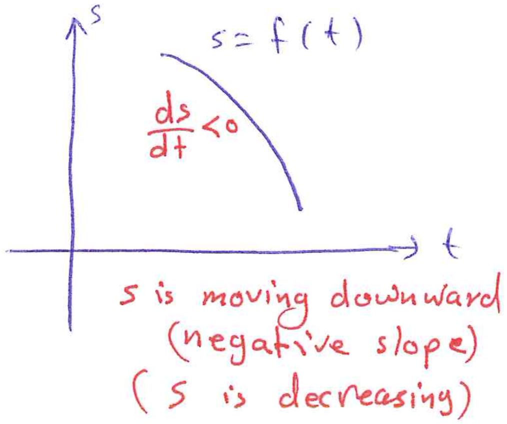
- The average velocity of the object over the time interval $[t_1, t_2]$ is

$$V_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t_1+\Delta t) - f(t_1)}{\Delta t}$$

- The velocity (instantaneous velocity) at time t is

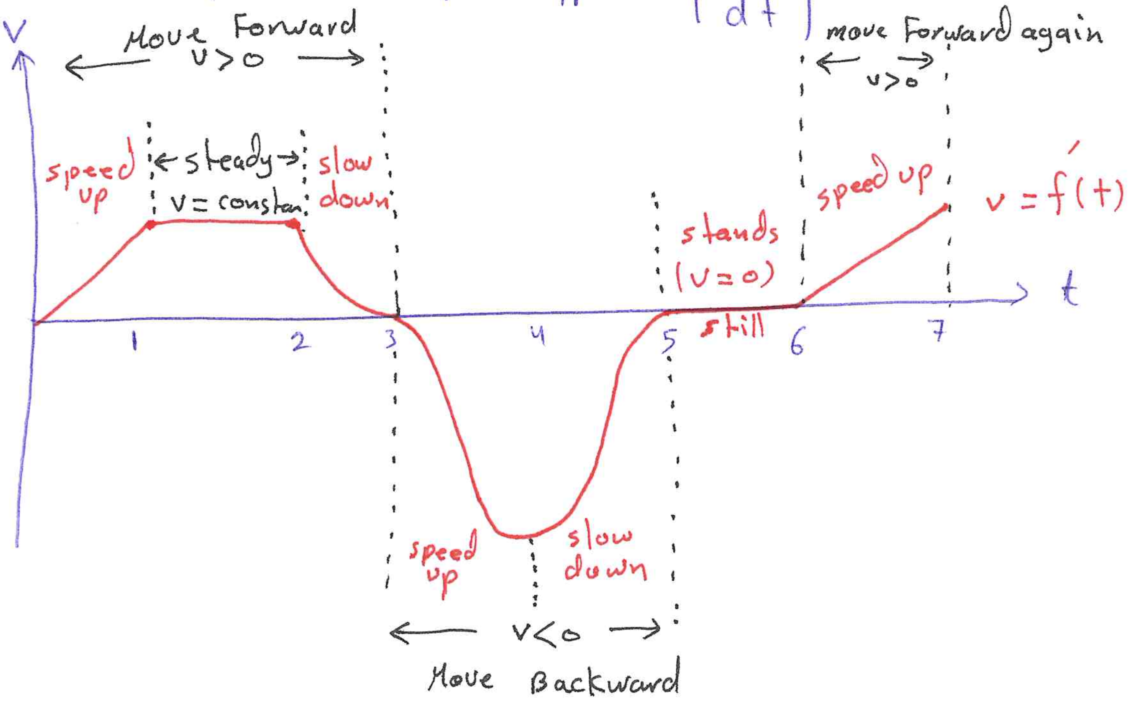
$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$





- Speed is the absolute value of velocity

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$



- Acceleration is the derivative of velocity w.r.t time.

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

- Jerk is the derivative of acceleration w.r.t. time

$$j(t) = \frac{da}{dt} = \frac{d^2 v}{dv^2} = \frac{d^3 s}{ds^3}$$

Example The free falling equation for an object falls from a rest is $s = \frac{1}{2} g t^2$ where g is the gravity acceleration $= 9.8\ m/s^2$

$$s = \frac{1}{2}(9.8) t^2 = 4.9\, t^2$$

[a] How many meters does the object fall in the first 2s

$$s(2) = 4.9\,(2)^2 = 4.9\,(4) = 19.6\ m$$

[b] What is its velocity at $t=2$?

$$v(t) = s'(t) = 2\,(4.9)\, t = 9.8\, t$$

$$v(2) = 9.8\,(2) = 19.6\ m/s$$

[c] what is its speed at $t=2$?

$$\text{speed}\Big|_{t=2} = |v(2)| = 19.6\ m/s$$

[d] what is its acceleration at $t=2$?

$$a(t) = \frac{dv}{dt} = 9.8$$

$$a(2) = 9.8\ m/s^2$$

(e) what is its jerk at $t=2$

$j(t) = a'(t) = 0$. Thus $j(2) = 0\ m/s^3$

Example: A rock straight up with a lunch velocity of 50 m/s. It reaches a height of $s = 49\,t - 4.9\,t^2$ m after t second.

(a) How high does the rock go?

velocity $= 0$

$v(t) = s'(t) = 49 - 9.8\,t = 0 \iff t = 5$ second

$\Rightarrow$ The rock's height at $t=5$ is $s(5) = 49(5) - 4.9\,(5)^2$
$= 122.5$ m

[b] what is the velocity and speed when the rock is 78.4 m above the ground on the way above and on the way down? (61)

$$s(t) = 49t - 4.9t^2 = 78.4$$
$$4.9t^2 - 49t + 78.4 = 0$$
$$4.9(t^2 - 10t + 16) = 0$$
$$(t-2)(t-8) = 0 \qquad t = 2 \text{ seconds}, \quad t = 8 \text{ seconds}$$

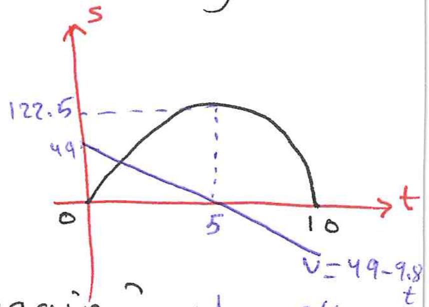
$$v(t) = 49 - 9.8t$$
$$v(2) = 49 - 9.8(2) = 29.4 \text{ m/s} \qquad \text{" The rock is moving up"}$$
$$v(8) = 49 - 9.8(8) = -29.4 \text{ m/s} \qquad \text{" " " " " down"}$$

The speed is $|v(t)| = 29.4$ m/s

[c] what is the acceleration of the rock at any time?

$$a(t) = v'(t) = -9.8 \text{ m/s}^2$$

The acceleration is always downward.
As the rock rises, it slows down.
As the rock falls, it speeds up.

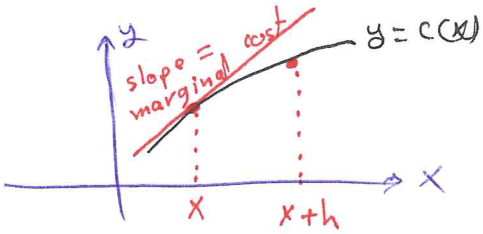


[d] When does the rock hit the ground again? when $s(t) = 0$

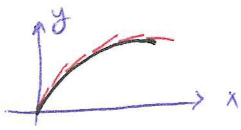
$$s(t) = 49t - 4.9t^2 = 0$$
$$4.9t(10 - t) = 0 \iff t = 0 \text{ sec}, \quad t = 10 \text{ sec.}$$

Derivative in Economics:

- $c(x)$: cost of production "to produce x"
- Marginal cost of production: $c'(x) = \lim_{h \to 0} \dfrac{c(x+h) - c(x)}{h}$



- $f'(x)$ measures the sensitivity
"small change in x produce large change in the value of a function $f(x)$"



$\boxed{3.5}$ Derivatives of Trigonometric Functions

$\boxed{1}$ If $f(x) = \sin x$, then $f'(x) = \cos x$

Proof: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \left(\cos x \frac{\sin h}{h}\right)$$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \sin x \ (0) + \cos x \ (1) = \cos x$$

Example: Find y' for the following functions:

$\boxed{a}$ $y = 3x - \sin x$ $\qquad y' = 3 - \cos x$

$\boxed{b}$ $y = 3x \sin x$ $\qquad y' = 3x \cos x + 3 \sin x$

$\boxed{c}$ $y = \dfrac{\sin x}{3x}$ $\qquad y' = \dfrac{3x \cos x - 3 \sin x}{9x^2}$

$\boxed{2}$ If $f(x) = \cos x$, then $f'(x) = -\sin x$

Proof: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0}\left(\cos x \, \frac{\cosh - 1}{h}\right) - \lim_{h\to 0}\left(\sin x \, \frac{\sin h}{h}\right)$$

$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \,(0) - \sin x \,(1) = -\sin x$$

Example: Find y' for the following functions:

[a] $y = 1 - \sin x + \cos x \quad \Rightarrow y' = -\cos x - \sin x$

[b] $y = (1 - \sin x)\cos x \quad \Rightarrow y' = \sin x(\sin x - 1) - \cos x \cos x$

$$= \sin^2 x - \sin x - \cos^2 x$$

[c] $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow y' = \dfrac{-(1-\sin x)\sin x + \cos x \cos x}{(1 - \sin x)^2}$

$$= \frac{\sin^2 x - \sin x + \cos^2 x}{(1-\sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$= \frac{1}{1 - \sin x}$$

Simple Harmonic Motion: الحركة التوافقية البسيطة

Example: A weight hanging (معلق) from a spring is stretched down 3 units under its rest position and relased at time $t = 0$.
It's position at any time later on is

$$s = 3\cos t$$

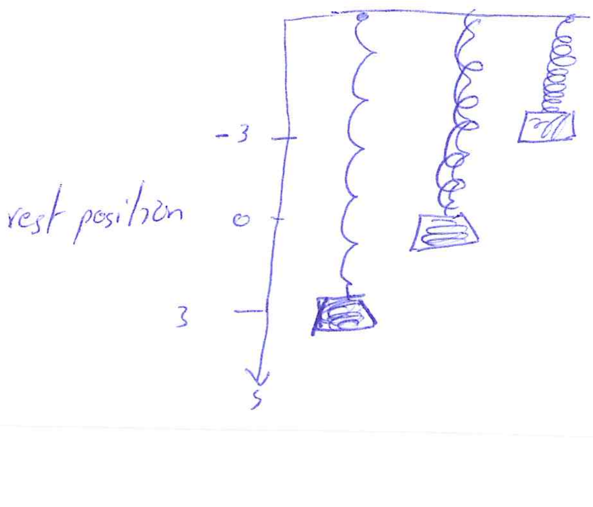
Find its velocity and acceleration, at time t? jerk

Position: $s = 3\cos t$

Velocity $v = \dfrac{ds}{dt} = -3\sin t$

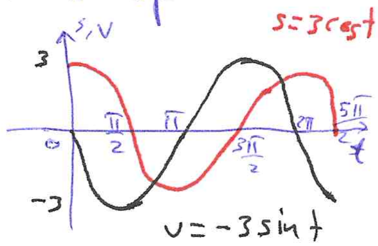
Acceleration $a = \dfrac{dv}{dt} = -3\cos t$

Jerk $j = \dfrac{da}{dt} = 3\sin t$



$\Rightarrow$

- As time passes, the weight moves down and up between $s=-3$ and $s=3$ on s-axis.



- The amplitude of the motion is 3
- The period of the motion is 2π
- $v=-3\sin t$ gets greatest magnitude when $\cos t=0$
- The speed $=|v|=3|\sin t|$
- The acceleration is always the exact apposite of the position value. when weight is above the rest position, gravity is pulling it back down, when the weight is below the rest position, the spring is pulling it back up.

---

3) If $f(x)=\tan x$, then $f'(x)=\sec^2 x$.

Proof $f(x)=\tan x=\dfrac{\sin x}{\cos x}$

$$f'(x)=\frac{\cos x\cos x+\sin x\sin x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

---

4) If $f(x)=\cot x$, then $f'(x)=-\csc^2 x$

Proof $f(x)=\cot x=\dfrac{1}{\tan x}$

$$f'(x)=-\frac{\sec^2 x}{\tan^2 x}=-\frac{\frac{1}{\cancel{\cos^2 x}}}{\frac{\sin^2 x}{\cancel{\cos^2 x}}}=-\frac{1}{\sin^2 x}=-\csc^2 x$$

---

5) If $f(x)=\sec x$, then $f'(x)=\sec x\tan x$

Proof $f(x)=\dfrac{1}{\cos x}$

$$f'(x)=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}\,\frac{\sin x}{\cos x}=\sec x\tan x$$

(6) If $f(x) = \csc x$, then $f'(x) = -\csc x \cot x$

Proof: $f(x) = \dfrac{1}{\sin x}$

$$f'(x) = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} = -\csc x \cot x$$

Example: Let $y = \sec x \tan x$. Find y'

$$y' = \sec x \sec^2 x + \tan x \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

Example: Find $\lim\limits_{x \to 0} \dfrac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \dfrac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)}$

$$= \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{\cos \pi} = -\sqrt{3}$$

$\boxed{3.7} + \boxed{3.6}$ The Chain Rule

Theorem If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x

$$y = (g \circ f)(x) = g(f(x))$$

$$y' = \frac{dy}{dx} = (g \circ f)'(x) = g'(f(x))\, f'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$y = g(u)$
$u = f(x)$

Example Find the derivatives of

$\boxed{a}$ $x(t) = \cos(t^2+1)$

$$\frac{dx}{dt} = -\sin(t^2+1)\,(2t) = -2t\sin(t^2+1)$$

$\boxed{b}$ $y = \sin(x^2+x)$

$$y' = \cos(x^2+x)\,(2x+1) = (2x+1)\cos(x^2+x)$$

$\boxed{c}$ $g(t) = \tan(5-\sin 2t)$

$$\frac{dg}{dt} = \sec^2(5-\sin 2t)\,\left(-\cos 2t\,(2)\right)$$

$$= -2\cos 2t\,\sec^2(5-\sin 2t)$$

$\boxed{d}$ $f(x) = (5x^3-x^4)^7$

$$f' = 7(5x^3-x^4)^6\,(15x^2-4x^3)$$

$\boxed{e}$ $y = \sin^5 x \Rightarrow y = [\sin x]^5$

$$y' = 5\sin^4 x\,\cos x$$

$\boxed{f}$ $h(x) = |x| = \sqrt{x^2} = (x^2)^{\frac{1}{2}}$

$h'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}}\, 2x$

$= \dfrac{x}{\sqrt{x^2}}$, $x \neq 0$

$= \dfrac{x}{|x|}$, $x \neq 0$

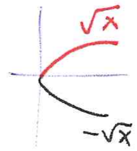
$\boxed{g}$ $y = \dfrac{1}{(1-2x)^3} = (1-2x)^{-3}$

$y' = -3(1-2x)^{-4}(-2)$

$= \dfrac{6}{(1-2x)^4}$

$\boxed{3.7}$ Implicit Differentiation

Example: Find $\frac{dy}{dx}$ if $y^2 = x$

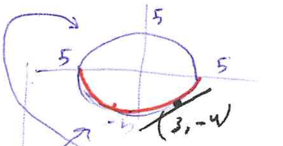


$$2y\,y' = 1 \Rightarrow y' = \frac{1}{2y}$$

Note that $y = \pm\sqrt{x} \Rightarrow y_1' = \frac{1}{2\sqrt{x}}$ for $y_1 = \sqrt{x}$

$$y_2' = \frac{-1}{2\sqrt{x}} \quad \text{for } y_2 = -\sqrt{x}$$

Example: Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.



$y = \pm\sqrt{25 - x^2}$

slope : $2x + 2y\,y' = 0 \Rightarrow y' = \frac{-x}{y}$

slope at $(3,-4)$ is $y' = \frac{-3}{-4} = \frac{3}{4}$

Example: Find $\frac{dy}{dx}$ for $y^2 = x^2 + \sin xy$

$$2y\,y' = 2x + \cos xy\,(x\,y' + y)$$

$$2y\,y' - xy'\cos xy = 2x + y\cos xy$$

$$y'\left[2y - x\cos xy\right] = 2x + y\cos xy$$

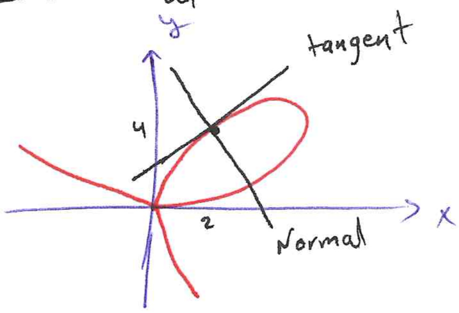
$$y' = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Example: Find $\frac{d^2y}{dx^2}$ for $2x^3 - 3y^2 = 8$

$$6x^2 - 6y\,y' = 0 \Rightarrow y' = \frac{x^2}{y}, \quad y \neq 0$$

$$y'' = \frac{2xy - x^2 y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\,y' = \frac{2x}{y} - \frac{x^4}{y^3}\,; \quad y \neq 0$$

Example: Find the tangent and the normal to the curve $x^3+y^3-9xy=0$ at the point $(2,4)$



$$3x^2+3y^2y'-9xy'-9y=0$$

$$y'(3y^2-9x)=9y-3x^2$$

$$y'=\frac{3y-x^2}{y^2-3x}$$

$$y'\Big|_{2,4}=\frac{3(4)-(2)^2}{(4)^2-3(2)}=\frac{12-4}{16-6}=\frac{8}{10}=\frac{4}{5}$$

- The tangent line at $(2,4)$ with slope $\frac{4}{5}$ is

$$y=\frac{4}{5}(x-2)+4=\frac{4}{5}x+\frac{12}{5}$$

- The normal line at $(2,4)$ is the line perpendicular to the tangent line at $(2,4)$

$$y=-\frac{5}{4}(x-2)+4=-\frac{5}{4}x+\frac{10}{4}+\frac{4(4)}{4}$$

$$y=-\frac{5}{4}x+\frac{13}{12}$$

3.8 Related Rates Equations

$\uparrow = +\ ,\ \downarrow = -$

Suppose we are pumping (ضخ) air into a spherical balloon:

The Volume (V) and radius (r) increase over time t:

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt} \quad \text{(Chain Rule)}$$

$\frac{m^3}{min}$ $\quad 8\pi \quad 4\,m^2$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

If we know r and $\frac{dV}{dt}$, then we can find $\frac{dr}{dt}$ ← how fast the radius increases over time

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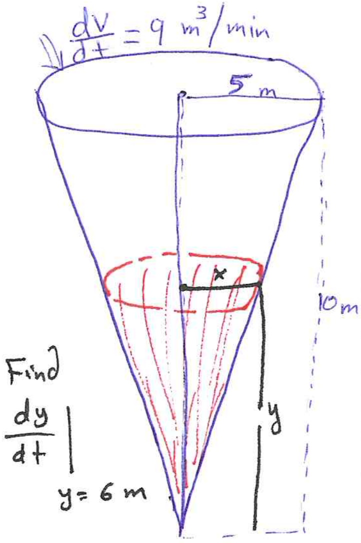
Example 1: Water runs into a conical tank at the rate $9\ m^3/min$. The tank stands point down and has a height 10 m and base radius 5 m. How fast is the water level rising when the water is 6 m deep?



The variables are: V, x, y. For red with relation: $V = \frac{1}{3}\pi x^2 y$

We must eliminate x because we have no information about x and $\frac{dx}{dt}$. Similar triangles ⟹

$$\frac{x}{y} = \frac{5}{10} \iff \boxed{x = \frac{y}{2}}$$

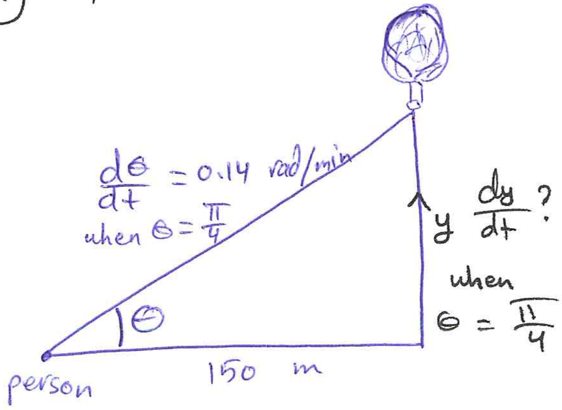
$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

$$\frac{dV}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt} \iff 9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt}$$

$$\iff \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32\ m/min \quad \text{(the level is rising)}.$$

Example 2: A balloon rising straight up from a level field that
حقل مستوي
تعقب
is tracked by a person whose is 150 m from the
زاوية الارتفاع
الاقلاع liftoff point. At the moment the person's elevation angle is $\frac{\pi}{4}$, the angle is increasing at rate of 0.14 rad/min
How fast is the balloon rising at that moment?

The variables are θ and y with relation $\tan\theta = \frac{y}{150}$



$$y = 150\tan\theta$$

$$\frac{dy}{dt} = 150\sec^2\theta\,\frac{d\theta}{dt} \quad \Big|_{\theta=\frac{\pi}{4}}$$

$$= 150(\sqrt{2})^2(0.14)$$

$$= 42 \text{ m/min}$$ "the balloon is rising"

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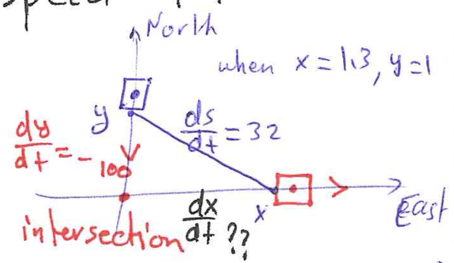
Example 3: A police car, approaching a right-angled intersection from
مطاردة
the north, is chasing a speeding car moving straight east. when the police is 1 Km north of the intersection and the car is 1.3 Km to the east, the police determine, using the radar, that the distance between them is increasing at rate 32 Km/hr. If the police is moving at 100 Km/hr at the instance of measurament, what is the speed of the car?



The variables are x, y, s and related by $s^2 = x^2 + y^2$

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$\sqrt{x^2+y^2}\,\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

$$\Leftrightarrow \sqrt{(1.3)^2+(1)^2}\,(32) = (1.3)\frac{dx}{dt} + (1)(-100)$$

$$\Leftrightarrow \frac{dx}{dt} = 117.3 \text{ Km/hr}$$

car's speed.

3.9 Linearization and Differentials

Def: If f is differentiable at $x = a$, then the approximating function

$$L(x) = f'(a)(x-a) + f(a)$$

is the linearization of f at a.

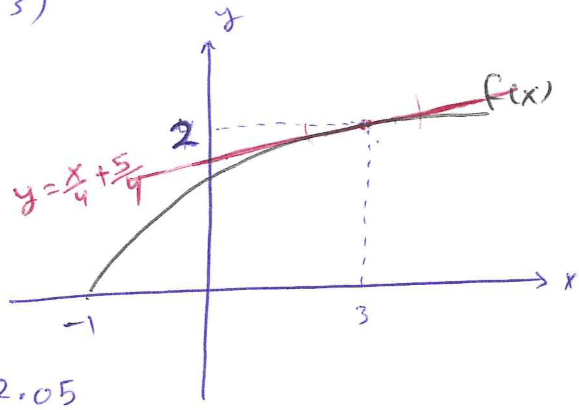
- We approximate f by L and we write $f(x) \approx L(x)$ is the standard linear approximation of f at a.
- The point $x = a$ is the center of the approximation.

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

$f(3) = \sqrt{1+3} = \sqrt{4} = 2$, $\quad f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f'(3) = \frac{1}{2} \frac{1}{\sqrt{1+3}} = \frac{1}{4}$

$$f(x) \approx L(x) = f'(3)(x-3) + f(3)$$
$$= \frac{1}{4}(x-3) + 2$$
$$= \frac{x}{4} + \frac{5}{4}$$



Take $x = 3.2 \Rightarrow L(3.2) = \frac{3.2}{4} + \frac{5}{4}$

$= \frac{8.2}{4} = 2.05$

$\Rightarrow f(3.2) = \sqrt{1+3.2} = \sqrt{4.2} \approx 2.04939$

Example: Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

$f(0) = \sqrt{1+0} = 1$, $f'(x) = \frac{1}{2\sqrt{1+x}}$

$f'(0) = \frac{1}{2}$

$L(x) = f'(0)(x-0) + f(0)$

$L(x) = \frac{x}{2} + 1$

- If $f(x) = (1+x)^k$, k any number, then the linearization of f at $x=0$ is $L(x) = 1 + kx$
- f can be any roots or powers

Example: $(1+x)^{\frac{1}{2}} \approx 1 + \frac{x}{2}$

$$\frac{1}{1-x} = (1-x)^{-1} \approx 1 + x$$

$$\sqrt[3]{1+5x^4} = (1+5x^4)^{\frac{1}{3}} \approx 1 + \frac{1}{3}(5x^4) = 1 + \frac{5}{3}x^4$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \approx 1 + (-\tfrac{1}{2})(-x^2) = 1 + \frac{x^2}{2}$$

Example: Estimate $(1.0001)^{1000}$

$$(1.0001)^{1000} = (1 + 0.0001)^{1000} \approx 1 + (0.0001)(1000) = 1 + 0.1 = 1.1$$

Estimate $\sqrt{1.004}$

$$(1.004)^{\frac{1}{2}} = (1 + 0.004)^{\frac{1}{2}} \approx 1 + 0.004\left(\tfrac{1}{2}\right) = 1 + 0.002 = 1.002$$

Def: Let $y = f(x)$ be differentiable function. The differential dy is

$$dy = f'(x)\,dx$$ where dx is the independent differential

Example: Find dy if $y = x^3 - 3\sqrt{x}$

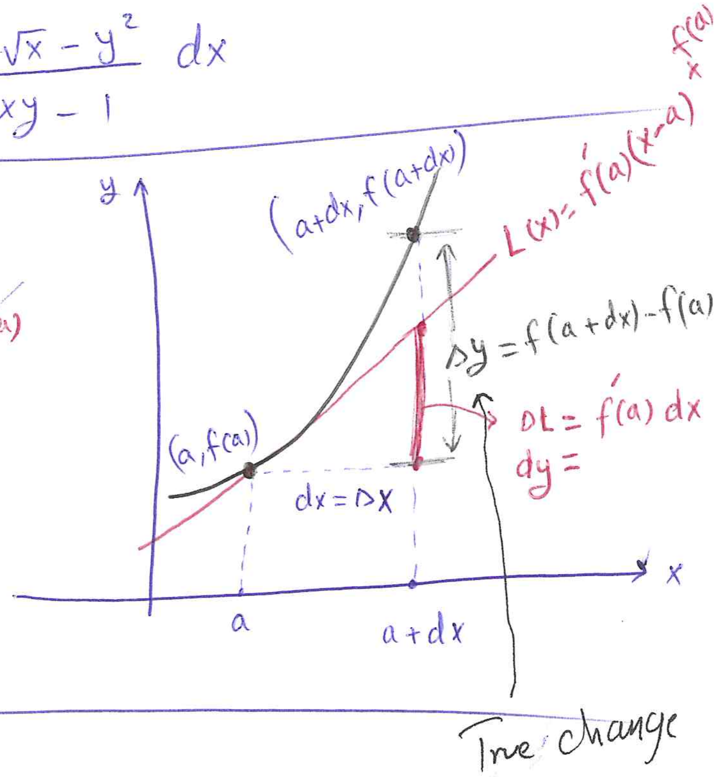
$$dy = 3x^2\,dx - \frac{3}{2}x^{-\frac{1}{2}}\,dx$$
$$= 3\left[x^2 - \frac{1}{2\sqrt{x}}\right]dx$$

Find the differential dy if

$$xy^2 - 4x^{3/2} - y = 0$$
$$y^2\,dx + 2yx\,dy - 6x^{\frac{1}{2}}\,dx - dy = 0$$
$$dy\left[2yx - 1\right] = \left(6x^{\frac{1}{2}} - y^2\right)dx$$
$$dy = \frac{6\sqrt{x} - y^2}{2xy - 1}\,dx$$

Estimated change

$$\Delta L = L(a+dx) - L(a)$$
$$= f'(a)\left((\cancel{a}+dx) - \cancel{a}\right) + \cancel{f(a)} - \cancel{f(a)}$$
$$= f'(a)\,dx$$
$$= dy$$



Estimating with Differentials

True change $\Delta f = f(a + \overset{\Delta x}{\overset{\shortparallel}{dx}}) - f(a)$

Estimated change $df = f'(a)\, \overset{\Delta x}{\overset{\shortparallel}{dx}}$

↳ Differential

Relative True change $\dfrac{\Delta f}{f(a)}$

Relative Estimated change $\dfrac{df}{f(a)}$

True Percentage change $\dfrac{\Delta f}{f(a)} \times 100$

Estimated Percentage change $\dfrac{df}{f(a)} \times 100$

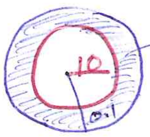
Sensitivity to change

$df = f'(x)\, dx$

how sensitive the output f is ~~to~~ a change in the input at different values x

Example: The radius of a circle increases from $a = 10$ m to 10.1 m.

[a] Estimate the increase in the circle's area.



$dA = A'(10)\, dr$

$= (20\pi)(0.1)$

$= 2\pi \ m^2$

$\Rightarrow A(r) = r^2 \pi$

$A' = 2r\pi$

$A'(10) = 2(10)\pi = 20\pi$

$\Rightarrow \underset{\Delta r}{dr} = r_2 - r_1 = 10.1 - 10 = 0.1$ m

[d] Estimate the enlarged circle area and compar it with the true area

- $A(10.1) \approx A(10) + dA$

$(10)^2\pi + 2\pi = 102\pi \ m^2$

- True area $A(10.1) = (10.1)^2\pi = 102.01\pi \ m^2$

[b] Find the true change in the area?

$\Delta A = A(10.1) - A(10) = (102.01 - 100)\pi = 2.01\pi \ m^2$

[c] Find the error? $\left|\overset{\Delta A}{2.01\pi} - \overset{dA}{2\pi}\right| = 0.01\pi \ m^2 = \epsilon \Delta x$

Error in Differential Approximation

Approximating error = The true change − The differential Estimated change

$= \Delta f - df$

$= f(a+\Delta x) - f(a) - f'(a)\Delta x$

$= \left[\dfrac{f(a+\Delta x)-f(a)}{\Delta x} - f'(a)\right]\Delta x$

$= \epsilon \cdot \Delta x$

as $\Delta x \to 0 \Rightarrow$.

$\dfrac{f(a+\Delta x)-f(a)}{\Delta x} \to f'(a)$

thus, $\epsilon \to 0$ which is very small.

$$\underbrace{\Delta f}_{\text{True change}} = \underbrace{df}_{\text{Estimated change}} + \underbrace{\epsilon \Delta x}_{\text{Error}}$$

$$\boxed{\Delta f = f'(a)\Delta x + \epsilon \Delta x}$$

In the previous example $\Rightarrow$ $\Delta A = 2.01\pi m^2$

$dA = 2\pi\ m^2$

$\Delta r = 0.1\ m$

$\Delta A = dA + \epsilon \Delta r$

$2.01\pi = 2\pi + \epsilon\ 0.1 \Leftrightarrow \epsilon\ 0.1 = 0.01\pi \Leftrightarrow \epsilon = 0.1\pi\ m$

approximating error

Example: How does a 10% decrease in r affect V if $V = kr^4$.

$dV = 4kr^3\, dr \Rightarrow$ The relative Estimated change $\dfrac{dV}{V} = \dfrac{4\cancel{k}r^3}{\cancel{k}r^4}\, dr$

$\Leftrightarrow \dfrac{dV}{V} = 4\dfrac{dr}{r}$

The relative change in V is 4 times the relative change in r.

Thus, a decrease of 10% r will decrease V by 40%.

4.1 Extreme Values of Functions

Def: Let f be a function with domain D.

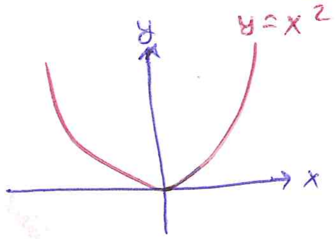
- f has an absolute or global maximum value on D at c if
$$f(c) \geq f(x) \quad \text{for all } x \in D.$$
- f has an absolute or global minimum value on D at c if
$$f(c) \leq f(x) \quad \text{for all } x \in D.$$

- Maximum and minimum values are also called extreme values.
- The function **might** not have a maximum or minimum if the domain is unbounded or is not a closed interval.

Example

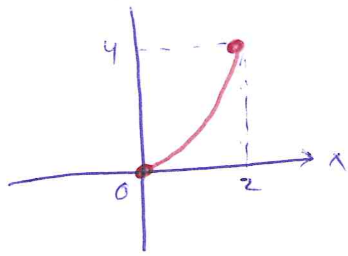
(a) $y = x^2$ on $(-\infty, \infty)$

No absolute maximum
Absolute minimum at $(0,0)$
(of 0 at $x=0$)



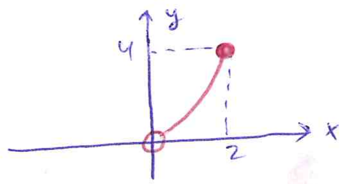
(b) $y = x^2$ on $[0, 2]$

Absolute maximum of 4 at $x = 2$ $(2,4)$
Absolute minimum of 0 at $x = 0$ $(0,0)$



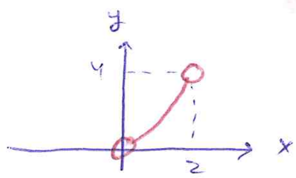
(c) $y = x^2$ on $(0, 2]$

Absolute maximum of 4 at $x = 2$
No absolute minimum



(d) $y = x^2$ on $(0, 2)$

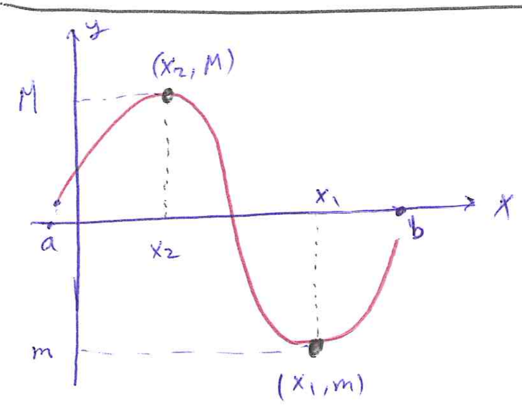
No absolute max
No absolute min



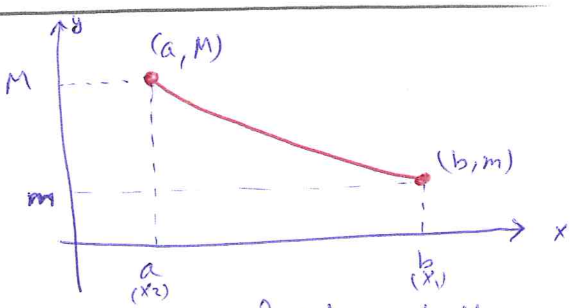
Theorem 1 (The Extreme Value Theorem)

If f is continuous on a closed interval $[a,b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a,b]$.

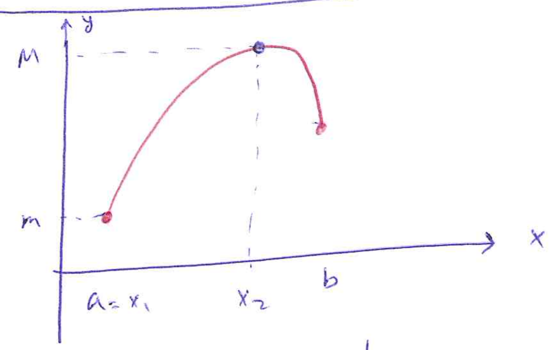
That is, there are numbers x_1 and $x_2 \in [a,b]$ with $f(x_1) = m$ and $f(x_2) = M$ and $m \le f(x) \le M$ for every other x in $[a,b]$



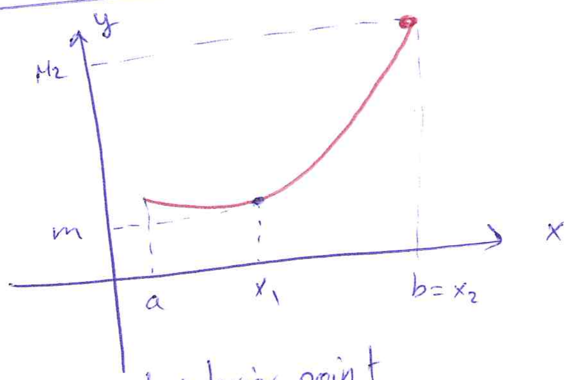
Max and min are interior points



Max and min at the end points

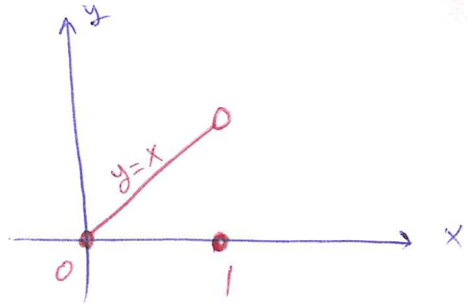


Max at interior point
min at endpoint



Min at interior point
max at end point

Example: $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$

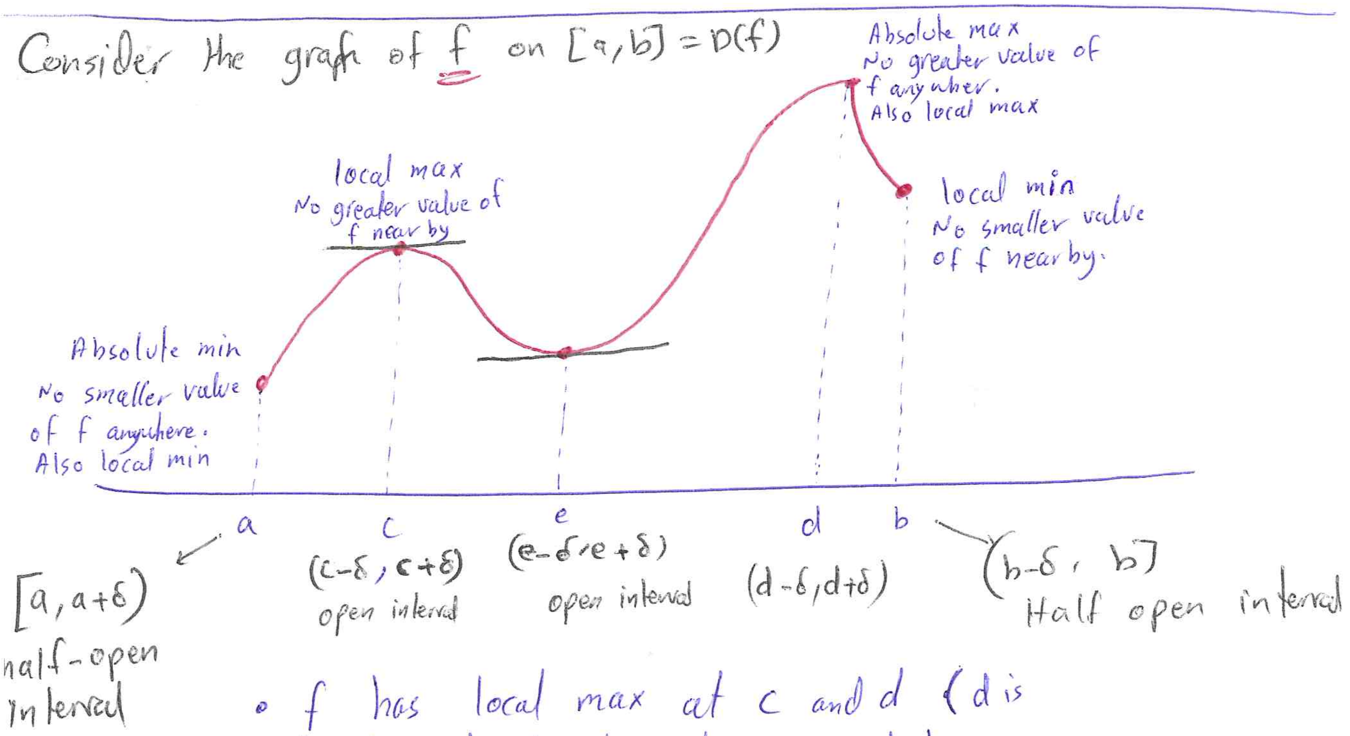


- Absolute minimum at $(0,0)$ and $(1,0)$ of 0 at $x = 0, 1$
- No absolute max because of discontinuity at 1.

Def • A function f has a local (or Relative) maximum at point $c \in D(f)$ if $f(c) \geq f(x)$ for all $x \in D$ lying in some open interval contains c

• A function f has a local (or Relative) minimum at point $c \in D(f)$ if $f(c) \leq f(x)$ for all $x \in D$ lying in some open interval contains c.

Consider the graph of f on $[a,b] = D(f)$



• f has local max at c and d (d is

• f has local min at a, e and b

• local extrema are also called relative extrema.

• Absolute max is also local max

• Absolute min is also local min

Theorem: If f has a local maximum or minimum value at an interior point $c \in D(f)$ and if $f'(c)$ is defined then $f'(c) = 0$

Proof: Suppose that f has a local max (for example) at $x = c$

$\Rightarrow f(c) \geq f(x)$ for all x near c

$\Rightarrow$ Since c is an interior point and $f'(c)$ is defined

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0 \quad \text{and} \quad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \Leftrightarrow \quad f'(c) = 0$$

Def: An interior point $c \in D(f)$ where $f'(c) = 0$ or $f'(c)$ is undefined is called a critical point.

Example: Find the critical points of $y = x^2 - 32\sqrt{x}$

$$y' = 2x - \frac{16}{\sqrt{x}} = 0 \Leftrightarrow x - \frac{8}{\sqrt{x}} = 0$$

$$\Leftrightarrow \frac{x^{\frac{3}{2}} - 8}{\sqrt{x}} = 0 \Leftrightarrow x^{\frac{3}{2}} - 8 = 0 \Leftrightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(2^3\right)^{\frac{2}{3}} \Leftrightarrow \boxed{x=4}$$

$\boxed{x=0}$

To find the Absolute Exterema of a continuous function f on $[a,b]$:

1. Evaluate f at all critical points and endpoint.
2. Take the largest and smallest of these values.

Example: Find absolute maximum and minimum of

① $f(x) = \frac{2}{3}x - 5 \qquad -3 \le x \le 6$

$f'(x) = \frac{2}{3} \neq 0$ no critical points

$f(-3) = \frac{2}{3}(-3) - 5 = -2 - 5 = -7$

$f(6) = \frac{2}{3}(6) - 5 = 4 - 5 = -1$

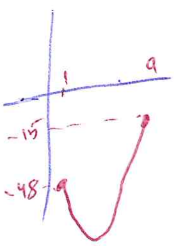
f has absolute max of -1 at $x = 6$

f has absolute min of -7 at $x = -3$

② $g(x) = x^2 - 32\sqrt{x}$ on $[1, 9]$

critical points ~~are $x=0$ and~~ is $x = 4$ since $x = 0 \notin D(g)$

~~$g(0) = 0$ (Absolute max of 0 at $x=0$).~~



✓ $g(4) = 16 - 32(2) = 16 - 64 = -48$ (Absolut min at $(4, -48)$)

$g(1) = 1 - 32 = -31$

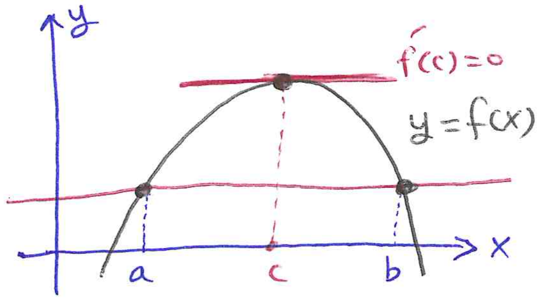
$g(9) = 81 - (32)(3) = 81 - 96 = -15$ (Absolut max of -15 at $x = 9$) ✓

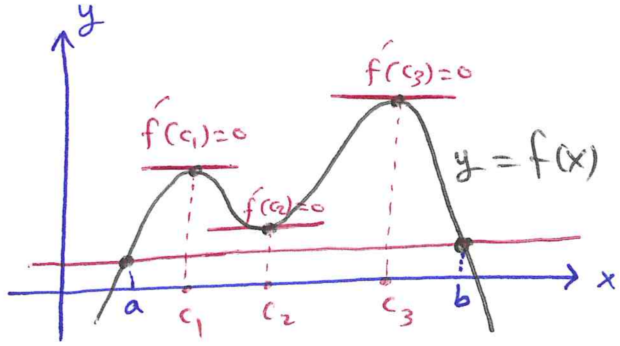
4.2 / 4.3 The Mean Value Theorem

Th3 (Rolle's Theorem)

Suppose $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$





Proof: Let f be a continuous function on $[a, b]$.

By Th1, f has max and min values on $[a, b]$.

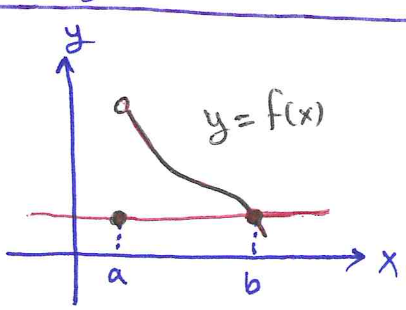
These values can occur only at

1) interior point $c \in (a, b)$ where $f'(c) = 0$
2) interior point $c \in (a, b)$ where $f'(c)$ DNE ✗ since f is differentiable on (a, b)
3) the endpoints a and b.

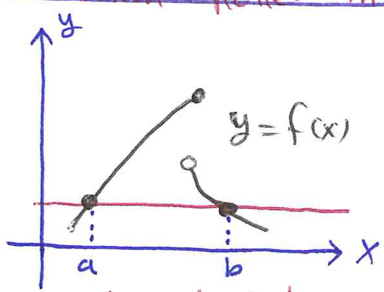
→ If the case is 1), then we are done.

→ If the max and the min occur at the endpoints, then f must be a constant function because we are given $f(a) = f(b)$. Thus, $f(x) = C$ and so $f'(x) = 0$. Hence, c can be taken any value in (a, b).

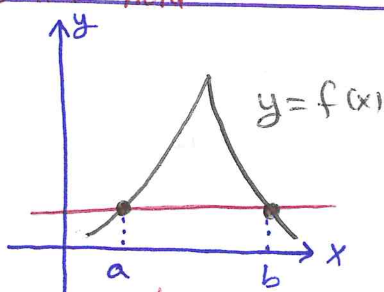
"When Rolle's Th. does not hold"



Dis. cont. at endpoints



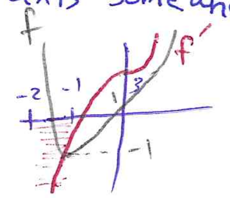
dis. cont. at interior



Continuous on $[a, b]$ but not diff. at interior point.

Example: show that the function $f(x) = x^4 + 3x + 1$ has exactly one real solution on $[-2,-1]$

- Note that f is continuous and
  $f(-2) = (-2)^4 + 3(-2) + 1 = 16 - 6 + 1 = 11 > 0$
  $f(-1) = (-1)^4 + 3(-1) + 1 = 1 - 3 + 1 = -1 < 0$
  } By the IVT, f crosses x-axis somewhere in $(-2,-1)$



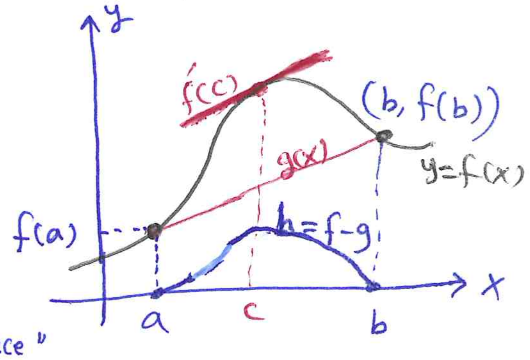
- $f'(x) = 4x^3 + 3$ is never zero on $(-2,-1)$ because it is always negative. There is no $c \in (-2,-1)$ where $f'(c) = 0$. Therefore, f has no more than one zero.

Th (The Mean Value Theorem)

Suppose $y = f(x)$ is continuous on $[a,b]$ and differentiable on (a,b).

Then, there is at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof: $g(x) = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$

Let $h(x) = f(x) - g(x)$ "vertical distance"

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$h(x)$ satisfies Rolle's Th: since 1) $h(x)$ is continuous on $[a,b]$
2) differentiable on (a,b)
3) $h(a) = h(b) = 0$

$\Rightarrow$ there is at least one point $c \in (a,b)$ such that $h'(c) = 0$

$$\Leftrightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Corollary 1 If $f'(x)=0$ for every $x\in(a,b)$, then $f(x)=c$ for all $x\in(a,b)$, where c is constant.

Proof: We need to show that f is constant value on (a,b).
Let x_1 and x_2 be two points in (a,b) with $x_1<x_2$.
We need to show that $f(x_1)=f(x_2)$.

Note that f satisfies the MVT on $[x_1,x_2]$. Hence, there is at least one point $c\in(x_1,x_2)$ such that $f'(c)=\dfrac{f(x_2)-f(x_1)}{x_2-x_1}=0$
$\Leftrightarrow f(x_2)=f(x_1)$.

Corollary 2 If $f'(x)=g'(x)$ for every $x\in(a,b)$, then there exists a constant C such that $f(x)=g(x)+C$ for all $x\in(a,b)$. That is $f-g=C$.

Proof Let $h(x)=f(x)-g(x)$. Then $h'(x)=f'(x)-g'(x)=0$
Thus, $h(x)=C$ on (a,b) by Corollary 1. That is $f-g=C$.

Example: Find the function $f(x)$ whose derivative is $\sin x$ and passes through the point $(0,2)$.

$f(x)=-\cos x+C$
$f(0)=-\cos 0+C=2 \Leftrightarrow -1+C=2 \Leftrightarrow \boxed{C=3}$
$f(x)=-\cos x+3$

Find the body's position if the body's velocity is $v=32t-2$ and the body passes through $(\frac{1}{2},4)$

$s(t)=16t^2-2t+C$
$s(\frac{1}{2})=16(\frac{1}{4})-2(\frac{1}{2})+C=4 \Leftrightarrow 4-1+C=4 \Leftrightarrow \boxed{C=1}$

$\boxed{s(t)=16t^2-2t+1}$

4.3 Monotonic Functions and the First Derivative Test

* A function that is increasing or decreasing on an interval is called monotonic on the interval.

Corollary 3 اللازمة Suppose f is continuous on $[a,b]$ and differentiable on (a,b).

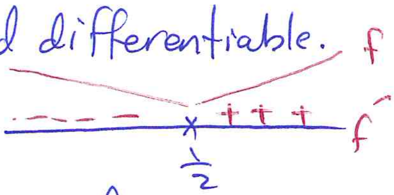
→ If $f'(x) > 0$ for every $x \in (a,b)$, then f is increasing on $[a,b]$

→ If $f'(x) < 0$ for every $x \in (a,b)$, then f is decreasing on $[a,b]$

Example: Find the critical points of $f(x) = x^2 - x$ and identify the intervals on which f is increasing or decreasing.

$f(x)$ is everywhere continuous and differentiable.

$f'(x) = 2x - 1 = 0 \Leftrightarrow x = \frac{1}{2}$

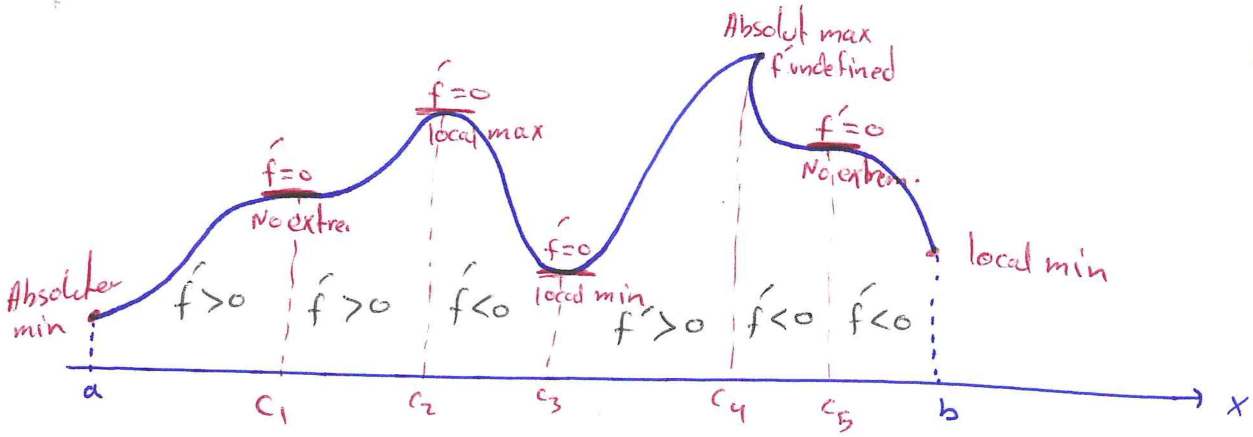


f is increasing on $(\frac{1}{2}, \infty)$ and decreasing on $(-\infty, \frac{1}{2})$

First Derivative Test for Local Extrema:

Suppose c is a critical point of a continuous function f, and f is differentiable on (a,b) except possible at c, where $c \in (a,b)$ then:

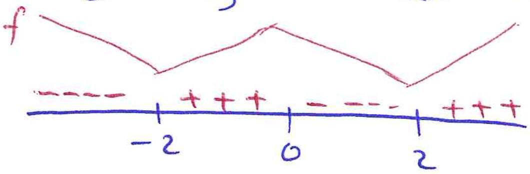
1. If f' changes from negative to positive at c, then f has a local min at c.
2. If f' " " positive to negative at c, " " " " " max at c.
3. If f' does not change sign at c, then f has no local extremum at c.



Example: Find the critical points of $f(x) = x^4 - 8x^2 + 16$
Identify the intervals on which f is increasing or decreasing
Find the local max/min and absolute max/min

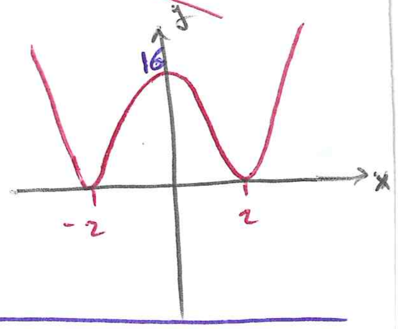
f is continuous for all $x \Rightarrow f'(x) = 4x^3 - 16x = 0$

$\Leftrightarrow 4x[x^2 - 4] = 0 \Leftrightarrow x = 0, 2, -2$ critical points



- f is increasing on $(-2, 0)$ and on $(2, \infty)$
- f is decreasing on $(-\infty, -2)$ and on $(0, 2)$

- f has local min at $(-2, f(-2)) = (-2, 0)$
  $(2, f(2)) = (2, 0)$
- f has local max at $(0, f(0)) = (0, 16)$
- f has absolute min at 0 when $x = \pm 2$
- f has no absolute max



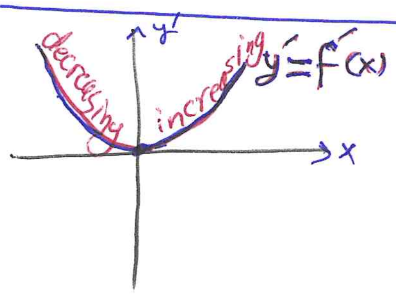
$\boxed{4.4}$ Concavity and Curve Sketching

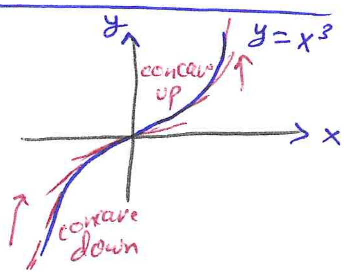
Def Let $f(x) = y$ be a differentiable function on interval I

(a) If f' is increasing on I, then f is concave up on the open interval I

(b) If f' is decreasing on I, then f is concave down on the open interval I.

Example $y = x^3$





The 2^{nd} Derivative Test for Concavity:

* Let $y = f(x)$ be twice-differentiable on an interval I.

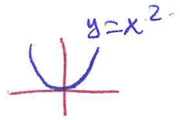
(a) If $f'' > 0$ on I, then f is concave up on I

(b) If $f'' < 0$ on I, then f is concave down on I.

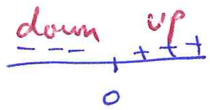
Example: Determine the concavity of

① $y = x^2 \Rightarrow y' = 2x \Rightarrow y'' = 2 > 0$

$\Rightarrow y$ is concave up on every I.



② $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow y'' = 6x$



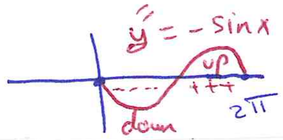
y is concave up on $(0, \infty)$

y is concave down on $(-\infty, 0)$

③ $y = 3 + \sin x$ on $[0, 2\pi]$

$y' = \cos x$

$y'' = -\sin x$



y concave up on $(\pi, 2\pi)$

y concave down on $(0, \pi)$

Def A point $x=c$ is called inflection point of the function f if the function f has a tangent line at $x=c$ and changes concavity.

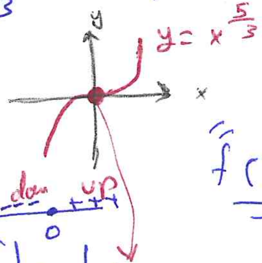
May be horizontal or vertical or oblique

If the function f has an inflection point at $(c, f(c))$ then either $f''(c)=0$ or $f''(c)$ fails to exist (undefined)

Example ① ($f'(c)$ exists but $f''(c)$ fails to exist) Find the inflection point?

$f(x) = x^{\frac{5}{3}}$, $f'(x) = \frac{5}{3} x^{\frac{2}{3}}$, $f''(x) = \frac{10}{9} x^{-\frac{1}{3}}$

$= \dfrac{10}{9\sqrt[3]{x}}$



$f''(0)$ fails to exist

at $x=0$ ① we have a tangent since $f'(0) = 0$ (exists)

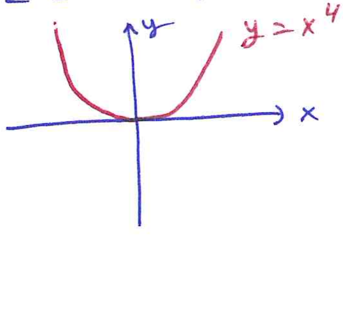
② f changes concavity since y'': $\underset{\text{down}}{- - -}\ \overset{0}{\bullet}\ \underset{\text{up}}{+ + +}$

Thus, $(0, f(0)) = (0,0)$ is an inflection point.

Example ② ($f'(c)$ and $f''(c)$ exist)

Find an inflection point of $y = x^4$

$y' = 4x^3$, $y'' = 12x^2 = 0 \iff x = 0$ so $f''(0) = 0$



at $x=0$, ① we have a tangent since $f'(0) = 0$ (exists)

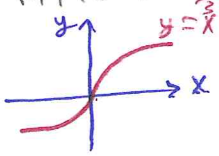
② f does not change concavity since y'': $\underset{\text{up}}{+ + +}\ \overset{0}{\bullet}\ \underset{\text{up}}{+ + +}$

Thus, $(0, f(0)) = (0,0)$ is not inflection point

Example ($f'(c)$ and $f''(c)$ do not exist) Find the inflection point of $y = x^{\frac{1}{3}}$

$y' = \frac{1}{3} x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$

$y'' = -\frac{2}{9} x^{-\frac{5}{3}} = \dfrac{-2}{9\sqrt[3]{x^5}}$



at $x=0$, ① we have a vertical tangent since $f'(0)$ fails to exist.

② f changes concavity at $x=0$ since y'': $\underset{\text{up}}{+ + +}\ \overset{0}{\bullet}\ \underset{\text{down}}{- - -}$

Thus $(0,0)$ is inflection point

Th (2^{nd} Derivative Test for local extrem)

Suppose f'' is continuous on an open interval that contain c.

(1) If $f'(c) = 0$ and $f''(c) < 0$, then f has local max at c.

(2) If $f'(c) = 0$ and $f''(c) > 0$, then f has local min at c.

(3) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails, and the function f may have local max or local min, or neither. (see example ①, ②)

Example ⓐ Find the local extrem of $f(x) = x^3 - 3x + 3$

$$f' = 3x^2 - 3 = 0 \iff x = \pm 1$$

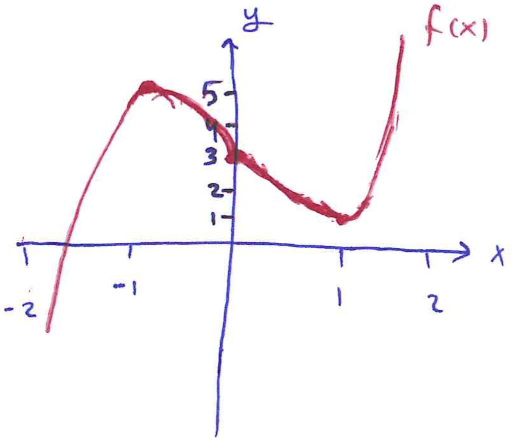
(critical points) $(1, f(1))$ and $(-1, f(-1))$

$$f''(x) = 6x$$

- $f'(1) = 0$ and $f''(1) = 6 > 0 \Rightarrow f$ has local min at $x = 1$
- $f'(-1) = 0$ and $f''(-1) = -6 < 0 \Rightarrow f$ has local max at $x = -1$.

ⓑ Graph the function f

- $f(0) = 3$
- local Max at $(-1, 5)$
- local Min at $(1, 1)$



- $f' = 3x^2 - 3 = 0 \iff x^2 - 1 = 0$

f': $+++$ | -1 | $----$ | 1 | $+++$ — f

- f is increasing on $(-\infty, -1)$ and $(1, \infty)$
- f is decreasing on $(-1, 1)$

- $f'' = 6x = 0$

f'': down $---$ | 0 | $+++$ up

at $x = 0$ we have inflection point $(0, 3)$ since $f'(0) = -3$ and so, we have a tangent and f changes concavity.

---

* To graph a function $y = f(x)$

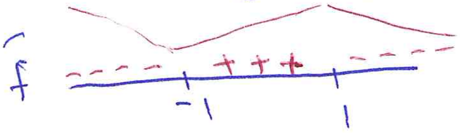
1) Find the Domain of $f(x)$.
2) Find y' and the critical points.
3) Find where f is increasing and decreasing, local Max & local Min.
4) Find y'' and the inflection points.
5) Find where f is concave up and concave down.
6) Find the asymptotes of f (horizontal and vertical)
7) Plot Key points : x- intercepts and y- intercepts.

Example: Sketch the graph $f(x) = \dfrac{(x+1)^2}{1+x^2}$

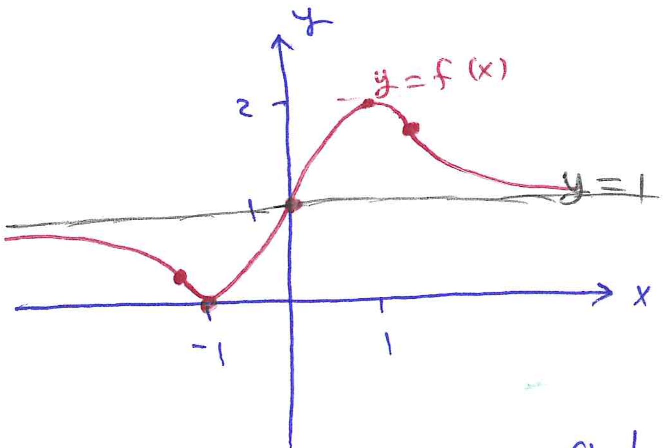
1) $D(f) = (-\infty, \infty)$

2) $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2} = 0 \iff x = \pm 1$, so the critical points are $(1,2)$, $(-1,0)$

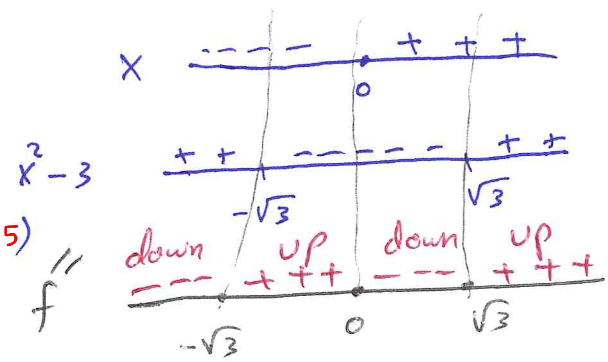
3)



- f is increasing on $(-1,1)$
- f is decreasing on $(-\infty,-1)$ and $(1,\infty)$
- f has local min $(-1,0)$
- f has local Max at $(1,2)$



4) $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3} = 0 \iff x = 0, \sqrt{3}, -\sqrt{3}$ are inflection points because f changes concavity



6) No vertical asymptotes
Horizontal asymptote at $x = 1$
because $\lim_{x \to \infty} f(x) = 1$

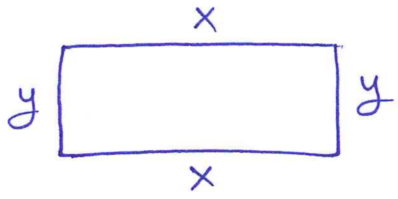
7) $(-1,0)$, $(0,1)$

4.5 Applied Optimization

Example: What is the smallest perimeter possible for a rectangle whose area is $100\ cm^2$, and what are its dimensions?

$P = 2x + 2y$, $A = xy = 100$

$\boxed{y = \frac{100}{x}}$



$P = 2x + \frac{200}{x}$

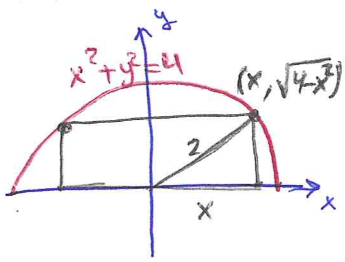
$\frac{dP}{dx} = 2 - \frac{200}{x^2} = 0 \iff x = \pm 10 \Rightarrow x = 10\ cm$ critical point

$P'' = \frac{400}{x^3}$ which is always positive (concave up) for all $x > 0$. Thus, at $x = 10$ we have min (actually abs. min).

$\Rightarrow y = \frac{100}{10} = 10\ cm$, so the perimeter becomes

$P = 2x + 2y = 2(10) + 2(10) = 20 + 20 = 40\ cm.$

Example: What is the largest area and dimensions that a rectangle can be inscribed in a semicircle of radius 2.



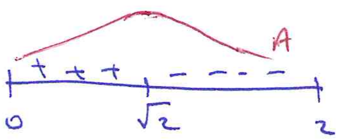
$A = 2x\sqrt{4 - x^2}$, $0 \le x \le 2$

$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$

$= \frac{8 - 4x^2}{\sqrt{4-x^2}} = 0 \iff x = \pm\sqrt{2}$

$\boxed{\Rightarrow x = \sqrt{2} \text{ critical point}}$

A'



$A(0) = A(2) = 0$

$A(\sqrt{2}) = 2\sqrt{2}\sqrt{2} = 4$

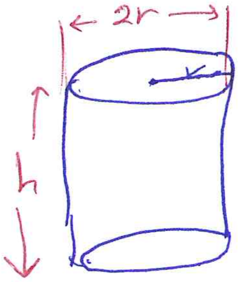
Max area is at $x = \sqrt{2}$

$\Rightarrow$ length is $2x = 2\sqrt{2}$

height is $\sqrt{4 - x^2} = \sqrt{2}$

__Example__ : What dimensions will you use for least material can be used to design a one liter can of cylinder shape.

- Let r be the radius and h be the height.



$$V = r^2 \pi h = 1000 \text{ cm}^3 \iff \boxed{h = \frac{1000}{\pi r^2}}$$

- least Materials

surface area: $A = 2r^2\pi + 2r\pi h$

$$A = 2r^2\pi + 2r\pi\left(\frac{1000}{\pi r^2}\right)$$

$$A = 2r^2\pi + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \iff r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \text{ cm}$$

critical point

$A'' = 4\pi + \dfrac{4000}{r^3}$ which is positive (concave up)

$\Rightarrow$ The value of A at $r = \sqrt[3]{\frac{500}{\pi}}$ is abs. min.

- The corresponding height is : $h = \dfrac{1000}{\pi (5.42)} \approx 10.84$ cm

---

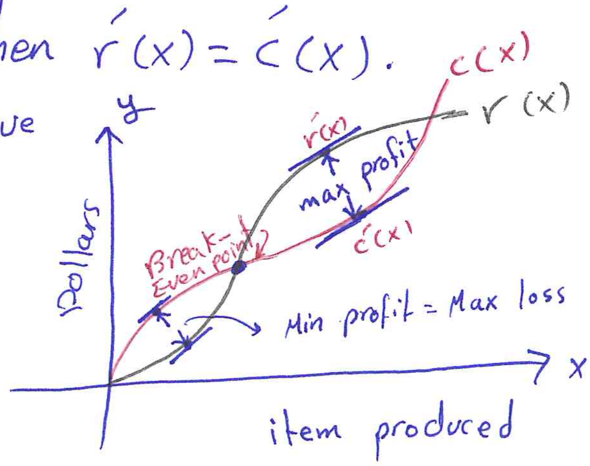
To solve an Applied Optimization problem:

1) Read the problem : what is given?
   what needs to be optimize?
2) Draw a picture : introduce variables, see if there is relation.
3) Write an equation for the unknown quantity in terms of single variable.
4) Find the critical points and test them, together with the endpoints in the Domain of the unknown, using the first and the second derivatives.

Examples from Economics

- Suppose that $r(x)$ = the revenue from selling x items
  $c(x)$ = the cost of producing the x items
  $p(x) = r(x) - c(x)$ is the profit from producing and selling x items.

- The Maximum profit occurs when $r'(x) = c'(x)$.
  where $r'(x)$ = marginal revenue
  $c'(x)$ = marginal cost



Example Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 18x$ where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

$r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 18$

$p(x) = r(x) - c(x)$. Thus $p'(x) = r'(x) - c'(x) = 0$

$r'(x) = c'(x) \iff 3x^2 - 12x + 18 = 9 \iff x^2 - 4x + 3 = 0$

$\iff (x-1)(x-3) = 0 \iff x = 1$ or $x = 3$ critical points.

- $p''(x) = r''(x) - c''(x)$
  $= 0 - 6x + 12 \iff p''(x) = 12 - 6x$

$p''(1) = 12 - 6 = 6 > 0$ "concave up" local min

$p''(3) = 12 - 18 = -6 < 0$ "concave down" local max.

- Max profit at level of production $x = 3$

$p(3) = r(3) - c(3)$
$= 9(3) - [(3)^3 - 6(3)^2 + 18(3)]$
$= 27 - [27 - \cancel{54} + \cancel{54}]$
$= 0$

4.7 Antiderivatives

Def A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Example: Find the antiderivative of

1. $f(x) = 2x$, $F(x) = x^2$
2. $g(x) = x^2 - 2x + 1$, $G(x) = \frac{x^3}{3} - x^2 + x$
3. $h(x) = \frac{5}{2\sqrt{x}}$, $H(x) = 5\sqrt{x}$
4. $r(x) = -\pi \sin \pi x$, $R(x) = \cos \pi x$

Therorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + c$, where c is an arbitrary constant.

Note that the collection of all antiderivati of f is called the indefinite integral of f with respect to x and defined by

$$\underbrace{\int}_{\text{integral}} \underbrace{f(x)}_{\text{integrand}}\, d\underbrace{x}_{\text{variable of integration}} = \underbrace{F(x)}_{\text{antiderivative}} + c .$$

Example: Find the most general antiderivatives (or indefinite integral) of

1. $\int \left(3x^2 + \frac{x}{2}\right) dx = x^3 + \frac{x^2}{4} + c$
2. $\int -2\cos t \, dt = -2\sin t + c$
3. $\int (1 + \tan^2\theta)\, d\theta = \int \sec^2\theta \, d\theta = \tan\theta + c$
4. $\int (2x^3 - 5x + 7)\, dx = \frac{x^4}{2} - \frac{5x^2}{2} + 7x + c$

Example: Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$

The general antiderivative is $F(x) = x^3 + c$

$F(1) = (1)^3 + c = -1 \Leftrightarrow c = -2$ Thus $F(x) = x^3 - 2$

So we need to find a function $y(x)$ that satisfies

Initial Value Problem $\Leftarrow$ $\begin{cases} \dfrac{dy}{dx} = f(x) & \text{.... "differential Equation" is equation that has derivatives"} \\ y(x_0) = y_0 & \text{--- "initial condition" To find } c \end{cases}$

* The general solution $y(x) = F(x) + c$
* when we find c, we find a particular solution $y(x) = x^3 - 2$

Example: Solve the initial value problem

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 2\sin 2x \quad , \quad y(0) = 0$$

$$\int dy = \int \left(\frac{1}{2\sqrt{x}} + 2\sin 2x\right) dx$$

$$y(x) = \sqrt{x} - \cos 2x + c \quad \text{...... the general solution}$$

$$y(0) = \sqrt{0} - \cos 0 + c = 0$$

$$0 - 1 + c = 0 \Leftrightarrow \boxed{c = 1}$$

The particular solution is $\boxed{y(x) = \sqrt{x} - \cos 2x + 1}$

$\boxed{5.2}$ Sigma Notation and Limits of Finite Sums

Finite Sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

Sigma "Sum"

← The index k ends at $k=n$

← The index k starts at $k=1$

a_k is the formula for the k^{th} term.

Example ① $\sum_{k=1}^{4} k = 1+2+3+4 = 10$

② write $\sum_{k=0}^{2} \frac{6k}{k+1}$ without sigma notation

$$\sum_{k=0}^{2} \frac{6k}{k+1} = 0 + \frac{6}{2} + \frac{12}{3} = 3 + 4 = 7$$

③ $\sum_{k=1}^{2} (-1)^{k+1} \sin\frac{\pi}{k} = \sin\pi - \sin\frac{\pi}{2} = 0 - 1 = -1$

Example: Express the sum $\frac{2}{3} + \frac{4}{3} + \frac{6}{3} + \frac{8}{3} + \frac{10}{3}$ in sigma notation

$$\frac{1}{3}\left[2+4+6+8+10\right] = \frac{1}{3}\sum_{k=1}^{5} 2k$$

Algebra Rules for Finite Sums:

$\boxed{1}$ Sum Rule: $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

$\boxed{2}$ Difference Rule: $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

$\boxed{3}$ Constant Multiple Rule: $\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$

$\boxed{4}$ Constant Value Rule: $\sum_{k=1}^{n} c = nc$

$\boxed{5}$ $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$\boxed{6}$ $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{7}$ $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

<u>Example</u>: Evaluate:

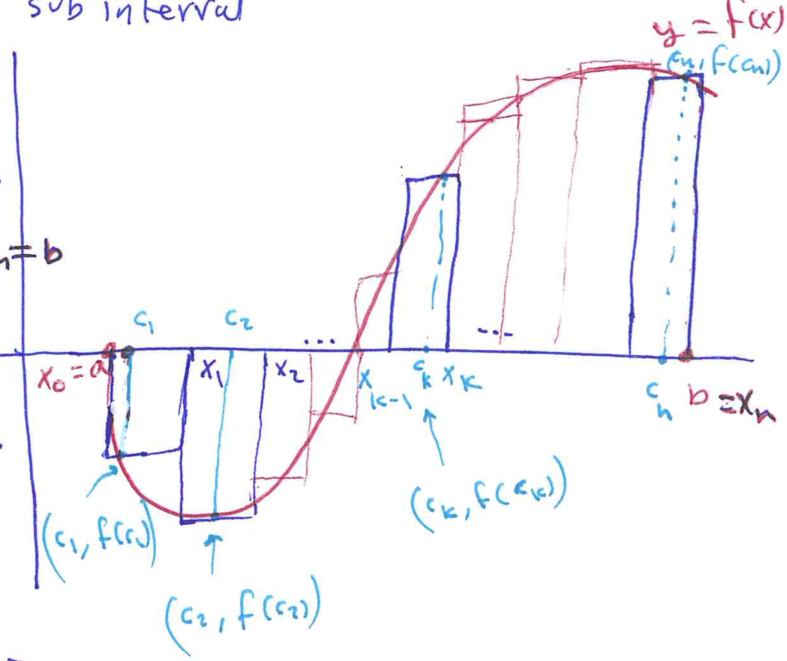
① $$\sum_{k=1}^{7}(-2k) = -2\sum_{k=1}^{7} k = -2\,\frac{7(7+1)}{2} = -7(8) = -56$$

② $$\sum_{k=1}^{6}(3-k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = (6)(3) - \frac{6(7)(13)}{6}$$
$$= 18 - 91 = -73$$

Riemann Sums

* Consider a bounded function f defined on $[a,b]$
* Divide $[a,b]$ into n closed sub interval by choosing $n-1$ points $\{x_1, x_2, \dots, x_{n-1}\}$ between a and b.

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

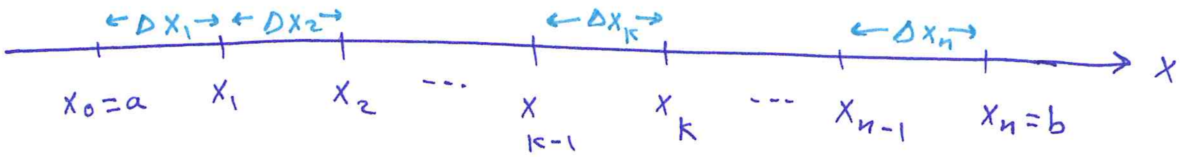


* The set $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ is called a partition of $[a,b]$.
* The partition P divides $[a,b]$ into n closed subintervals:

$$[x_0, x_1], [x_1, x_2], \dots, \underbrace{[x_{k-1}, x_k]}_{k^{th}\text{ sub interval}}, \dots [x_{n-1}, x_n]$$

* $\Delta x_k = x_k - x_{k-1}$ is the width of the k^{th} subinterval.
* $\text{Max}\,|\Delta x_k|$ is the norm of the partition P denoted by $\|P\|$.

* If all subintervals have equal width, then the width of any subinterval is $\Delta x = \frac{b-a}{n}$



* select a point c_k in Δx_k for all $k = 1, 2, \cdots, n$
* On each subinterval we find $f(c_k)\,\Delta x_k$. This product is
  - positive when $f(c_k)$ is positive. This gives area above x
  - negative when $f(c_k)$ is negative. This gives area below x
* The Riemann sum S_p for f on the interval $[a, b]$ is

$$S_p = \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

---

* If we choose n subintervals all having equal width, $\Delta x = \frac{b-a}{n}$, and we choose the point c_k to be the right hand endpoint of each subinterval, Then Riemann Sum formula becomes $S_n = \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)$

__Example__ Let $f(x) = 1 - x^2$ on $[0, 1]$

[a] Find formula for the Riemann sum by dividing $[0,1]$ into n equal subintervals and using c_k to be the right hand endpoint. $a = 0$, $b = 1$

$\Delta x_k = \frac{1-0}{n} = \frac{1}{n}$ the width of the k^{th} subinterval $k = 1, 2, \cdots, n$

$$S_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\frac{1}{n} = \sum_{k=1}^{n}\left[1 - \left(\frac{k}{n}\right)^2\right]\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{n} - \sum_{k=1}^{n}\frac{k^2}{n^3}$$

$$= \frac{n}{n} - \frac{1}{n^3}\sum_{k=1}^{n} k^2 = 1 - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$S_n = 1 - \frac{2n^3+3n^2+n}{6n^3} = 1 - \left[\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$S_n = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

[b] Calculat the area under the curve over [0,1]
"Take limit for S_n as $n \to \infty$"

$$\lim_{n\to\infty} S_n = \boxed{\frac{2}{3}}$$

[c] Find $\int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \boxed{\frac{2}{3}}$

Example: Find the norm of the partition

$P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$

The subintervals are: $[0, 1.2]$, $[1.2, 1.5]$, $[1.5, 2.3]$, $[2.3, 2.6]$, $[2.6, 3]$

$\Delta x_1 = 1.2$ $\quad \Delta x_2 = 0.3$ $\quad \Delta x_3 = 0.8$ $\quad \Delta x_4 = 0.3$ $\quad \Delta x_5 = 0.4$

The norm of the partition P is $\|P\| = 1.2$

5.3 The Definite Integral

Let $f(x)$ be a function defined on a closed interval $[a,b]$.

We say a number J is the definite integral of f over $[a,b]$ and we write $J = \lim_{\|P\| \to 0} \underbrace{\sum_{k=1}^{n} f(c_k)\,\Delta x_k}_{\text{Riemann sum}}$ if

Given $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have $\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon$

So, we say the Riemann sums of f on $[a,b]$ converge to the definite integral J. That is

$$\lim_{\|P\| \to 0} S_n = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = J = \int_a^b f(x)\,dx$$

upper limit of integration

the integrand

the variable of integration "dummy variable"

integral

lower limit of integration

- When the partition P has n equal subintervals each of width $\Delta x = \dfrac{b-a}{n}$, then $\lim_{n \to \infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(\underbrace{c_k}_{a+k\left(\frac{b-a}{n}\right)}) \left(\dfrac{b-a}{n}\right) = J = \int_a^b f(x)\,dx$

- Note that $\int_a^b f(t)\,dt = \int_a^b f(x)\,dx = \int_a^b f(u)\,du$

Example Express the following limits as definite integrals:

1. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k^2\,\Delta x_k$, where P is a partition of $[0,2]$ $= \int_0^2 x^2\,dx$

2. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\,\Delta x_k$, where P is a partition of $[-7,5]$ $= \int_{-7}^{5} (x^2 - 3x)\,dx$

Th (Integrability of Continuous Functions)

If f is continuous on $[a,b]$ or

If f has finitely many jump discontinuities on $[a,b]$,

then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable on $[a,b]$.

Th If f and g are integrable on $[a,b]$, then

[1] $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

[2] $\int_a^a f(x)\,dx = 0$

[3] $\int_a^b K\,f(x)\,dx = K\int_a^b f(x)\,dx$

[4] $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

[5] $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

[6] If f has max value M and min value m on $[a,b]$, then

$$m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$$

[7] If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

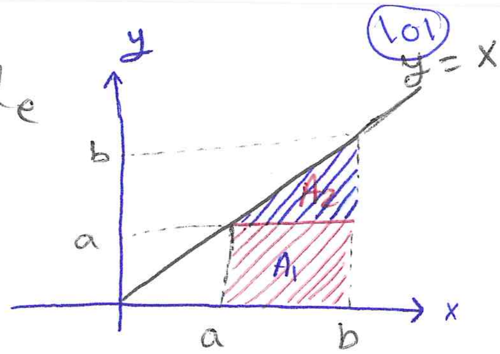
If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$

Example show that $\int_0^1 \sqrt{1+\sin x}\,dx \le \sqrt{2}$

$$\int_0^1 \sqrt{1+\sin x}\,dx \le \int_0^1 \sqrt{1+1}\,dx = \int_0^1 \sqrt{2}\,dx \le \sqrt{2}\,(1-0) = \sqrt{2}$$

Def If $y = f(x)$ is nonnegative and integrable on $[a,b]$, then the area under the curve $y = f(x)$ on $[a,b]$ is the integral of f from a to b: $A = \int_a^b f(x)\,dx$.

Example: Use areas to evaluate



① $\int_a^b x\,dx = A_1 + A_2$

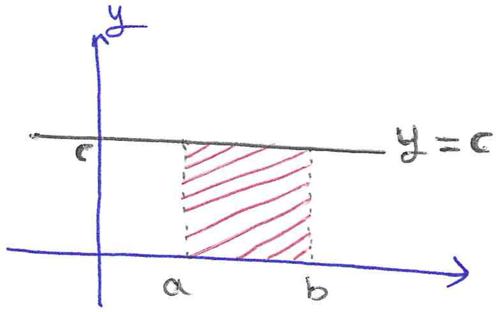
$= (b-a)\,a + \frac{1}{2}(b-a)(b-a)$

شبه المنحرف or $A = \frac{(b+a)(b-a)}{2} = \frac{b^2-a^2}{2}$ ✓

$$\int_a^b x\,dx = ab - a^2 + \frac{1}{2}(b^2 - 2ab + a^2)$$

$$= \cancel{ab} - a^2 + \frac{b^2}{2} - \cancel{ab} + \frac{a^2}{2}$$

$$\boxed{\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}}$$



② $\int_a^b c\,dx = (b-a)c$

③ $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$ $\quad a < b$

Example: Use Riemann sum to calculate $\int_0^b x\,dx$

- Consider the partition P that divids $[0,b]$ into n subintervals of eqval width $\Delta x = \frac{b-0}{n} = \frac{b}{n}$
- $P = \{0, \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}, \dots, \frac{nb}{n} = b\}$

  $[0, \frac{b}{n}], [\frac{b}{n}, \frac{2b}{n}], [\frac{2b}{n}, \frac{3b}{n}], \dots [\frac{(n-1)b}{n}, b]$
- choose c_k to be the right endpoint $c_k = \frac{kb}{n}$

$$\int_0^b x\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{kb}{n}\right) \Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb}{n} \frac{b}{n}$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} \frac{b^2}{n^{\cancel{2}}} \frac{\cancel{n}(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{b^2}{2}\left(1 + \frac{1}{n}\right)$$

$$= \frac{b^2}{2}$$

Def If f is integrable on $[a, b]$, then the average value of f on $[a, b]$ "Mean" is

$$av(f) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Example Find the average value of $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$.

$$av(f) = \frac{1}{\sqrt{3}-0} \int_0^{\sqrt{3}} (x^2 - 1)\, dx = \frac{1}{\sqrt{3}} \left[\int_0^{\sqrt{3}} x^2\, dx - \int_0^{\sqrt{3}} dx \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{(\sqrt{3})^3}{3} - \frac{(0)^3}{3} - (\sqrt{3} - 0) \right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{3\sqrt{3}}{3} - \sqrt{3} \right]$$

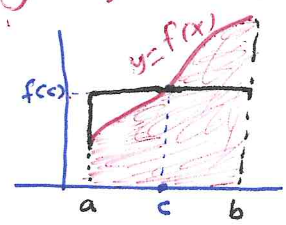
$$= \frac{1}{\sqrt{3}} \left[\sqrt{3} - \sqrt{3} \right]$$

$$= 0$$

$\boxed{5.4}$ The Fundamental Theorem of Calculus + $\boxed{5.5}$

Theorem (The Mean Value Theorem for Definite Integrals)

If f is continuous on $[a,b]$, then at some point $c \in [a,b]$, we have: $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$



* $f(c)$ is the average height "Mean" of f on $[a,b]$

The area of the rectangle = area under f

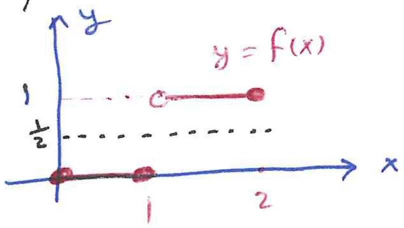
$$f(c)(b-a) = \int_a^b f(x)\,dx$$

---

* If f is discontinuous, then f may never equals its average value

$$av(f) = \frac{1}{2-0}\int_0^2 f(x) = \frac{1}{2}\int_1^2 f(x) = \frac{1}{2}(1) = \frac{1}{2}$$

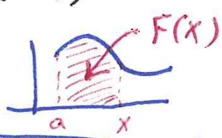
but $\nexists\, c \in [0,2]$ s.t $f(c) = \frac{1}{2}$



Th ① (The fundamental Theorem of Calculus)

If f is continuous on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ and differentiable on (a,b) with

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



Example: Find $\frac{dy}{dx}$ for

① $y = \int_3^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

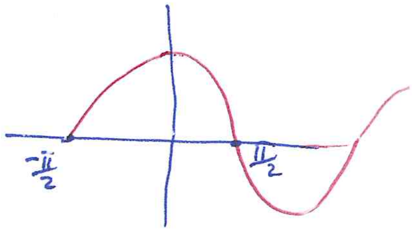
② $y = \int_5^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

③ $y = \int_a^x (t^2 - 5t)\,dt \Rightarrow x^2 - 5x$

④ $y = \int_x^5 (t^2 - 5t)\,dt \Rightarrow -\int_5^x (t^2 - 5t)\,dt = -(x^2 - 5x) = 5x - x^2$

⑤ $y = \int_x^0 \sqrt{1+t^2}\,dt \Rightarrow -\sqrt{1+x^2}$

⑥ $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, \quad |x| < \frac{\pi}{2}$



$y' = \frac{\cos x}{\sqrt{1-\sin^2 x}} = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{\cos x} = 1$

⑦ $y = \int_3^{x^3} \sin t\,dt \Rightarrow 3x^2 \sin x^3$

$u = x^3$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \int_{27}^{u} \sin t\,dt \iff y' = \left(\int_{27}^{u} \sin t\,dt\right)' u' = \sin u \,(3x^2) = 3x^2 \sin x^3$

Th ② (The Fundamental Theorem of Calculus)

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

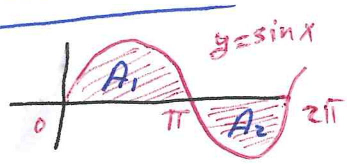
Example: Evaluate the integrals:

① $\int_{-2}^{0} (2x+5)\,dx = x^2 + 5x \Big|_{-2}^{0} = -[4-10] = 6$

② $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan x \Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1$

* To Find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a,b]$:
  - Find the zeros of f on $[a,b]$
  - Divid $[a,b]$ at the zeros of f.
  - Integrate f over each subinterval in absolute value.

Example: let f be as defined in the graph:



a) Find the definit integral of f on $[0, 2\pi]$

$\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1-1] = 0$

b) Find the area between f and the x-axis on $[0, 2\pi]$

Area $= |A_1| + |A_2|$

where $A_1 = \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$

$A_2 = \int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi}^{2\pi} = -[\cos 2\pi - \cos\pi] = -[1+1] = -2$

$\text{Area} = |A_1| + |A_2| = |2| + |-2| = 4$

Th (The Net Change Theorem)

The net change in the function $F(x)$ on $[a,b]$ is the integral of its rate of change: $F(b) - F(a) = \int_a^b F'(x)\,dx$

---

* If $c(x)$ is the cost for producing x units, then $c'(x)$ is the marginal cost and

  $\int_{x_1}^{x_2} c'(x)\,dx = c(x_2) - c(x_1)$ is the cost of increasing the production from x_1 units to x_2 units.

* If $s(t)$ is the position of an object, then $s'(t) = v(t)$ is its velocity and

  $\int_{t_1}^{t_2} \underbrace{v(t)}_{\text{velocity}}\,dt = s(t_2) - s(t_1)$ is the displacement over $[t_1, t_2]$ and

  $\int_{t_1}^{t_2} \overbrace{|v(t)|}^{\text{speed}}\,dt$ is the total distance traveled on $[t_1, t_2]$

* $\underbrace{F(b)}_{\substack{\text{Final}\\ \text{value}}} = \underbrace{F(a)}_{\substack{\text{initial}\\ \text{value}}} + \underbrace{\int_a^b F'(x)\,dx}_{\text{Net change}}$

5.5 Indifinite Integrals and Substitution Method

Example: Evaluate the indifinite integrals:

① $\int (x^3+x)^5 (3x^2+1)\,dx$ $u = x^3 + x$, $du = (3x^2+1)\,dx$

$= \int u^5\,du = \frac{u^6}{6} + C = \frac{1}{6}(x^3+x)^6 + C$

② $\int (3x+2)(3x^2+4x)^4\,dx$ $u = 3x^2+4x$, $du = (6x+4)\,dx$, $\frac{du}{2} = (3x+2)\,dx$

$\frac{1}{2}\int u^4\,du = \frac{1}{2}\frac{u^5}{5} + C$

$= \frac{1}{10}(3x^2+4x)^5 + C$

③ $\int 7\sqrt{7x-1}\,dx$ $u = 7x-1$, $du = 7\,dx$

$\int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(7x-1)^{\frac{3}{2}} + C$

④ $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2}$ $u = 1+\sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$

$2\int u^{-2}\,du = \frac{-2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$

⑤ $\int \frac{-6\tan^2 x \sec^2 x}{(2+\tan^3 x)^3}\,dx$ $u = 2+\tan^3 x$, $du = 3\tan^2 x \sec^2 x\,dx$

$-2\int \frac{du}{u^3} = -2\frac{u^{-2}}{-2} + C = \frac{1}{(2+\tan^3 x)^2} + C$

Example: Solve the IVP $\frac{ds}{dt} = 12t(3t^2-1)^3$, $s(1)=3$ $u = 3t^2-1$, $du = 6t\,dt$

$s(t) = \int 12t(3t^2-1)^3\,dt = 2\int u^3\,du$

$s(t) = 2\frac{u^4}{4} + C = \frac{(3t^2-1)^4}{2} + C$ $s(1) = \frac{2^4}{2} + C = 3 \Leftrightarrow C = -5$

$s(t) = \frac{1}{2}(3t^2-1)^4 - 5$

Example ⓐ $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$ ⓑ $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

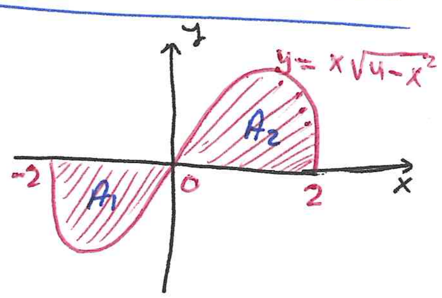
$\boxed{5.6}$ Substitution and Area Between Curves

Example: Evaluate the definite Integral $\int_0^1 (t^5+2t+1)^{-\frac{1}{2}}(5t^4+2)\,dt$

$$\int_0^1 (t^5+2t+1)^{-\frac{1}{2}}(5t^4+2)\,dt = \int_1^4 u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}}\Big|_1^4 = 2\sqrt{u}\Big|_1^4 = 2(2)-2(1) = 4-2 = 2$$

$u = t^5+2t+1$
$du = (5t^4+2)\,dt$

Example: Consider the graph of $y = x\sqrt{4-x^2}$



[a] Find $\int_{-2}^{2} x\sqrt{4-x^2}\,dx = 0$ since y is odd

[b] Find the total area of the shaded regions

$$A = |A_1| + |A_2| = 2\int_0^2 x\sqrt{4-x^2}\,dx$$

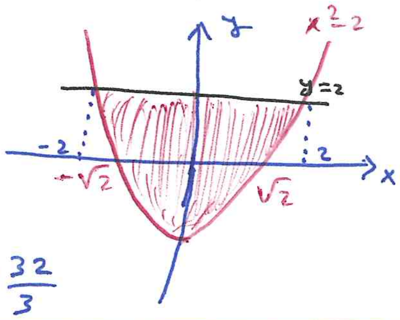
$u = 4-x^2$
$du = -2x\,dx$

$$= -\int_4^0 u^{\frac{1}{2}}\,du = \frac{-2}{3}u^{\frac{3}{2}}\Big|_4^0 = -\frac{2}{3}\left[\sqrt{0^3}-\sqrt{4^3}\right] = -\frac{2}{3}(-8) = +\frac{16}{3}$$

Theorem Let f be continuous on the symmetric interval $[-a,a]$.

- If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
- If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example: Find the area of the region enclosed by the curve $y = x^2-2$ and $y=2$. $x^2-2=2 \Leftrightarrow x^2=4 \Leftrightarrow x=\pm 2$

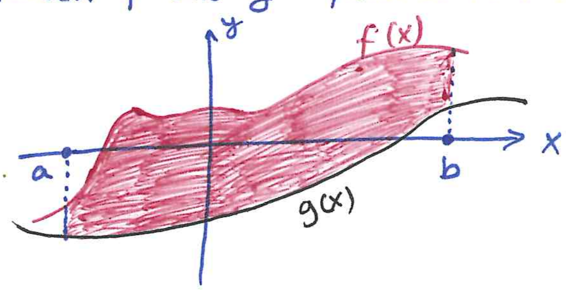


$$\text{Area} = 2\int_0^2 (2-x^2+2)\,dx = 2\int_0^2 (4-x^2)\,dx$$

$$= 2\left[4x-\frac{x^3}{3}\Big|_0^2\right] = 2\left[8-\frac{8}{3}\right] = 2\left(\frac{16}{3}\right) = \frac{32}{3}$$

Definition: If f and g are continuous on $[a,b]$ with $f(x) \geq g(x)$, then the area of the region between f and g from a to b is

$$A = \int_a^b [f(x)-g(x)]\,dx$$



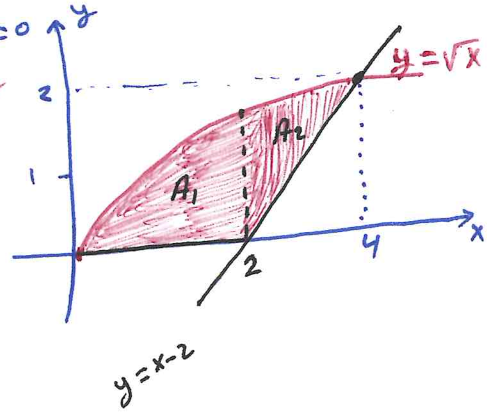
Example*: Find the area of the region in the 1^{st} quadrant that is bounded above by $y=\sqrt{x}$ and below by the x-axis and the line $y=x-2$ 108

$\sqrt{x} = x-2 \iff x = x^2-4x+4 \iff x^2-5x+4=0$

$\iff (x-4)(x-1)=0 \iff \boxed{x=4}$ ✓

Does not satisfy ← $\cancel{x=1}$

$\sqrt{x} = x-2$



The total area $= A_1 + A_2$

$$= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 + \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^2}{2} + 2x\right]\Big|_2^4 = \frac{10}{3}$$

Example: Find the area of the region in Example* by integrating w.r.t. y.

$x = y+2$ upper curve

$x = y^2$ lower curve

$y+2 = y^2 \iff$

$y^2 - y - 2 = 0 \iff$

$(y-2)(y+1) = 0 \iff$

$\boxed{y=2}$ and $\cancel{y=-1}$ below x-axis

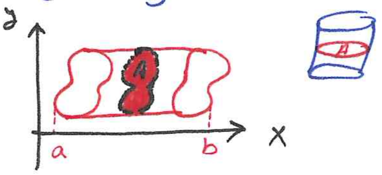
$$A = \int_0^2 (y+2-y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

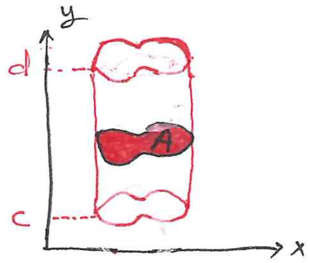
$$= 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{10}{3}$$

6.1 Volumes Using Cross-Sections

* The volume of cylindrical solid is $V = \text{base area} \times \text{height} = Ah$



* To Find Volume of a Solid:
  1- Graph the solid
  2- Determine a cross-section of the solid



(a) • If the cross-sectional is perpendicular to the x-axis, then the volume is

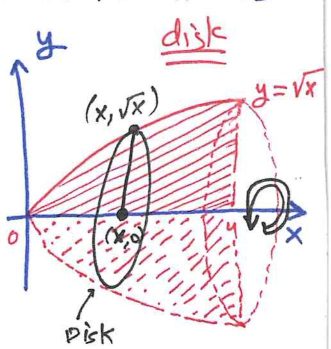
$$V = \int_a^b A(x)\,dx$$

(b) • If the cross-sectional is perpendicular to the y-axis, then the volume is

$$V = \int_c^d A(y)\,dy$$

Disk Method (c) • If the cross-sectional is perpendicular to the x-axis results by rotation about x-axis, then

$$V = \int_a^b \underbrace{A(x)}_{\text{Disk}}\,dx = \int_a^b \pi\,[\underbrace{R(x)}_{\text{radius}}]^2\,dx$$



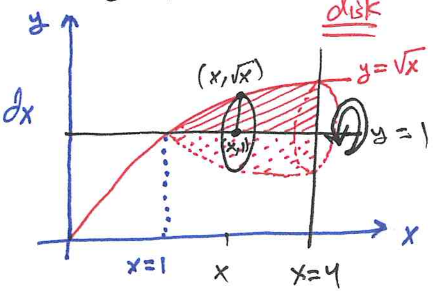
Example: Find the volume of the region between the curve $y = \sqrt{x}$, $0 \le x \le 4$ and the x-axis that is revolved about the x-axis.

$$V = \int_0^4 \pi(\sqrt{x})^2\,dx = \pi\int_0^4 x\,dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$$

Example: Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

$$V = \int_1^4 \pi(\sqrt{x} - 1)^2\,dx = \pi\int_1^4 (x - 2x^{\frac{1}{2}} + 1)\,dx$$

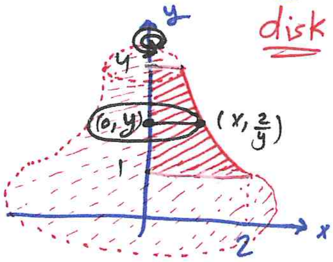
$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{\frac{3}{2}} + x\right]_1^4 = \frac{7\pi}{6}$$



Disk Method (d) • If the cross-sectional is perpendicular to the y-axis results by rotation about y-axis, then

disk

$$V = \int_c^d \underbrace{A(y)}_{\Downarrow \text{ Disk}}\, dy = \int_c^d \pi\,[R(y)]^2\, dy$$



Example: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y-axis.

when $y = 1 \Rightarrow x = 2$

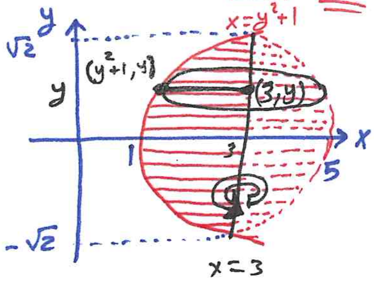
$$V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi \left.\frac{y^{-1}}{-1}\right|_1^4 = 3\pi$$

Example: Find the volume of the solid generated by revolving the region between $x = y^2+1$ and the line $x=3$ about the line $x=3$.

disk

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\,[3-y^2-1]^2\, dy = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2-y^2)^2 dy$$

$$= \pi\left(4 - 4y^2 + y^4\right)\Big|_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\pi\sqrt{2}}{15}$$



$y^2+1 = 3 \Leftrightarrow y^2 = 2 \Leftrightarrow y = \pm\sqrt{2}$

Washer Method (e) • If the cross-sectional is perpendicular to x-axis results by rotation about x-axis with outer radius $R(x)$ and inner radius $r(x)$, then

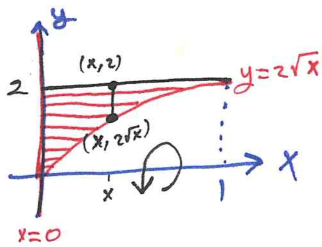
$$V = \int_a^b A(x)\, dx = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx$$

Example: Find the volume of the solid generated by revolving the region bounded by $y = 2\sqrt{x}$, $y=2$ and $x=0$ about the x-axis.

$$V = \pi \int_0^1 \left[(2)^2 - (2\sqrt{x})^2\right] dx = \pi \int_0^1 [4 - 4x]\, dx$$

$$= 4\pi \int_0^1 (1-x)\, dx = 4\pi\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\pi$$



no disk

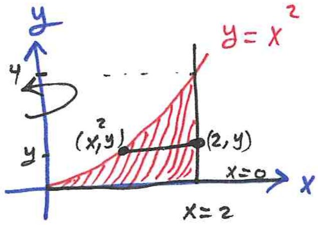
Washer Method (f)

If the cross-sectional is perpendicular to y-axis results by rotation about y-axis with outer radius $R(y)$ and inner radius $r(y)$, then

$$V=\int_c^d A(y)\,dy=\int_c^d \pi\left[R^2(y)-r^2(y)\right]dy$$

Example: Find the volume of the solid generated by revolving the region, bounded by $y=x^2$, x-axis, and $x=2$ about the y-axis.

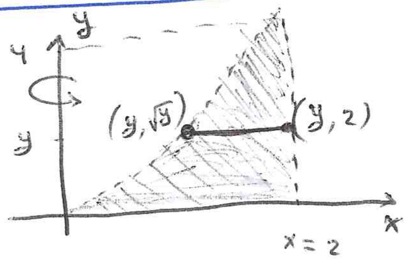
in the first quadrant



$R(y)=2\quad,\quad r(y)=\sqrt{y}$

$$V=\pi\int_0^4\left[2^2-(\sqrt{y})^2\right]dy$$

$$=\pi\int_0^4(4-y)\,dy=\pi\left[4y-\frac{y^2}{2}\right]\Big|_0^4=8\pi$$



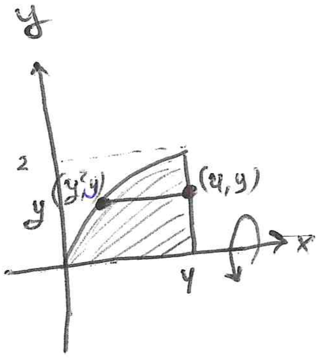
6.2 Volumes Using Shell Method

a The volume of the solid generated by revolving the region about x-axis is

$$V = \int_c^d 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy \quad , \quad \text{where}$$

- shell radius: is the distance from the axis of revolution and the shell length.
- shell length: is the segment's length parallel to the axis of revolution.

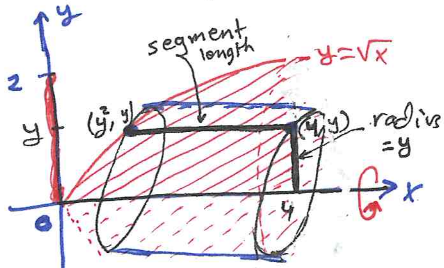
Example: Find the volume of the solid generated by revolving the region bounded by the curve $y=\sqrt{x}$, x-axis and the line $x=4$, about x-axis. Use the shell Method.



$$V = 2\pi \int_0^2 (y)(4-y^2)\, dy$$

$$= 2\pi \int_0^2 (4y - y^3)\, dy$$

$$= 2\pi \left[2y^2 - \frac{y^4}{4}\right]\Big|_0^2 = 8\pi$$



The shell thickness variable is y

b The volume of the solid generated by revolving the region about y-axis is

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx \quad , \quad \text{where}$$

- shell radius: is the distance from the axis of revolution and the shell height.
- shell height: is the segmant's height parallel to the axis of revolution.

Example: Use the shell method to find the volume of the solid generated by revolution the region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x=4$ about y-axis.

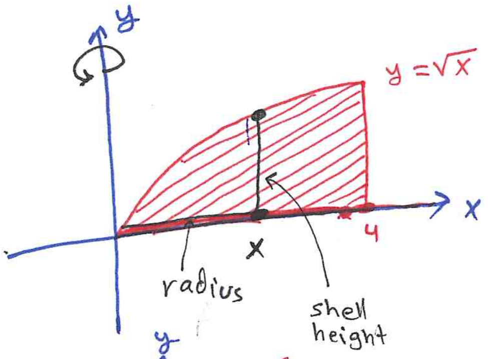
Use Washer Method $\Leftarrow$

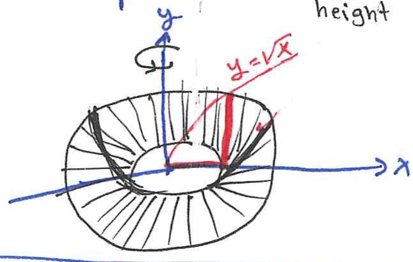
$$V=\pi\int_0^2 [16-y^4]\,dy = \frac{128}{5}\pi$$

$$V = 2\pi\int_0^4 (x)(\sqrt{x})\,dx$$

$$= 2\pi\int_0^4 x^{\frac{3}{2}}\,dx = 2\pi\,\frac{2}{5}\,x^{5/2}\Big|_0^4$$

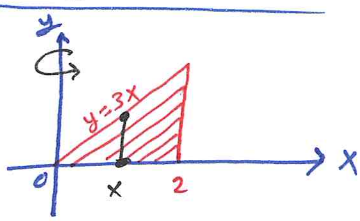
$$= \frac{128}{5}\pi$$





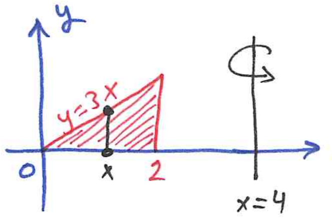
Example Q23 (a) $V = 2\pi\int_a^b \left(\text{shell radius}\right)\left(\text{shell height}\right)dx$

$$= 2\pi\int_0^2 (x)(3x)\,dx = 16\pi$$



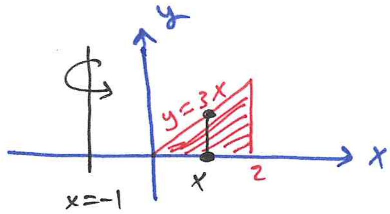
(b) $V = 2\pi\int_a^b \left(\text{shell radius}\right)\left(\text{shell height}\right)dx$

$$= 2\pi\int_0^2 (4-x)(3x)\,dx = 32\pi$$



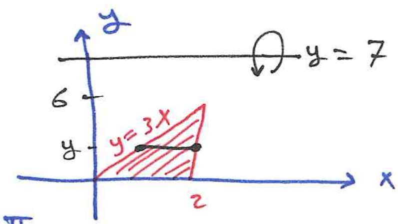
(c) $V = 2\pi\int_a^b \left(\text{shell radius}\right)\left(\text{shell height}\right)dx$

$$= 2\pi\int_0^2 (x--1)(3x)\,dx = 28\pi$$



(e) $V = 2\pi\int_c^d \left(\text{shell radius}\right)\left(\text{shell length}\right)dy$

$$= 2\pi\int_0^6 (7-y)\left(2-\frac{y}{3}\right)dy = 60\pi$$



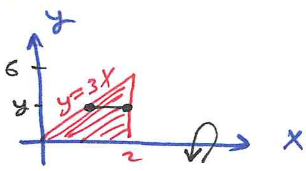
use disk method

$$\int_0^2 (3x)^2\pi\,dx \rightarrow$$

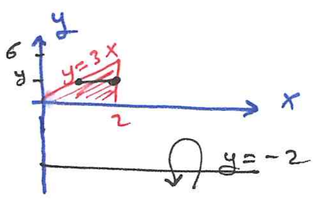
$$= 9\pi\int_0^2 x^2\,dx = 24\pi$$

(d) $V = 2\pi\int_c^d \left(\text{shell radius}\right)\left(\text{shell length}\right)dy$

$$= 2\pi\int_0^6 (y)\left(2-\frac{y}{3}\right)dy = 24\pi$$



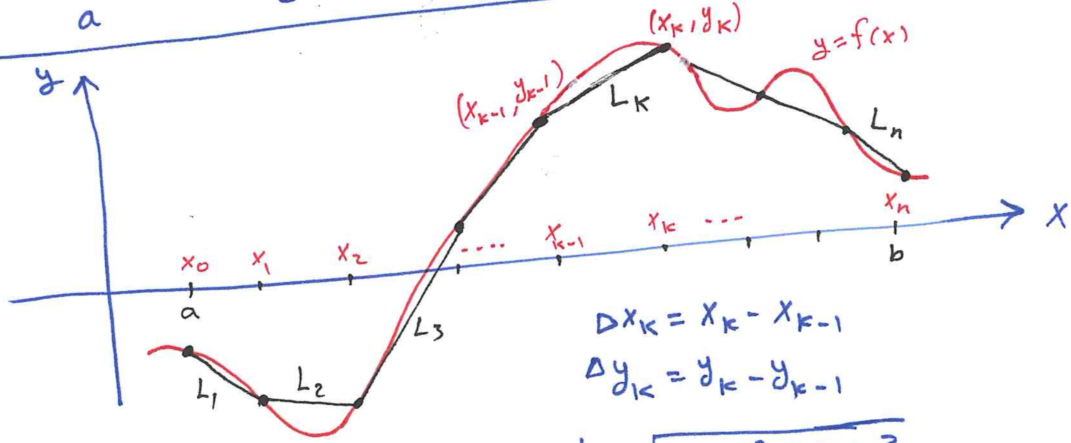
(f) $V = 2\pi\int_0^6 (y--2)\left(2-\frac{y}{3}\right)dy = 48\pi.$



6.3 Arc Length

Def: If f' is continuous on $[a, b]$, then the arc length of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$\Delta x_k = x_k - x_{k-1}$$

$$\Delta y_k = y_k - y_{k-1}$$

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

- The Length of the curve is approximated by:

$$L = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

- By Mean Value Theorem, there is a point c_k with $x_{k-1} < c_k < x_k$ such that

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k} \iff \Delta y_k = f'(c_k)\, \Delta x_k$$

- Therefore, $L = \sum_{k=1}^{n} \boxed{\sqrt{1 + [f'(c_k)]^2}}\, \Delta x_k$ This is continuous on $[a, b]$ $\Rightarrow$

- The approximation L improves as the partition of $[a,b]$ becomes finner:

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example: Find the length of the curve $f(x) = \frac{x^3}{12} + \frac{1}{x}$, $1 \le x \le 4$.

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \Rightarrow 1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$L = \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12}$$

$$= 6.$$

Example: Find the length of the curve $y=\left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x=0$ to $x=2$.

$$y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\frac{1}{2} = \frac{1}{3}\left(\frac{2}{x}\right)^{\frac{1}{3}}$$

" y' is not defined at $x=0$ "

$\Rightarrow$ we write x interms of y

$$L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$\Rightarrow y^{\frac{3}{2}} = \frac{x}{2} \Rightarrow x = 2y^{\frac{3}{2}}$$

$x=0 \Rightarrow y=0$

$x=2 \Rightarrow y=1$

$$\Rightarrow x' = 3y^{\frac{1}{2}}$$

$$\Rightarrow (x')^2 = 9y$$

$$\Rightarrow L=\int_0^1 \sqrt{1+9y}\,dy$$

$u=1+9y$

$du = 9\,dy$

$y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=10$

$$= \int_1^{10} \frac{1}{9}u^{\frac{1}{2}}\,du$$

$$= \frac{1}{9}u^{\frac{3}{2}}\frac{2}{3}\Big|_1^{10} = \frac{2}{27}\left[(10)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right] = \frac{2}{27}\left[10\sqrt{10}-1\right]$$

$$\approx 2.27.$$

Example Find the length of the curve $y=\int_0^x \sqrt{\cos 2t}\,dt$ from $x=0$ to $x=\frac{\pi}{4}$

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$y' = \sqrt{\cos 2x} \Rightarrow (y')^2 = \cos 2x$

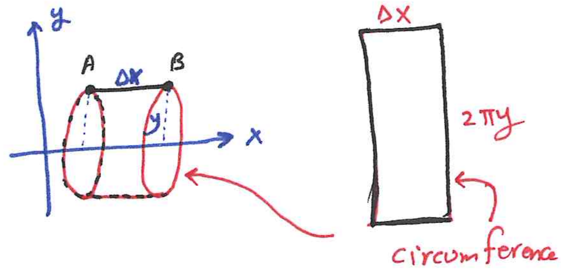
$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\cos 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{1+2\cos^2 x-1}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos x\,dx$$

$$= \sqrt{2}\sin x\Big|_0^{\frac{\pi}{4}} = \sqrt{2}\left[\frac{1}{\sqrt{2}}-0\right] = 1$$

6.4 Surfaces Areas of Revolution

* The surface area generated by rotating the horizontal line AB of length Δx about x-axis is $2\pi y\,\Delta x$



Def: If $y=f(x)\geq 0$ is continuously differentiable on $[a,b]$, then the area of surface generating by revolving the graph of $y=f(x)$ about the x-axis is

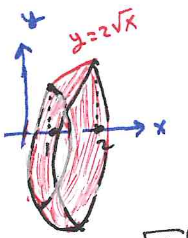
$$S=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

Def: If $x=g(y)\geq 0$ is 〃 〃 〃 $[c,d]$, 〃
〃 〃 〃 〃 〃 〃 〃 〃 〃 〃 $x=g(y)$
〃 〃 y-axis is

$$S=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_c^d 2\pi g(y)\sqrt{1+\left[g'(y)\right]^2}\,dy$$

Example: Find the area of the surface generated by revolving the curve

1. $y=2\sqrt{x}$, $1\leq x\leq 2$ about x-axis



$$S=2\int_1^2 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx=4\pi\int_1^2\sqrt{x}\sqrt{\frac{x+1}{x}}\,dx=4\pi\int_1^2\sqrt{x+1}\,dx$$

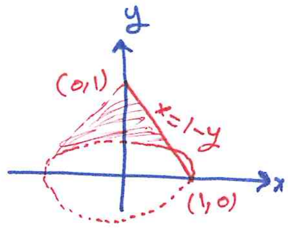
$$=4\pi\int_2^3 u^{\frac{1}{2}}du=\frac{8\pi}{3}u^{\frac{3}{2}}\Big|_2^3=\frac{8\pi}{3}\left(3\sqrt{3}-2\sqrt{2}\right).$$

$u=x+1$
$du=dx$

2. $x=1-y$, $0\leq y\leq 1$ about y-axis.

$$S=2\pi\int_0^1(1-y)\sqrt{1+(-1)^2}\,dy=2\pi\int_0^1\sqrt{2}\,(1-y)\,dy$$

$$=2\sqrt{2}\pi\left[y-\frac{y^2}{2}\right]\Big|_0^1=2\pi\sqrt{2}\left(1-\frac{1}{2}\right)=\pi\sqrt{2}$$



cone
The base not included
"Only the Lateral surface area".

مساحه مثلث = ½ x القاعدة x الارتفاع

$$S=\frac{1}{\cancel{2}}\cancel{2}(1)(\pi)\ \sqrt{2}=\pi\sqrt{2}$$

$\boxed{3}$ $x = \sqrt{2y-1}$, $\frac{5}{8} \leq y \leq 1$, y- axis

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{2}(2y-1)^{-\frac{1}{2}}(2)$$

$$= \frac{1}{\sqrt{2y-1}}$$

$$(x')^2 = \frac{1}{2y-1}$$

$$1 + (x')^2 = \frac{2y-1+1}{2y-1} = \frac{2y}{2y-1}$$

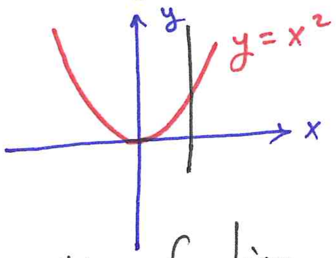
$$= \int_{\frac{5}{8}}^1 2\pi \sqrt{2y-1} \sqrt{\frac{2y}{2y-1}}\, dy$$

$$= \int_{\frac{5}{8}}^1 2\pi \sqrt{2y}\, dy$$

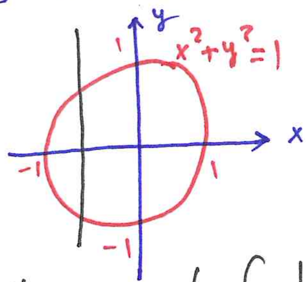
$$= 2\pi \sqrt{2}\, y^{\frac{3}{2}} \frac{2}{3} \Big|_{\frac{5}{8}}^1 = \frac{4\sqrt{2}\pi}{3}\left[1 - \sqrt{\left(\frac{5}{8}\right)^3}\right]$$

7.1 Inverse Functions and Their Derivatives

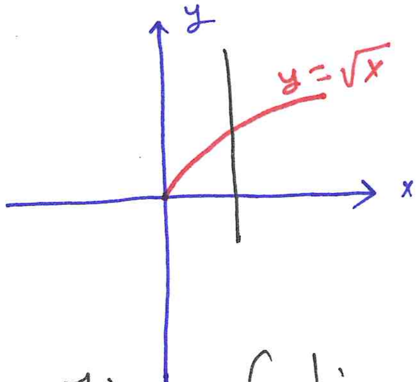
* A function is a rule that assigns a value from its range to each element in its domain.



This is function



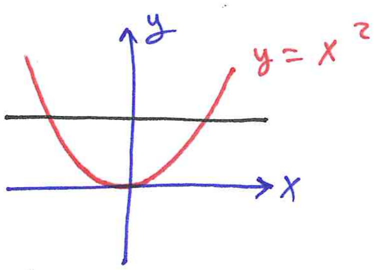
This is not a function



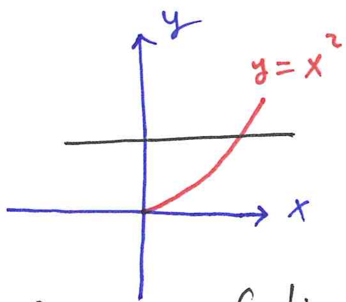
This is a function

---

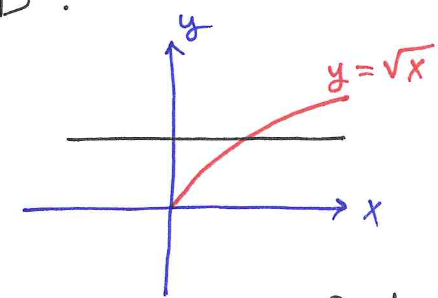
* A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.



Not one-to-one function



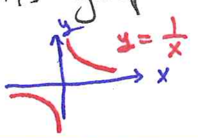
One-to-One function on $D=[0,\infty)$



one-to-one function for any Domain

---

* A function $f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



is one-to-one function

---

* Def: Let $f: D \to R$ be one-to-one. The inverse function $f^{-1}: R \to D$ is defined by $f^{-1}(b)=a$ if $f(a)=b$.

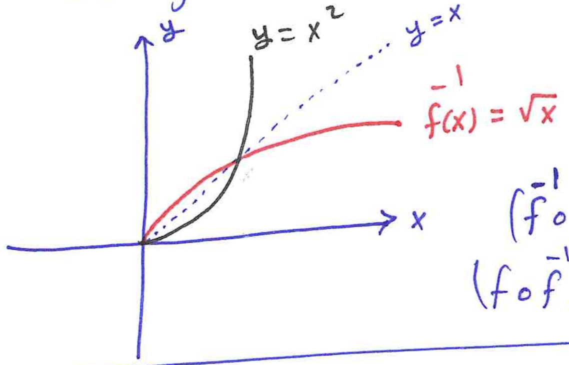
---

* The domain of f^{-1} is R and the range is D
* $f^{-1}(x) \neq \frac{1}{f(x)}$ but $[f(x)]^{-1} = \frac{1}{f(x)}$.
* Only one-to-one functions have inverse.
* $(f^{-1} \circ f)(x) = x$ for all $x \in D(f)$
* $(f \circ f^{-1})(y) = y$ for all $y \in D(f^{-1}) = R(f)$

Example Find the inverse of the function $y = x^2$, $x \geq 0$

$\sqrt{y} = \sqrt{x^2} = |x| = x$ since $x \geq 0$

interchange x and y to obtain $y = \sqrt{x} \Rightarrow f^{-1}(x) = \sqrt{x}$



y: $D = [0, \infty)$, $R = [0, \infty)$

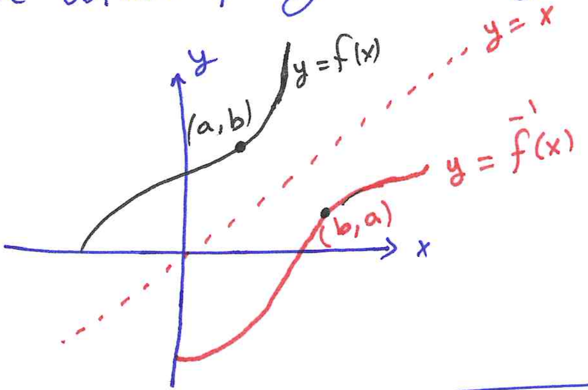
f^{-1}: $D = [0, \infty)$, $R = [0, \infty)$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

---

* We obtain f^{-1} by reflecting the graph of f about the line $y = x$.



x	-2	0	1	5
$y = f(x)$	4	3	2	6

y	4	3	2	6
$f^{-1}(y)$	-2	0	1	5

Example Find the inverse of the function $f(x) = x^2 - 2x$, $x \leq 1$

$f(x) = (x-1)^2 - 1 \Leftrightarrow y = (x-1)^2 - 1 \Leftrightarrow (x-1)^2 = y + 1$

$|x - 1| = \sqrt{y+1} \Leftrightarrow 1 - x = \sqrt{y+1} \Leftrightarrow x = 1 - \sqrt{y+1}$

$y = 1 - \sqrt{x+1} \Leftrightarrow f^{-1}(x) = 1 - \sqrt{x+1}$ with $D = [-1, \infty)$

$R = (-\infty, 1]$

Th 1 If $f: D \to R$ is one-to-one with f' exists and never zero on D then $f^{-1}: R \to D$ is differentiable on R and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \text{or} \quad \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x = f^{-1}(b)}}$$

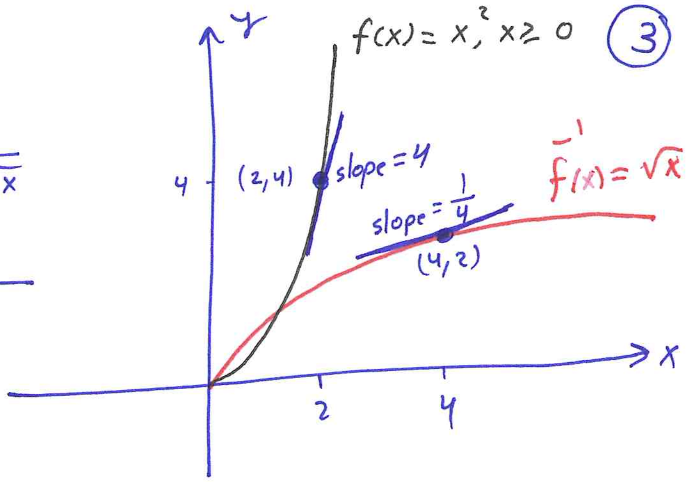
$f(x) = x^2, \quad x \geq 0$

$f'(x) = 2x$

$f'(2) = 2(2) = 4$

$f^{-1}(x) = \sqrt{x}$

$(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$



$$(f^{-1})'(4) = \frac{1}{f'\left(f^{-1}(4)\right)}$$

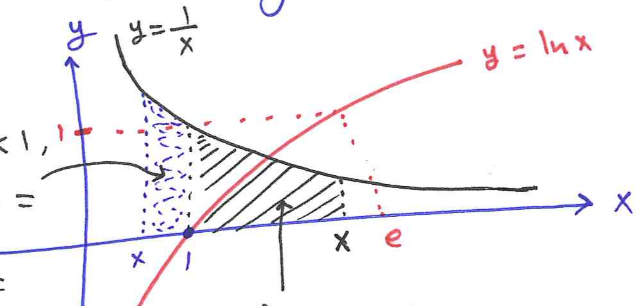
$$= \frac{1}{f'(\sqrt{4})} = \frac{1}{f'(2)} = \frac{1}{2(2)} = \frac{1}{4}$$

Example Let $f'(x) = 3x^2$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})}$

$$\frac{df^{-1}}{dx}\left(f(\sqrt{2})\right) = \frac{1}{f'\left(f^{-1}(f(\sqrt{2}))\right)} = \frac{1}{f'(\sqrt{2})} = \frac{1}{3(2)} = \frac{1}{6}$$

$\boxed{7.2}$ Natural logarithms

* The natural logarithm function $\ln x = \int_1^x \frac{1}{t}\,dt \quad , x > 0$



If $0<x<1$, then $\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$ gives the negative of this area

If $x>0$, then $\ln x = \int_1^x \frac{1}{t}\,dt$ gives this area

- $\ln 1 = 0 = \int_1^1 \frac{1}{t}\,dt$
- $D = (0, \infty)$
- $R = (-\infty, \infty)$
- $\ln e = 1$ where $e \approx 2.718$ is just a number.

---

* If $y = \ln x = \int_1^x \frac{1}{t}\,dt \quad , x > 0,$ then $y' = \frac{dy}{dx} = \frac{1}{x}$

- If $y = \ln|u(x)| = \int_1^{|u(x)|} \frac{1}{t}\,dt ,$ then $y' = \frac{u'(x)}{u(x)} \quad , u(x) \neq 0$ and different
- If $y = \ln|x| \quad , x \neq 0,$ then $y' = \frac{dy}{dx} = \frac{1}{x}$

---

* If u is differentiable that is never zero, then

$$\int \frac{1}{u}\,du = \ln|u| + C$$

Th2 (Properities of the Natural logarithm)

For any positive numbers a and b :

1. $\ln ab = \ln a + \ln b$ "Product Rule"
2. $\ln \frac{a}{b} = \ln a - \ln b$ "Quotient Rule"
3. $\ln \frac{1}{b} = -\ln b$ "Reciprocal Rule"
4. $\ln b^r = r \ln b$ "Power Rule" r is rational.

Example: Express $\ln\sqrt{13.5}$ in terms of $\ln 2$ and $\ln 3$

$$\ln\sqrt{13.5} = \ln\left(\frac{27}{2}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{27}{2}\right) = \frac{1}{2}\left[\ln 27 - \ln 2\right]$$

$$= \frac{1}{2}\left[\ln 3^3 - \ln 2\right] = \frac{1}{2}\left[3\ln 3 - \ln 2\right]$$

* The integrals of $\tan x$, $\cot x$, $\sec x$, $\csc x$

[1] $\int \tan x \, dx = \ln|\sec x| + c$

[2] $\int \cot x \, dx = \ln|\sin x| + c$

[3] $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

[4] $\int \csc x \, dx = -\ln|\csc x + \cot x| + c$

Remember:

$\int \sec x \tan x = \sec x + c$

$\int \csc x \cot x = -\csc x$

$\int \sec^2 x \, dx = \tan x + c$

$\int \csc^2 dx = -\cot x + c$

Proof [1] $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u}$

$u = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$du = -\sin x \, dx$

$$= -\ln|u| + c = -\ln|\cos x| + c = \ln\frac{1}{|\cos x|} + c = \ln|\sec x| + c$$

[3] $\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} dx$

$u = \sec x + \tan x$

$du = \sec x \tan x + \sec^2 x \, dx$

$$= \int \frac{du}{u} = \ln|u| + c$$

$$= \ln|\sec x + \tan x| + c$$

Example: Find [1] $\int_{-3}^{-2} \frac{dx}{x} = \ln|x| \Big|_{-3}^{-2} = \ln 2 - \ln 3 = \ln\frac{2}{3}$

[2] $\int_0^{\frac{\pi}{2}} \tan\frac{x}{2} \, dx = 2\ln\left|\sec\frac{x}{2}\right| \Big|_0^{\frac{\pi}{2}} = 2\ln\frac{1}{\cos\frac{x}{2}} \Big|_0^{\frac{\pi}{2}}$

$$= -2\ln\cos\frac{x}{2} \Big|_0^{\frac{\pi}{2}} = -2\left[\ln\cos\frac{\pi}{4} - \ln\cos 0\right]$$

$$= -2\left[\ln\frac{1}{\sqrt{2}} - \ln 1\right] = +2\ln\sqrt{2} = \ln 2$$

Example: Use logarithmic differentiation to find $\frac{dy}{dx}$ for

① $y = \sqrt{x(x+1)}$

$$\ln y = \ln \sqrt{x}\sqrt{x+1} = \ln\sqrt{x} + \ln\sqrt{x+1}$$

$$\ln y = \frac{1}{2}\ln x + \frac{1}{2}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2}\left[\frac{1}{x} + \frac{1}{x+1}\right] \Leftrightarrow y' = \frac{1}{2}y\left[\frac{1}{x} + \frac{1}{x+1}\right]$$

$$= \frac{1}{2}\sqrt{x(x+1)}\left[\frac{1}{x} + \frac{1}{x+1}\right]$$

---

② $y = t(t+1)(t+2)$

$$\ln y = \ln t + \ln(t+1) + \ln(t+2)$$

$$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$$

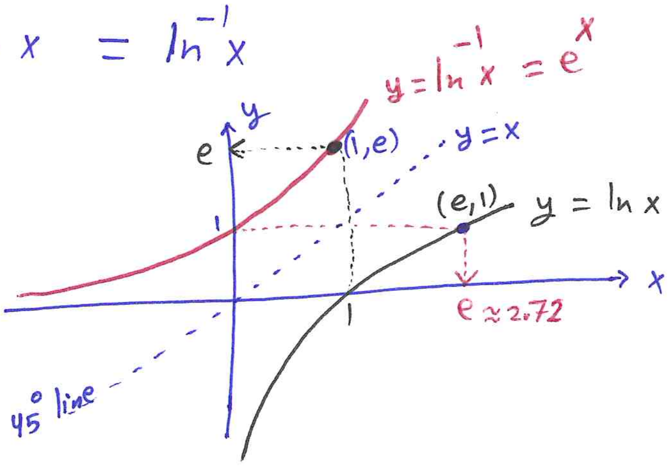
$$y' = y\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right]$$

$$y' = t(t+1)(t+2)\left[\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right]$$

7.3 Exponential Functions

Def: For every real number x, the natural exponential function is $e^x = \exp x = \ln^{-1} x$

- $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$
- $\ln e = 1$ and $\ln^{-1} 1 = e^1 = e$



$$e^{\ln x} = x \quad \text{for all } x > 0$$
$$\ln e^x = x \quad \text{for all } x$$

Inverse Equations for e^x and $\ln x$

Example: Solve the equation for x: [a] $e^{\ln(0.2x)} = 0.4$

$0.2x = 0.4 \Rightarrow x = 2$

[b] $e^{(\ln 0.2)x} = 0.4$

$(\ln 0.2)x = \ln 0.4 \Rightarrow x = \dfrac{\ln 0.4}{\ln 0.2}$

---

* If $u(x)$ is differentiable function of x and $y = e^{u(x)}$,

Then $y' = \dfrac{dy}{dx} = e^{u(x)} \dfrac{du}{dx}$

Example: Find y' for [1] $y = e^x \Rightarrow y' = e^x$

[2] $y = e^{5-7x} \Rightarrow y' = -7e^{5-7x}$

[3] $y = e^{\cos x} \Rightarrow y' = -\sin x \, e^{\cos x}$

---

* The general antiderivative of the exponential function

$$\int e^u \, du = e^u + C$$

Example: Find

1. $\displaystyle\int_{\ln 2}^{\ln 3} e^{x}\,dx = e^{x}\Big|_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2} = 3 - 2 = 1$ 8

2. $\displaystyle\int 2t\,e^{-t^2}\,dt = -\int -2t\,e^{-t^2}\,dt = -e^{-t^2} + C$

3. $\displaystyle\int \frac{e^{\sqrt{r}}}{\sqrt{r}}\,dr = 2\int \frac{e^{\sqrt{r}}}{2\sqrt{r}}\,dr = 2e^{\sqrt{r}} + C$

Th For all x_1, x_2, and x_3 we have

1. $e^{x_1} e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^{x}}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $\left[e^{x_1}\right]^{r} = e^{r x_1}$, $r \in \mathbb{Q}$

Proof 1. let $y_1 = e^{x_1} \Rightarrow x_1 = \ln y_1$
let $y_2 = e^{x_2} \Rightarrow x_2 = \ln y_2$

$$x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$$

$$e^{x_1+x_2} = e^{\ln y_1 y_2}$$

$$e^{x_1+x_2} = y_1 y_2 = e^{x_1} e^{x_2}$$

---

* The general Exp. Function a^{x}, $\underset{\text{base}}{a} > 0$ is given by

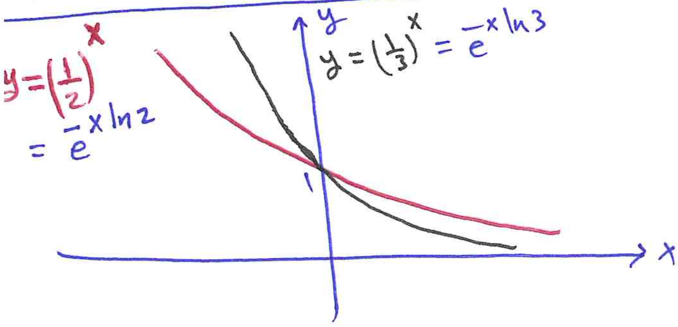
$$a^{x} = e^{x \ln a}$$

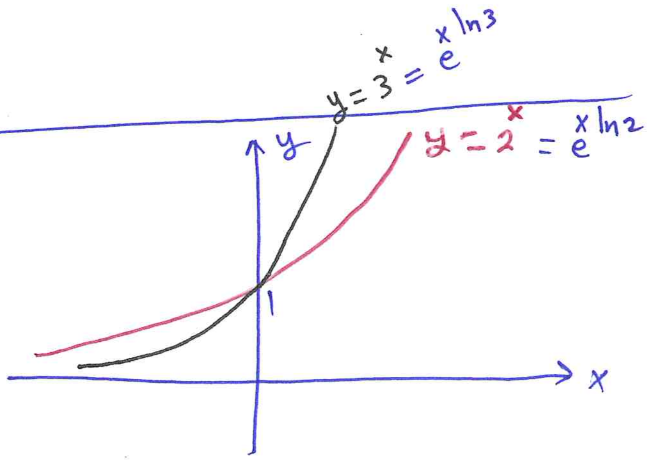
- $a = e^{\ln a} \Rightarrow a^{x} = \left(e^{\ln a}\right)^{x} = e^{x \ln a}$
- when $a = e \Rightarrow e^{x} = e^{x \ln e} = e^{x}$

* If $y = a^{x}$, then $y' = a^{x} \ln a$ i.e $y' = \ln a \, e^{x \ln a} = \ln a \, a^{x}$

* If $y = a^{u(x)}$, then $y' = a^{u(x)} \ln a \; u'(x)$

* $\displaystyle\int a^{u}\,du = \frac{a^{u}}{\ln a} + C$





Example Find y' for

1. $y = 5^x \Rightarrow y' = 5^x \ln 5$

2. $y = 5^{\sqrt{x}} \Rightarrow y' = 5^{\sqrt{x}} \ln 5 \; \frac{1}{2\sqrt{x}}$

3. $y = x^\pi \Rightarrow y' = \pi x^{\pi - 1}$

3. $y = 2^{\sin 3t} \ln 2 \Rightarrow y' = 3\, 2^{\sin 3t} (\ln 2)^2 \cos 3t$

Example Find

1. $\int 7^{\sec\theta} \ln 7 \sec\theta \tan\theta \, d\theta = 7^{\sec\theta} + C$

take $u = \sec\theta \Rightarrow du = \sec\theta \tan\theta \, d\theta$

$$\int 7^u \ln 7 \, du = \ln 7 \int 7^u du = \ln 7 \frac{7^u}{\ln 7} + C = 7^u + C$$
$$= 7^{\sec\theta} + C$$

2. $\int 7^x dx = \frac{7^x}{\ln 7} + C$

---

* For $x > 0$, we have $x^n = e^{n \ln x}$, $n \in \mathbb{R}$

If $y = x^n$, then $y' = e^{n \ln x} \frac{n}{x} = x^n \frac{n}{x} = n x^{n-1}$

Example Find $f'(x)$ if $f(x) = x^x$

$$f(x) = x^x = e^{x \ln x}$$
$$f'(x) = e^{x \ln x} \left(x \frac{1}{x} + \ln x \right)$$
$$= x^x (1 + \ln x)$$

Th $e = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$ "The number e as a limit"

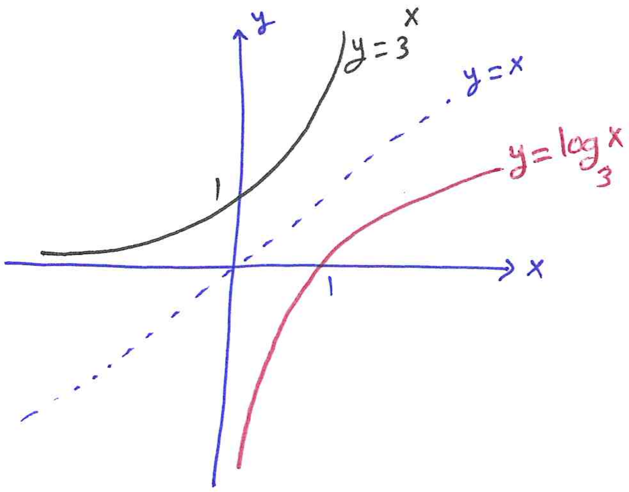
Proof: let $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$ with $f'(1) = 1$

But $f'(1) = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x}$

$$= \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln (1+x)^{\frac{1}{x}}$$

$$1 = \ln \left[\lim_{x \to 0} (1+x)^{\frac{1}{x}} \right] \Leftrightarrow e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

* Inverse Equations for a^x and $\log_a x$ "more general"



$a^{\log_a x} = x$ for $x > 0$

$\log_a a^x = x$ for all x

since when $a = e \Rightarrow$

$e^{\log_e x} = e^{\ln x} = x$ ✓

$\log_e e^x = \ln e^x = x$ ✓

$$\log_a x = \frac{\ln x}{\ln a}$$

So $\log_e x = \dfrac{\ln x}{\ln e} = \ln x$

For any $x > 0$ and $y > 0$:

$\boxed{1}$ $\log_a xy = \log_a x + \log_a y$ "Product Rule"

$\boxed{2}$ $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ "Quotient Rule"

$\boxed{3}$ $\log_a \dfrac{1}{y} = -\log_a y$ "Reciprocal Rule"

$\boxed{4}$ $\log_a x^y = y \log_a x$ "Power Rule"

* If $y = \log_a x$, then $y' = \dfrac{1}{\ln a} \dfrac{1}{x}$

* If $y = \log_a u(x)$, then $y' = \dfrac{1}{\ln a} \dfrac{u'(x)}{u(x)}$

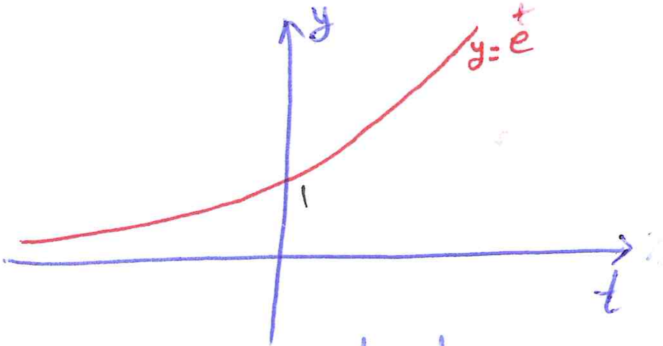
Example: Find y' for $\boxed{1}$ $y = \log_4 x^2 \Rightarrow y' = \dfrac{1}{\ln 4} \dfrac{2x}{x^2} = \dfrac{1}{x \ln 2}$

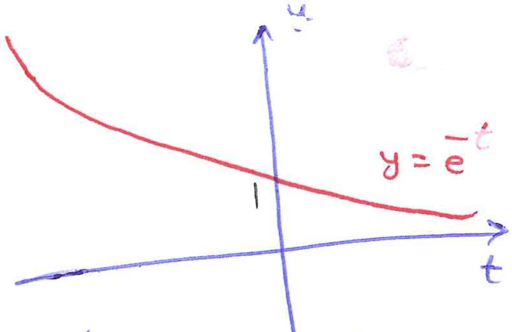
$\boxed{2}$ $y = \log_2 (8 x^{\ln 2}) \Rightarrow y' = \dfrac{1}{\ln 2} \dfrac{8 \ln 2\, x^{\ln 2 - 1}}{8 x^{\ln 2}} = \dfrac{1}{x}$

$\boxed{3}$ $y = \displaystyle\int_0^{\log_4 x} 2 \ln 2\, 4^t \, dt = \cancel{2 \ln 2}\, 4^{\log_4 x} \left(\frac{1}{\cancel{\ln 4}} \frac{1}{x} \right) = \frac{x}{x} = 1$

$\boxed{7.4}$ Exponential change and Separable Differential Equations

- Recall that the exponential function is





e^t describes growth natural

e^{-t} describes natural decay

- Recall that
  - Differential Equations are equations (relations) with derivatives (rates)
  - DE's that describes growth or decay may have the form

The changes in the amount of y(t) with time is proportional to the amount present

$$\frac{dy}{dt} = k\,y \quad \text{----} * ^1$$

  - DE's may have initial condition:

$$y(0) = y_0 \quad \text{-----} * ^2$$

- IVP is a DE together with initial condition

$$\frac{dy}{dt} = k\,y, \quad y(0) = y_0 \quad \text{----} *$$

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* How to solve the IVP $*$?

we use the method of calculus to find the solution $y(t)$ of $*$:

$$\frac{1}{y}\frac{dy}{dt} = K \iff \int \frac{y'}{y} = \int K$$

$$\ln|y| = Kt + C \iff |y| = e^{Kt+C}$$

$$|y| = e^{C} e^{Kt} \iff y = \pm e^{C} e^{Kt}$$

$$y(t) = D e^{Kt} \quad \text{where } D = \pm e^{C}$$

To find D, we use the initial condition:

$$y(0) = D e^{K(0)} = y_0 \iff D = y_0$$

Thus, the solution becomes

$$\boxed{y(t) = y_0 e^{Kt}} \quad *^3$$

* Note that if $K > 0$, the solution grows exponentially
  if $K < 0$, the solution decays exponentially

The growth equation is $y(t) = y_0 e^{Kt}$ / The decay eq. is $y(t) = y_0 e^{-Kt}$

* Check $*^3$ is a solution for $*$

* Note that $\boxed{\text{Half-life} = \frac{\ln 2}{K}}$ because $\frac{y_0}{2} = y_0 e^{-Kt}$

$$\frac{1}{2} = e^{-Kt} \iff -\ln 2 = -Kt \iff \boxed{t = \frac{\ln 2}{K}}$$

Separable Differential Equation

Example: Solve the DE $2\sqrt{xy}\,\frac{dy}{dx} = 1$, $x, y > 0$.

$$2\sqrt{x}\sqrt{y}\,dy = dx \iff 2\,y^{\frac{1}{2}}\,dy = x^{-\frac{1}{2}}\,dx$$

$$2\,\frac{2}{3}\,y^{\frac{3}{2}} = \frac{2}{1}\,x^{\frac{1}{2}} + c \iff \frac{4}{3}\,y^{\frac{3}{2}} - 2\sqrt{x} = c$$

implicit solution

$$\frac{4}{3}\,y^{\frac{3}{2}} = 2\sqrt{x} + c \iff y = \left[\frac{3}{4}\left(2\sqrt{x} + c\right)\right]^{\frac{2}{3}}$$

explicit solution

Example: Solve the IVP $\frac{dy}{dx} = \frac{y\cos x}{1+3y^3}$, $y(0) = 1$

$$(1+3y^3)\,dy = y\cos x\,dx$$

$$\left(\frac{1}{y} + 3y^2\right)dy = \cos x\,dx$$

$$\ln|y| + y^3 = \sin x + C$$

$$\ln 1 + (1)^3 = \sin(0) + C \iff \boxed{1 = C}$$

The solution becomes $\ln|y| + y^3 = \sin x + 1$ implicit solution

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Example (Radioactive) The half-life of the plutonium is 24,360 years. If 10g of plutonium is relased into the atmosphere, how many years will it take for 80% of the plutonium to decay.

$P_0 = 10\,g$

Equation for decay is $P(t) = P_0\,e^{-kt}$

Half-life $= \frac{\ln 2}{k} \iff k = \frac{\ln 2}{24{,}360} \simeq 0.00002845$

We need to find the time t when $P(t) = 20\%\,P_0$

$$0.2\,\cancel{P_0} = \cancel{P_0}\, e^{-kt} \Leftrightarrow -kt = \ln 0.2 = -1.61$$

$$\Leftrightarrow t = \frac{1.61}{0.00002845} \approx 56690 \text{ year}$$

Heat Transfer "Newton's law of Cooling"

- If $H(t) = H$ is the temperature of the object at time t,

  H_s is the constant surrounding tempreature

  then the DE that describes the heat transfer is

$$\frac{dH}{dt} = -k\,(H - H_s) \quad \cdots\cdots \; *$$

* To solve * Let $y = H - H_s$

$$\frac{dy}{dt} = \frac{dH}{dt} = -k\,(H - H_s)$$

$$\frac{dy}{dt} = -ky \qquad \Leftrightarrow y(t) = y_0\, e^{-kt}$$

$$\Leftrightarrow H - H_s = (H_0 - H_s)\, e^{-kt}$$

$$H(t) = H_s + (H_0 - H_s)\, e^{-kt}$$

Example A boiled egg at 98°C is put in a sink of 18°C water. After 5 min the egg's temperature is 38°C. How long will it take the egg to reach 20°C?

$H(0) = H_0 = 98°C$, $H_s = 18°C$, $H(5) = 38°C$

We need to find t such that $H(t) = 20°C$.

$$H(t) = 18 + (98 - 18)\, e^{-kt}$$

$$\boxed{H(t) = 18 + 80\, e^{-kt}}$$

$$H(5) = 18 + 80\, e^{-k(5)} = 38 \qquad \Leftrightarrow 80\, e^{-k(5)} = 20$$

$$e^{-5k} = \frac{1}{4} \Leftrightarrow -5k = -\ln 4 \Leftrightarrow k = \frac{\ln 4}{5} \approx 0.28$$

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$$H(t) = 18 + 80\, e^{-0.28t}$$

Thus,

$$20 = 18 + 80\, e^{-0.28t}$$

$$2 = 80\, e^{-0.28t}$$

$$e^{-0.28t} = \frac{1}{40} \Leftrightarrow -0.28t = -\ln 40$$

$$\Leftrightarrow t = \frac{\ln 40}{0.28} \approx 13 \text{ min.}$$

7.5 Indeterminate Forms and L'Hopital Rule

- Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty\cdot 0$, $\infty-\infty$, 0^0, 1^∞, ∞^0

Th (L'Hopital Rule): suppose f and g are differentiable on an open interval I s.t
$\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

$f(a)=g(a)=0$ where $a\in I$

and $g'(x)\neq 0$ on I if $x\neq a$

Then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ assuming this limit exists.

Example ① $$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4} \qquad \left(\frac{0}{0}\right)$$

② $$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

③ $$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} = \lim_{x\to 0}\frac{\sin x}{1+2x} = \frac{0}{1} = 0 \qquad \left(\frac{0}{0}\right)$$

④ $$\lim_{x\to 0^+}\frac{\sin x}{x^2} = \lim_{x\to 0^+}\frac{\cos x}{2x} = \frac{1}{\text{very small}^+} = \infty$$

⑤ $$\lim_{x\to 0^-}\frac{\sin x}{x^2} = \lim_{x\to 0^-}\frac{\cos x}{2x} = \frac{1}{\text{very small}^-} = -\infty$$

⑥ $$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0 \qquad \left(\frac{\infty}{\infty}\right)$$

⑦ $$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty \qquad \left(\frac{\infty}{\infty}\right)$$

⑧ $$\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{h\to 0^+} \frac{\sin h}{h} = 1 \qquad (\infty \cdot 0)$$

⑨ $$\lim_{x\to 0^+} \sqrt{x} \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}} \qquad (\infty \cdot 0)$$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$

⑩ $$\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \qquad (\infty - \infty)$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{\frac{x-1}{x} + \ln x} = \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x}$$

$$= \lim_{x\to 1^+} \frac{-\frac{1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = -\frac{1}{2}$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a} \ln f(x)}$$

Example ⑪ $$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln (1+x)^{\frac{1}{x}}} \qquad (1^\infty)$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}} = e^1 = e$$

(12) $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \lim\limits_{x\to\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim\limits_{x\to\infty} \frac{\ln x}{x}}$ (18) (∞^0)

$= e^{\lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1}} = e^{\lim\limits_{x\to\infty} \frac{1}{x}} = e^0 = 1$

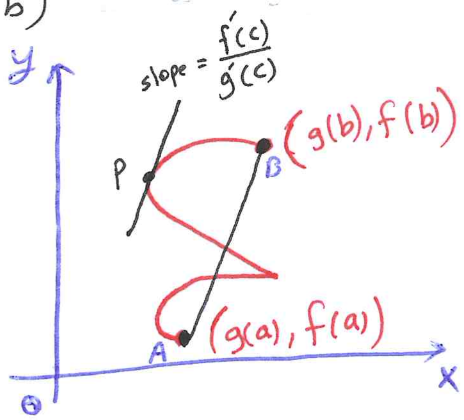
Th (Cauchy's Mean Value Theorem) CMVT

Suppose
- the functions f and g are continuous on $[a, b]$
- " " " " " " differentiable on (a, b)
- $g'(x) \neq 0$ on (a, b)

Then there exists a number $c \in (a, b)$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

slope of tangent = slope of secant



__Example__ : Find values of c that satisfy CMVT for the function $f(x) = x$, $g(x) = x^2$ on $(-2, 0)$ $a = -2$, $b = 0$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{1}{2c} = \frac{f(0) - f(-2)}{g(0) - g(-2)}$$

$$\frac{1}{2c} = \frac{0 - -2}{0 - 4}$$

$$2c = \frac{-4}{2} = -2$$

$$\boxed{c = -1}$$

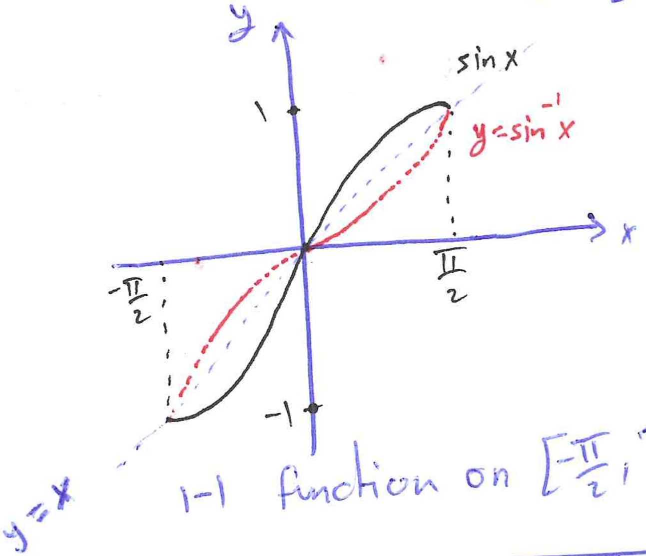
$f(0) = 0$
$f(-2) = -2$

$g(0) = 0$
$g(-2) = 4$

$f'(x) = 1$
$g'(x) = 2x$

$\boxed{7.6}$ Inverse Trigonometric Functions

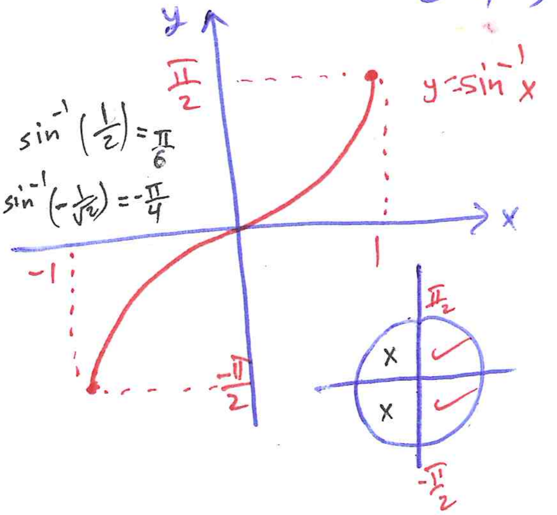
If $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then $f^{-1}(x) = y = \sin^{-1} x$ on $[-1, 1]$
$= \arcsin x$



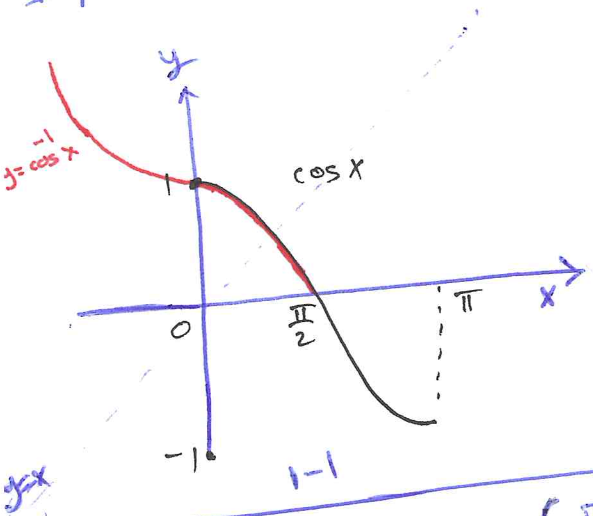
1-1 function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$



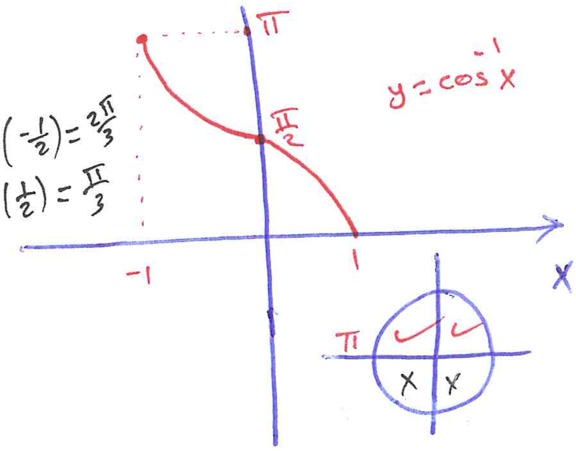
If $f(x) = \cos x$ on $[0, \pi]$, then $f^{-1}(x) = y = \cos^{-1} x$ on $[-1, 1]$
$= \arccos x$



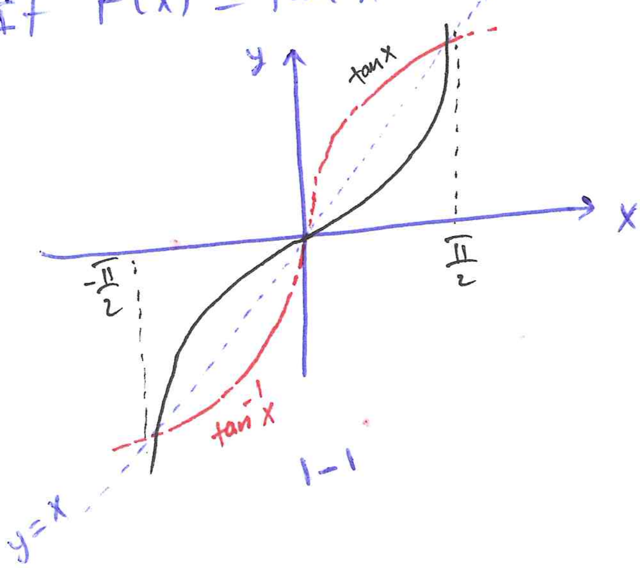
1-1

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$



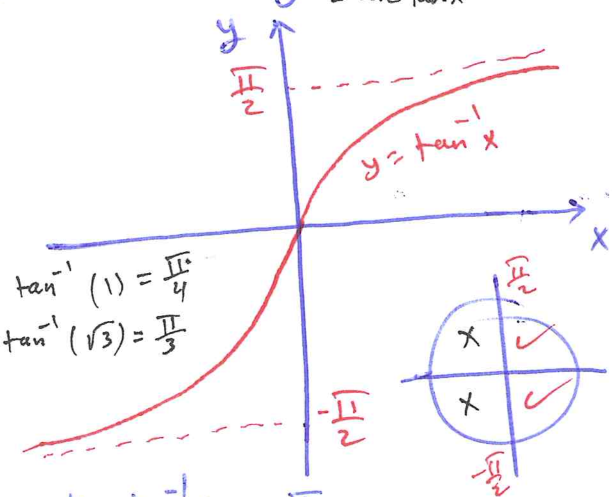
If $f(x) = \tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, then $f^{-1}(x) = y = \tan^{-1} x$ on $\mathbb{R}$
$= \arctan x$



1-1

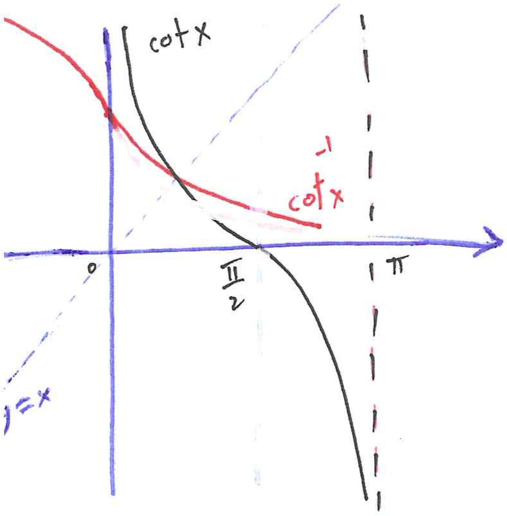
$\tan^{-1}(1) = \frac{\pi}{4}$

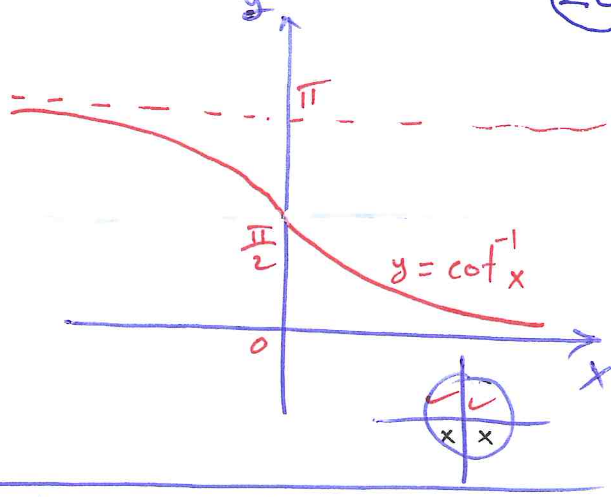
$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

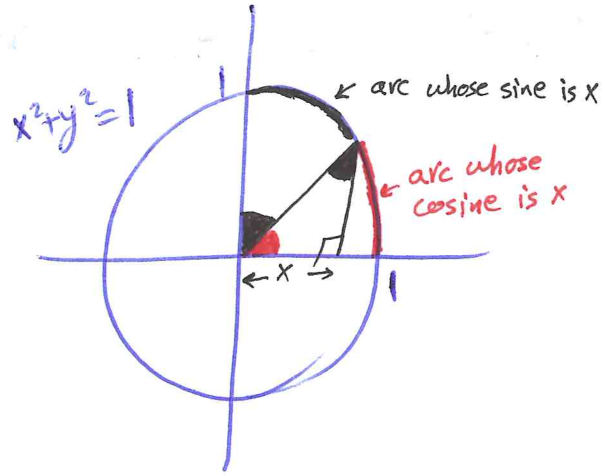


$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

If $f(x) = \cot x$ on $(0, \pi)$, then $f^{-1}(x) = y = \cot^{-1} x$ on $\mathbb{R}$
$= \text{arc} \cot x$

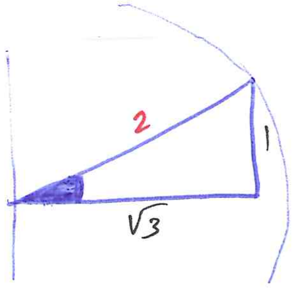




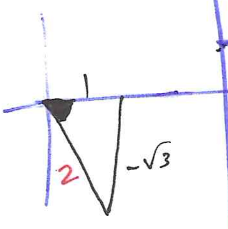


$$s = r\theta = \theta$$

$$\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$



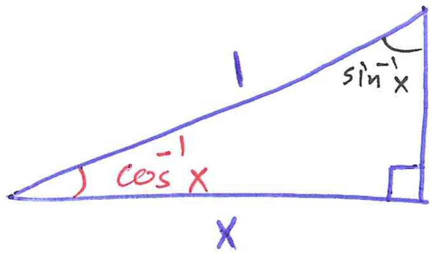
- $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

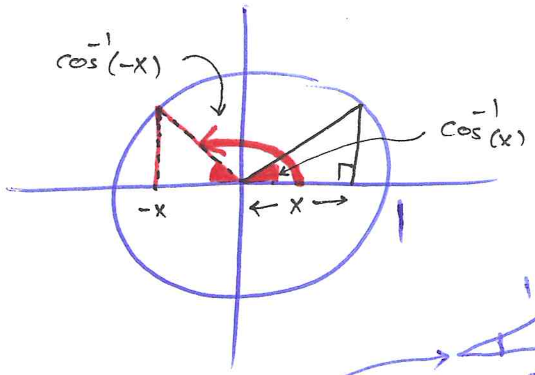


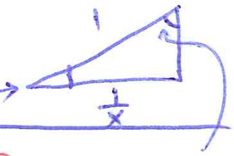
Note that

$$\cos^{-1} x + \cos^{-1}(-x) = \pi$$

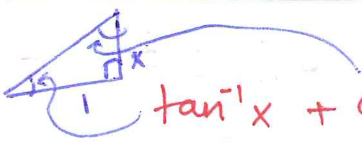
$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$







$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$



$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$

* If $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$, then

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

$$= \frac{1}{\sqrt{1-\sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1-x^2}}$$

$\sin(\sin^{-1} x) = x$

$\sin^2(\sin^{-1} x) = \sin(\sin^{-1} x)\,\sin(\sin^{-1} x)$
$= x \quad x = x^2$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u|<1$$

Example $\frac{d}{dx}(\sin^{-1}\sqrt{2}\,x) = \frac{1}{\sqrt{1-(\sqrt{2}x)^2}}\sqrt{2} = \frac{\sqrt{2}}{\sqrt{1-2x^2}}$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$

$f(x) = \tan x$ with $f^{-1}(x) = \tan^{-1} x$

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1+\tan^2(\tan^{-1} x)}$$

$$= \frac{1}{1+x^2}$$

$\tan(\tan^{-1} x) = x$

Example $y = \ln \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\tan^{-1} x}\left(\frac{1}{1+x^2}\right)$

Similarly

$$\frac{d}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx} \quad , \ |u|<1$$

$$\frac{d}{dx}\left(\cot^{-1} u\right) = \frac{-1}{1+u^2} \frac{du}{dx}$$

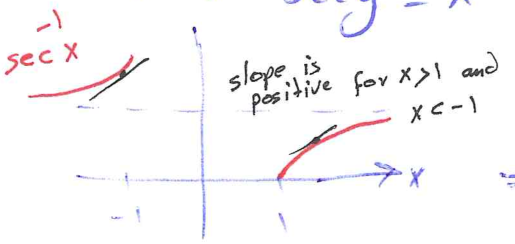
$$\frac{d}{dx}\left(\sec^{-1} u\right) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \quad , \ |u|>1$$

Let $y = \sec^{-1} x \implies \sec y = x$

$$\implies \sec y \tan y \frac{dy}{dx} = 1$$

$$\implies \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

But $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1}$
$= \pm\sqrt{x^2-1}$



$$\implies \frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

Example $y = \sec^{-1}(2x+1) \implies \frac{dy}{dx} = \frac{2}{|2x+1|\sqrt{(2x+1)^2-1}} = \frac{1}{|2x+1|\sqrt{x^2+x}}$

Similarly

$$\frac{d}{dx}\left(\csc^{-1} u\right) = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \quad , \ |u|>1$$

Now for any constant $a \neq 0$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \quad , \quad |u| > a > 0$$

Exp. $\displaystyle\int_{1/\sqrt{2}}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{1/\sqrt{2}}^{\sqrt{3}/2} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Exp $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$

$$4x - x^2 = -(x^2 - 4x)$$
$$= -(x-2)^2 + 4$$
$$= 4 - (x-2)^2$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 2$
$u = x - 2$
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

$\boxed{7.7}$ Hyperbolic Functions

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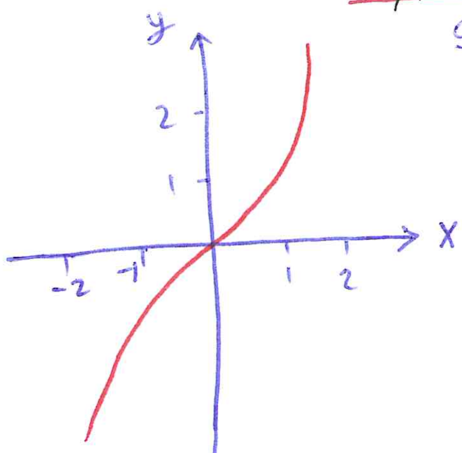
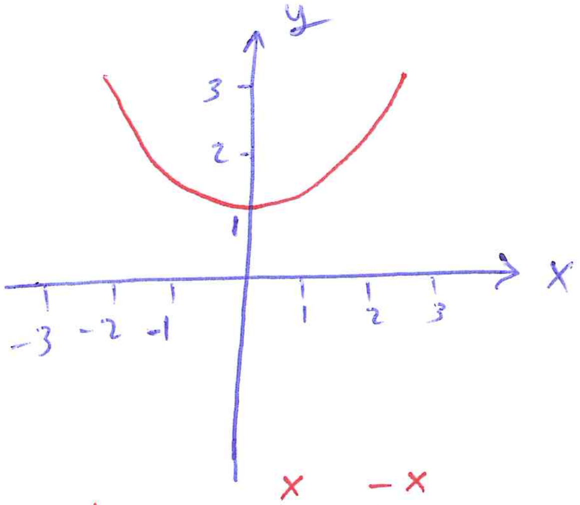
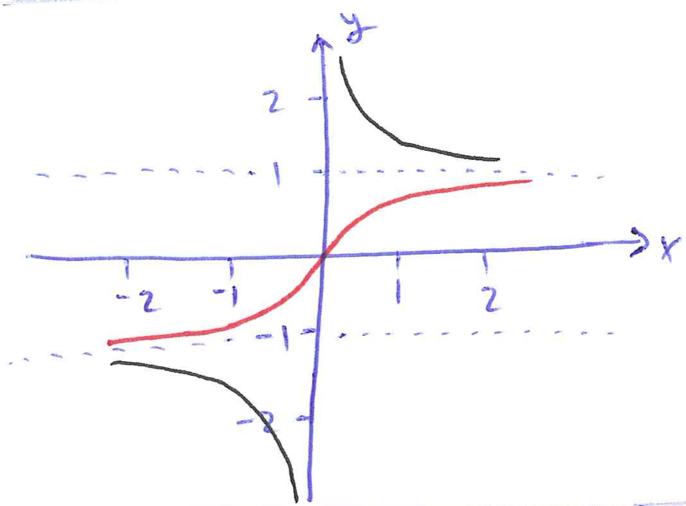


$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

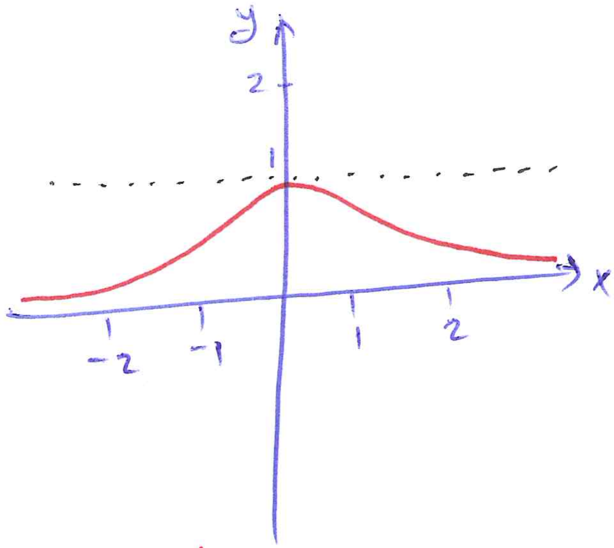


$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



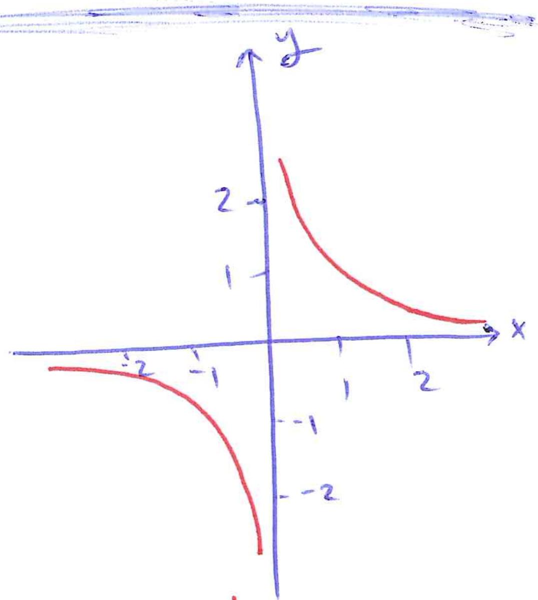
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$



$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$= \frac{2}{e^{x} + e^{-x}}$$



$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$= \frac{2}{e^{x} - e^{-x}}$$

* Identities For Hyperbolic Functions

- $\cosh^2 x - \sinh^2 x = 1$ $\Rightarrow$ Proof $\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - \left(\frac{e^{x}-e^{-x}}{2}\right)^2 =$ $\frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4} = \frac{4}{4} = 1$
- $\sinh 2x = 2\sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
  $= 2\cosh^2 x - 1$
  $= 2\sinh^2 x + 1$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$
- $\coth^2 x - \operatorname{csch}^2 x = 1$

---

* Derivatives of Hyperbolic Functions

✓ • $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

• $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$ $\Rightarrow$ Proof

$\cosh u = \frac{e^{u}+e^{-u}}{2} \Rightarrow$

$\frac{d}{dx}(\cosh u) = \left(\frac{e^{u}-e^{u}}{2}\right)\frac{du}{dx}$

$= \sinh u \frac{du}{dx}$

✓ • $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$

• $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$ $\quad (\cot u)' = -\csc^2 u$

• $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$

✓ • $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} u \coth u \frac{du}{dx}$

Exp. Find y' for ① $y = \ln(\sinh x) \Rightarrow y' = \frac{\cosh x}{\sinh x} = \coth x$

② $y = 4\cosh\frac{x}{2} \Rightarrow y' = 2\sinh\frac{x}{2}$

* Integrals of Hyperbolic Functions

- $\int \sinh u \, du = \cosh u + c$
- $\int \cosh u \, du = \sinh u + C$
- $\int \operatorname{sech}^2 u \, du = \tanh u + c$
- $\int \operatorname{csch}^2 u \, du = -\coth u + c$
- $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$
- $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + c$

Exp: ① $\int \sinh 2x \, dx = \frac{1}{2}\int \sinh u \, du = \frac{1}{2}\cosh 2x + c$

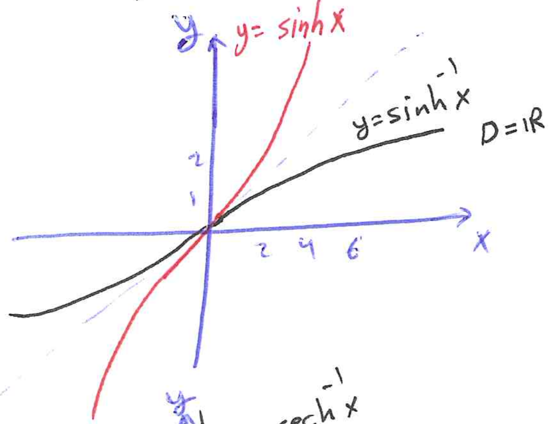
$u = 2x$
$du = 2\,dx$

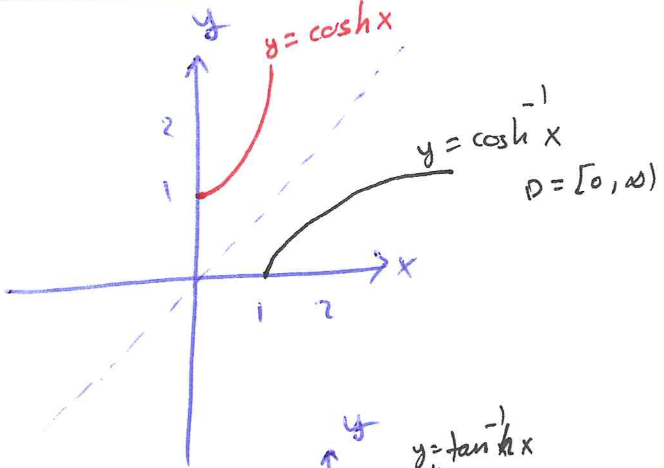
② $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx$

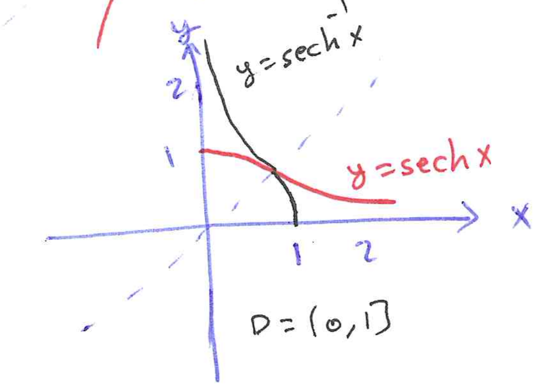
$= \int \frac{du}{u} = \ln|u| + c = \ln|\sinh x| + c$

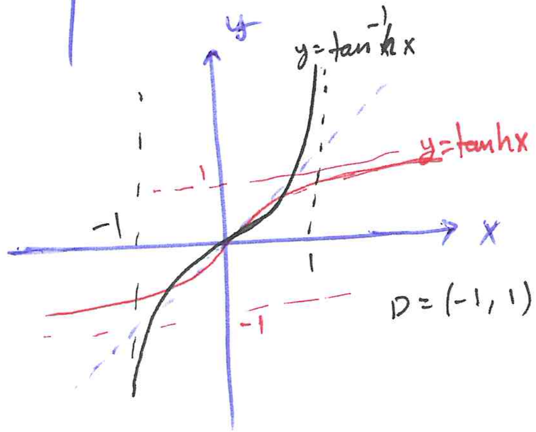
$u = \sinh x$
$du = \cosh x \, dx$

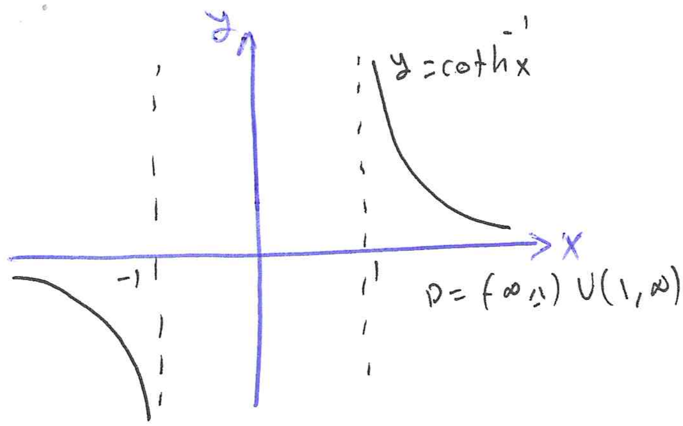
* Inverse Hyperbolic Functions:

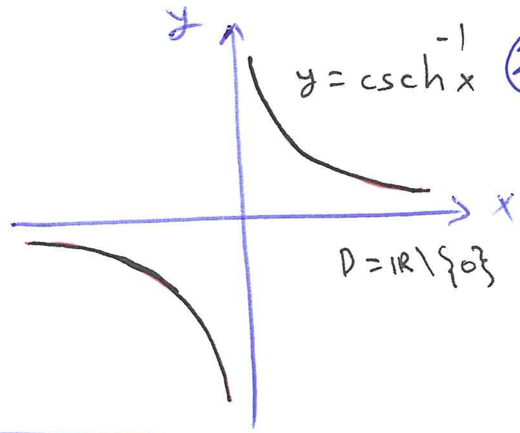












* Identities for inverse hyperbolic functions:

- $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$
- $\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$
- $\coth^{-1} x = \tanh^{-1} \frac{1}{x}$

$\Rightarrow$ Proof

$$\operatorname{sech}\left(\cosh^{-1}\left(\tfrac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x$$

$$= \operatorname{sech}\left(\operatorname{sech}^{-1} x\right)$$

* Derivatives of inverse hyperbolic functions

- $\dfrac{d}{dx}\left(\sinh^{-1} u\right) = \dfrac{1}{\sqrt{1+u^2}} \dfrac{du}{dx}$
- $\dfrac{d}{dx}\left(\cosh^{-1} u\right) = \dfrac{1}{\sqrt{u^2-1}} \dfrac{du}{dx}$, $u > 1$
- $\dfrac{d}{dx}\left(\tanh^{-1} u\right) = \dfrac{1}{1-u^2} \dfrac{du}{dx}$, $\underline{|u| < 1}$
- $\dfrac{d}{dx}\left(\coth^{-1} u\right) = \dfrac{1}{1-u^2} \dfrac{du}{dx}$, $\underline{|u| > 1}$
- $\dfrac{d}{dx}\left(\operatorname{sech}^{-1} u\right) = \dfrac{-1}{u\sqrt{1-u^2}} \dfrac{du}{dx}$, $0 < u < 1$
- $\dfrac{d}{dx}\left(\operatorname{csch}^{-1} u\right) = \dfrac{-1}{|u|\sqrt{1+u^2}} \dfrac{du}{dx}$, $u \neq 0$

$f(x) = \cosh x$

$f^{-1}(x) = \cosh^{-1} x$

Proof

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sinh(\cosh^{-1} x)} = \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} = \frac{1}{\sqrt{x^2-1}}$$

* Integrals leading to inverse hyperbolic functions

- $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad a>0$

- $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u>a>0$

- $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C \quad , & u^2<a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C \quad , & u^2>a^2 \end{cases}$

- $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C \quad , \quad 0<u<a$

- $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad , \quad u\neq 0$ and $a>0$

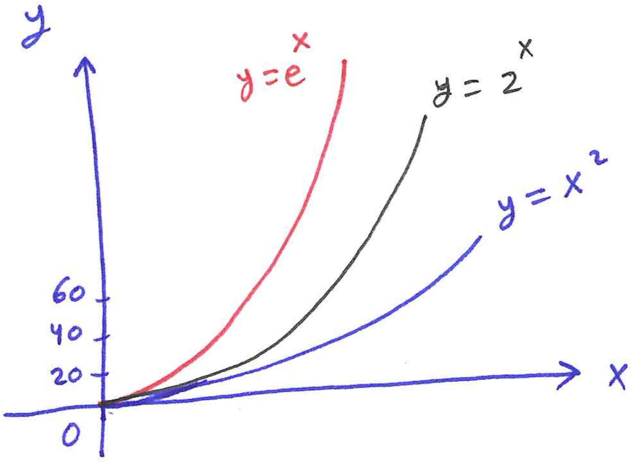
Exp. $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{du}{\sqrt{1+u^2}} = 0$ $\qquad u=\sin x$, $du=\cos x\,dx$

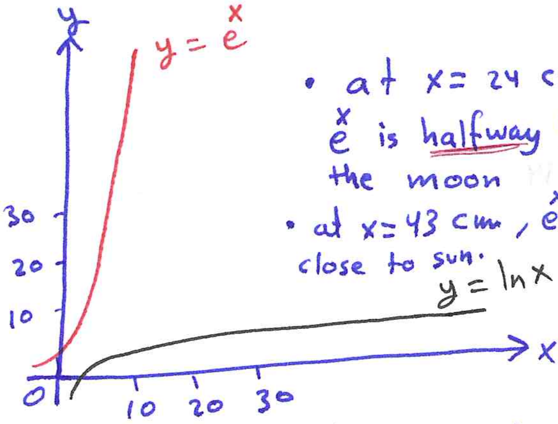
$\displaystyle\int_{\frac{1}{5}}^{\frac{3}{13}} \frac{dx}{x\sqrt{1-16x^2}} = \int_{\frac{4}{5}}^{\frac{12}{13}} \frac{du}{u\sqrt{1-u^2}}$ $\qquad u=4x$, $du=4\,dx$

$\displaystyle = -\operatorname{sech}^{-1}(u)\Big|$ $\qquad a=1$, $0<\frac{12}{13}<1$, $0<\frac{4}{5}<1$

$\displaystyle = -\operatorname{sech}^{-1}\left(\frac{12}{13}\right) + \operatorname{sech}^{-1}\left(\frac{4}{5}\right)$

7.8 Relative Rates of Growth





- at $x = 24$ cm e^x is halfway to the moon
- at $x = 43$ cm, e^x is close to sun.

we need ≈ 5 light-years on x-axis to find point where $y = \ln x = 43$ cm

Def: Let $f(x)$ and $g(x)$ be positive for large x:

- If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$, then f grows faster than g as $x\to\infty$

$\equiv$

- If $\lim\limits_{x\to\infty} \dfrac{g(x)}{f(x)} = 0$, then g grows slower than f as $x\to\infty$.

- If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ " $0<L<\infty$ ", then f and g grow at the same rate as $x\to\infty$

Exp. ① 4^x grows faster than e^x as $x\to\infty$ since

$$\lim_{x\to\infty} \frac{4^x}{e^x} = \left(\frac{4}{e}\right)^x = \infty \qquad e \approx 2.718$$

② $\left(\frac{3}{2}\right)^x$ grows slower than e^x as $x\to\infty$ since

$$\lim_{x\to\infty} \frac{\left(\frac{3}{2}\right)^x}{e^x} = \lim_{x\to\infty}\left(\frac{3}{2e}\right)^x = 0$$

③ $\log_3 x$ grows same as $\ln x$ as $x\to\infty$ since

$$\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 3}}{\ln x} = \lim_{x\to\infty} \frac{1}{\ln 3} = \frac{1}{\ln 3}$$

④ $\ln x$ grows slower than e^x as $x\to\infty$ since

$$\lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0.$$

8.1 Integration by Parts

① $$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Exp ① $\int x\cos x\,dx = \int u\,dv$

$u = x \qquad dv = \cos x\,dx$

$du = dx \implies v = \sin x$

$$= uv - \int v\,du$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x + \cos x + c$$

② $$\int_0^{\pi} x\cos x\,dx = x\sin x\Big|_0^{\pi} - \int_0^{\pi}\sin x\,dx$$
$$= 0 + \cos x\Big|_0^{\pi} = -1 - 1 = -2$$

③ $$\int_0^{\pi} x\cos x\,dx$$

$f(x)$ and its derivatives		$g(x)$ and its integrals
$f(x) = x$	$\xrightarrow{(+)}$	$g(x) = \cos x$
1	$\xrightarrow{(-)}$	$\sin x$
0		$-\cos x$

$$= x\sin x + \cos x + c$$

Exp $\int \ln x\,dx = uv - \int v\,du$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \implies v = x$

$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

Exp $\int x^2 e^x \, dx$

$f(x)$		$g(x)$
$f(x) = x^2$		$g(x) = e^x$
$2x$	$(+)$	e^x
2	$(-)$	e^x
0	$(+)$	e^x

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C$$

$\int x^2 e^x \, dx$ $\qquad u = x^2 \quad dv = e^x \, dx$

$\qquad\qquad\qquad du = 2x \, dx \quad V = e^x$

$= x^2 e^x - \int 2x e^x \, dx$ $\qquad$ we need $\int x e^x \, dx$

$\qquad u = x \quad dv = e^x \, dx$

$\qquad du = dx \quad v = e^x$

$= x^2 e^x - 2 \int x e^x \, dx$

$= x^2 e^x - 2\left[x e^x - \int e^x \, dx\right]$

$= x^2 e^x - 2x e^x + 2\int e^x \, dx \quad = x^2 e^x - 2x e^x + 2e^x + C$

Exp* $\int e^x \cos x \, dx$

$f(x) = e^x$		$g(x) = \cos x$
e^x	$(+)$	$\sin x$
e^x	$(-)$ stop	$-\cos x$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C$$

$\int e^x \cos x \, dx$ $\qquad u = e^x \quad dv = \cos x \, dx$

$\qquad du = e^x \, dx \quad V = \sin x$

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x$ $\qquad u = e^x \quad dv = \sin x \, dx$

$\qquad du = e^x \, dx \quad V = -\cos x$

$= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right]$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{1}{2} e^x [\sin x + \cos x] + C$$

Exp $\displaystyle\int_1^2 x \ln x \, dx$

$u = \ln x \qquad dv = x \, dx$

$du = \dfrac{dx}{x} \qquad v = \dfrac{x^2}{2}$

$$\int_1^2 x \ln x \, dx = \frac{x^2}{2} \ln x \Big|_1^2 - \int_1^2 \frac{1}{x} \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2} \ln x \Big|_1^2 - \frac{1}{2}\int_1^2 x \, dx$$

$$= 2\ln 2 - 0 - \frac{1}{2} \frac{x^2}{2} \Big|_1^2 = 2\ln 2 - 1 + \frac{1}{4}$$

$$= \ln 4 - \frac{3}{4}$$

$$\int x \ln x \, dx = x^2 \ln x - x^2 - \int x \ln x \, dx + \frac{x^2}{2}$$

$$2\int x \ln x \, dx = x^2 \ln x - \frac{x^2}{2}$$

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_1^2$$

$$= [2\ln 2 - 1] - [0 - \tfrac{1}{4}]$$

$$= 2\ln 2 - \frac{3}{4} \checkmark$$

$$\begin{array}{ccc} f(x) = x & & g(x) = \ln x \\ & \xrightarrow{(+)} & \\ 1 & & x\ln x - x \\ & \xrightarrow{(-)} & \\ 0 & & \int x \ln x \, dx - \frac{x^2}{2} \end{array}$$

8.2 Trigonometric Integrals

$$\int \sin^m x \, \cos^n x \, dx$$

Case 1 m is odd $\Rightarrow m = 2k+1$ we use $\sin^2 x = 1 - \cos^2 x$

$$\sin^m x = \sin^{2k+1} x = [\sin^2 x]^k \sin x = [1-\cos^2 x]^k \sin x$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

Cas2 m is even and n is odd $\Rightarrow n = 2k+1$ we use $\cos^2 x = 1 - \sin^2 x$

$$\cos^n x = \cos^{2k+1} x = [\cos^2 x]^k \cos x = [1-\sin^2 x]^k \cos x$$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

Case3 m and n are both even: we use

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Exp $\int \sin^3 x \cos^3 x \, dx = \int \sin^2 x \cos^3 x \sin x \, dx$ $\quad u = \cos x$, $du = -\sin x \, dx$

$= \int (1-\cos^2 x)\cos^3 x \sin x \, dx$

$= -\int (1-u^2) u^3 \, du = -\int (u^3 - u^5) \, du = \frac{u^6}{6} - \frac{u^4}{4} + c$

$= \frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + c$

Exp $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$ $\quad u = \sin x$, $du = \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx = \int u^2 (1-u^2) \, du$

$= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

---

3 **Exp** $\int 16\sin^2 x\cos^2 x\,dx = 16\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx$

$= 4\int(1-\cos^2 2x)\,dx = 4x - 4\int\cos^2 2x\,dx$

$= 4x - 4\int\left(\frac{1+\cos 4x}{2}\right)dx = 4x - \frac{4}{2}x - 2\int\cos 4x\,dx$

$= 4x - 2x - \frac{1}{2}\sin 4x + c = 2x - \frac{\sin 4x}{2} + c$

Product of sin and cos

$\int \sin mx\,\sin nx\,dx$

$\int \sin mx\,\cos nx\,dx$

$\int \cos mx\,\cos nx\,dx$

$\sin mx\,\sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$

$\sin mx\,\cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$

$\cos mx\,\cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$

Exp $\int \cos 3x\,\cos 4x\,dx = \int\frac{1}{2}\left[\cos(-x) + \cos 7x\right]dx$

$= \frac{1}{2}\int(\cos x + \cos 7x)\,dx = \frac{1}{2}\sin x + \frac{1}{14}\sin 7x + c$

---

* Eliminating Square Roots:

Exp $\int_0^{\pi}\sqrt{1-\sin^2 x}\,dx = \int_0^{\pi}\sqrt{\cos^2 x}\,dx = \int_0^{\pi}|\cos x|\,dx$

$= \int_0^{\frac{\pi}{2}}\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos x\,dx$

$= \sin x\Big|_0^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\pi}$

$= \sin\frac{\pi}{2} + \sin\frac{\pi}{2} = 1 + 1 = 2$

Power of tanx and secx

Exp* $\int 4\tan^3 x \, dx = 4\int \tan^2 x \tan x \, dx$

$= 4\int (\sec^2 x - 1)\tan x \, dx = 4\int \sec^2 x \tan x \, dx - 4\int \tan x \, dx$

$= 2\tan^2 x - 4\int \frac{\sin x}{\cos x} dx$

$= 2\tan^2 x + 4 \ln|\cos x| + c$

$= 2\tan^2 x - 2\ln|\sec^2 x| + c$

$= 2\tan^2 x - 2\ln(1+\tan^2 x) + c$

Exp $\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$

$= \int (1+\tan^2 x)\sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int (1+u^2) \, du$

$= u + \frac{u^3}{3} + c$

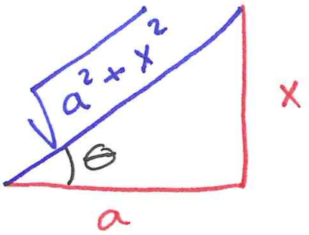
$= \tan x + \frac{1}{3}\tan^3 x + c$

$= \tan x + \frac{1}{3}[\sec^2 x - 1]\tan x + c$

$= \tan x - \frac{1}{3}\tan x + \frac{1}{3}\sec^2 x \tan x + c$

$= \frac{2}{3}\tan x + \frac{1}{3}\sec^2 x \tan x + c.$

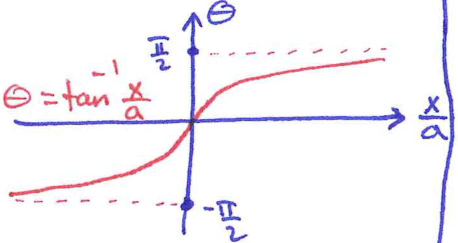
$\boxed{8.3}$ Trigonometric Substitution



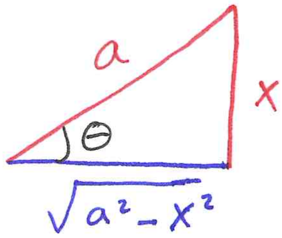
$$\boxed{x = a\tan\theta}$$ requires

$\theta = \tan^{-1}\frac{x}{a}$ with

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ♡[1]



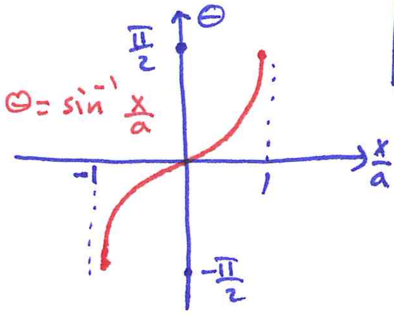
$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$$
$$= a\sqrt{1+\tan^2\theta}$$
$$= a\sqrt{\sec^2\theta}$$
$$= a\,|\sec\theta| \quad ♡[1]$$
$$= a\sec\theta$$



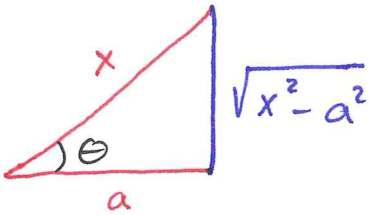
$$\boxed{x = a\sin\theta}$$ requires

$\theta = \sin^{-1}\frac{x}{a}$ with

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ ♡[2]



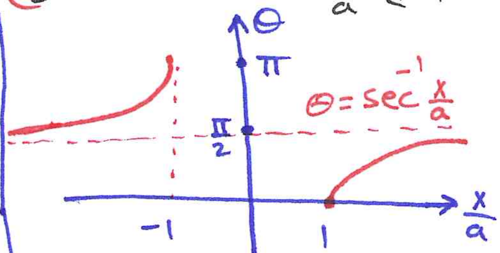
- $$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$$
$$= a\sqrt{1-\sin^2\theta}$$
$$= a\sqrt{\cos^2\theta}$$
$$= a\,|\cos\theta| \quad ♡[2]$$
$$= a\cos\theta$$



$$\boxed{x = a\sec\theta}$$ requires

$\theta = \sec^{-1}\frac{x}{a}$ with

$$\begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1 \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1 \end{cases}$$ ♡[3]

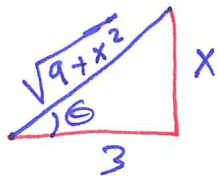


- $$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$$
$$= a\sqrt{\sec^2\theta - 1}$$
$$= a\sqrt{\tan^2\theta}$$
$$= a\,|\tan\theta| \quad ♡[3]$$
$$= a\tan\theta$$

Exp $\displaystyle\int \frac{dx}{\sqrt{x^2+9}}$ $\qquad x = 3\tan\theta \Rightarrow dx = 3\sec^2\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{|\sec\theta|}\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$



$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C = \ln\left|\sqrt{9+x^2} + x\right| + C.$$

$$\underline{Exp} \quad \int \sqrt{25-t^2}\,dt \qquad t = 5\sin\theta \Rightarrow dt = 5\cos\theta\,d\theta, \; -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

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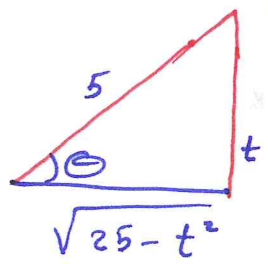
$$\int \sqrt{25-t^2}\,dt = \int \sqrt{25-25\sin^2\theta}\;5\cos\theta\,d\theta$$

$$= 25\int \sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta = 25\int \sqrt{\cos^2\theta}\;\cos\theta\,d\theta$$

$$= 25\int \cos^2\theta\,d\theta = \frac{25}{2}\int (1+\cos 2\theta)\,d\theta = \frac{25}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{25}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$



$$= \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \frac{t}{5}\;\frac{\sqrt{25-t^2}}{5}\right] + C$$

$$= \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2}\Big] + C$$

$$\underline{Exp} \quad \int \frac{\sqrt{y^2-25}}{y^3}\,dy \qquad y = 5\sec\theta \Rightarrow dy = 5\sec\theta\tan\theta\,d\theta$$

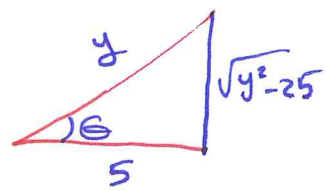
$$\boxed{0 < \theta < \frac{\pi}{2}}$$

$$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{\sqrt{25\sec^2\theta - 25}}{125\sec^3\theta}\;5\sec\theta\tan\theta\,d\theta$$

$$= \frac{1}{5}\int \frac{\sqrt{\sec^2\theta-1}}{\sec^2\theta}\tan\theta\,d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta$$

$$= \frac{1}{10}\int (1-\cos 2\theta)\,d\theta = \frac{1}{10}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{10}\left[\theta - \sin\theta\cos\theta\right] + C$$



$$= \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \frac{\sqrt{y^2-25}}{y}\left(\frac{5}{y}\right)\right] + C$$

Exp $\int_{-2}^{2} \frac{dx}{4+x^2}$ $\quad x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta \, d\theta$ (38)

$$\boxed{-\frac{\pi}{4} < \theta < \frac{\pi}{4}}$$

when $x = -2 \Rightarrow \theta = \tan^{-1}(-1) = -\frac{\pi}{4}$

$x = 2 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$\int_{-2}^{2} \frac{dx}{4+x^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2\theta \, d\theta}{4 + 4\tan^2\theta} = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta}{1+\tan^2\theta} \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{1}{2}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \frac{\pi}{4}$$

Note that $\int_{-2}^{2} \frac{dx}{4+x^2} = \frac{1}{2} \tan^{-1}\frac{x}{2} \Big|_{-2}^{2}$

$$= \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} -1\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - -\frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

8.4 Integration of Rational Functions By Partial Fractions

Ex ①
$$\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x+6)(x-1)}\,dx$$

Heaviside "cover up" method

$$\frac{x+4}{\underbrace{(x+6)}_{A}\underbrace{(x-1)}_{B}} = \frac{A}{x+6} + \frac{B}{x-1}$$

$$x+4 = A(x-1) + B(x+6)$$
$$= (A+B)x + 6B - A$$

$A+B=1$, $-A+6B=4$ $\Rightarrow$ $A=\frac{2}{7}$, $B=\frac{5}{7}$

$$\int \frac{x+4}{(x+6)(x-1)}\,dx = \int \frac{\frac{2}{7}}{x+6}\,dx + \int \frac{\frac{5}{7}}{x-1}\,dx$$
$$= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C$$
$$= \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$$

---

* The partial fraction Method: is a method for writing $\frac{f(x)}{g(x)}$ "rational functions" as a sum of simpler fractions.

* The Heaviside "cover up" method can be used when $g(x)$ can be written as product of distinct linear factors.
  (distinct: non repeated)

* The degree of f must be less than the degree of g.
  If not we use long dividion:

Exp $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

cover up method

$$A = \frac{1+4+1}{(2)(4)} = \frac{6}{(2)(4)} = \frac{3}{4}$$

$$B = \frac{1-4+1}{(-2)(2)} = \frac{-2}{(-2)(2)} = \frac{1}{2}$$

$$C = \frac{9-12+1}{(-4)(-2)} = \frac{-2}{8} = \frac{-1}{4}$$

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int\left(\frac{\frac{3}{4}}{x-1} + \frac{\frac{1}{2}}{x+1} - \frac{\frac{1}{4}}{x+3}\right)dx$$

$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + C$$

Exp $\displaystyle\int \frac{dx}{x^3+x^2-2x} = \int \frac{dx}{x(x^2+x-2)} = \int \frac{dx}{x(x+2)(x-1)}$

"cover up"

$$\frac{1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$A = \frac{-1}{2}$

$B = \frac{1}{6}$

$C = \frac{1}{3}$

$$\int \frac{dx}{x(x+2)(x-1)} = \int\left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{6}}{x+2} + \frac{\frac{1}{3}}{x-1}\right)dx$$

$$= -\frac{1}{2}\ln|x| + \frac{1}{6}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

Exp $\displaystyle\int \frac{x^3}{x^2+2x+1}\,dx = \int\left(x-2+\frac{3x+2}{x^2+2x+1}\right)dx$

$$\begin{array}{r} x-2 \\ x^2+2x+1 \overline{)\, x^3 \qquad\qquad} \\ \underline{x^3+2x^2+x} \\ -2x^2-x \\ \underline{-2x^2-4x-2} \\ 3x+2 \end{array}$$

$$= \int (x-2)\,dx + \int \frac{3x+2}{x^2+2x+1}\,dx$$

$$= \frac{x^2}{2} - 2x + \int \frac{3x+2}{(x+1)^2}$$

Repeated linear factor

$$\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$3x+2 = A(x+1) + B$$
$$= Ax + A + B$$

$\boxed{A=3}$ $\boxed{B=-1}$

$$= \frac{x^2}{2} - 2x + \int\left(\frac{3}{x+1} - \frac{1}{(x+1)^2}\right)dx$$

$$= \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C$$

Exp (irreducible Quadratic Factors + repeated linear Factor) $\displaystyle\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx$

$$\frac{4-2x}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{Ax+B}{x^2+1} + \frac{C(x-1)+D}{(x-1)^2}$$

$$4-2x = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$
$$= (Ax+B)(x^2-2x+1) + (x-1)(x^2+1) + D(x^2+1)$$
$$4-2x = A(x^3-2x^2+x) + B(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$A+C=0$, $-2A+B-C+D=0$, $-2=A-2B+C$, $4=B-C+D$

$\boxed{A=2}$, $\boxed{B=1}$, $\boxed{C=-2}$, $\boxed{D=1}$

$$\int \frac{4-2x}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1} - 2\ln|x-1| - \frac{1}{x-1} + C$$

$\ln(x^2+1) \leftarrow$; $\tan^{-1}x$

Exp $\displaystyle\int \frac{dx}{x(x^2+1)^2}$ 　　　　　　　　　　　　　　　　　　42*

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A}{x} + \frac{(Bx+C)(x^2+1) + (Dx+E)}{(x^2+1)^2}$$

$$1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x$$

$$* \quad 1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$

$A+B=0$, $\boxed{C=0}$, $2A+B+D=0$, $C+E=0$, $\boxed{A=1}$

$\boxed{B=-1}$

$\boxed{D=-1}$

$\boxed{E=0}$

$$\int \frac{dx}{x(x^2+1)^2} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} \right) dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\left(\frac{1}{x^2+1}\right) + C$$

$$= \ln \frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C$$

---

* can be differentiated to find the coefficients

- $A=1$ by cover up method
- $0 = A(4x^3+4x) + B(4x^3+2x) + C(3x^2+1) + 2Dx + E$ 　　$\boxed{E+C=0}$

$0 = A(12x^2+4) + B(12x^2+2) + C(6x) + 2D$ 　　$4A+2B+2D=0$, $\boxed{2A+B+D=0}$

$0 = A(24x) + B(24x) + 6C$ 　　$\boxed{C=0}$

$A+B=0$