1.2 Elementary Functions

Definition:
A function f is a rule that assigns each element x in the set A exactly one element y in the set B.
The element y is called the image of x under f and is denoted by $f(x)$.

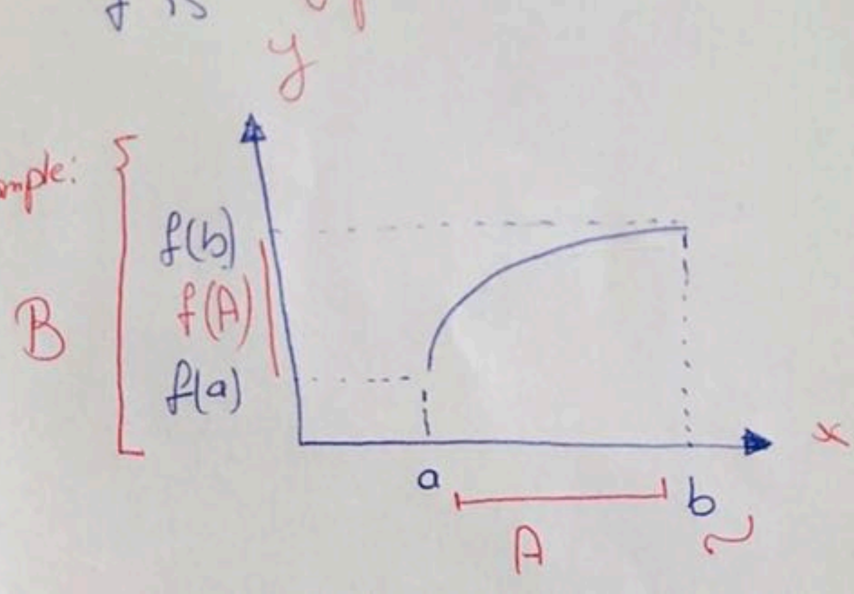
The set A is called the Domain of f.
The set B is called the Codomain of f.
The set $f(A) = \{y \in B : y = f(x) \text{ for some } x \in A\}$ is called the range of f.

To define a function we use the notation

$$f : A \longrightarrow B$$
$$x \longrightarrow f(x)$$

x is independent variable. متغير مستقل
y is dependent variable متغير تابع

Example:



Example: Find Domain and range of $f(x)=x^2$

$f(x)=x^2$

Domain $= \mathbb{R} = (-\infty,\infty)$

Range $= [0,\infty)$

* Two functions f and g are $\boxed{\text{equal}}$ iff

1. f and g are defined on the same domain
2. $f(x) = g(x)$ $\forall x \in$ Domain

$\in$ Belongs to

Example:

Let $f_1: [0,1] \longrightarrow R$, $x \longrightarrow x^2$ $\quad$ $f_2: [0,1] \longrightarrow R$, $x \longrightarrow \sqrt{x^4}$ $\quad$ $f_3: R \longrightarrow R$, $x \longrightarrow x^2$

which of these functions are equal?

f_1 and f_2

since

1. f_1 and f_2 are defined on the same domain
2. $f_1(x) = f_2(x) = x^2$ $\forall x \in [0,1]$

$(\sqrt{x})^2 =$

$\sqrt{x^2} =$

Even and Odd function

Def:

1. $f: A \longrightarrow B$ is called even if

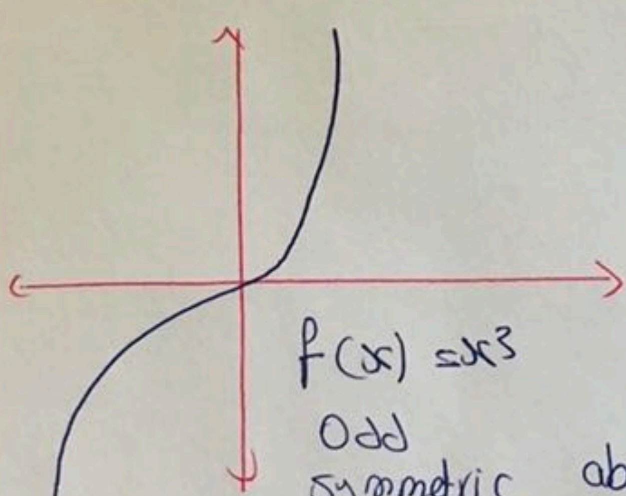
$$f(x) = f(-x)$$

"graphically: f is symmetric about the y-axis"

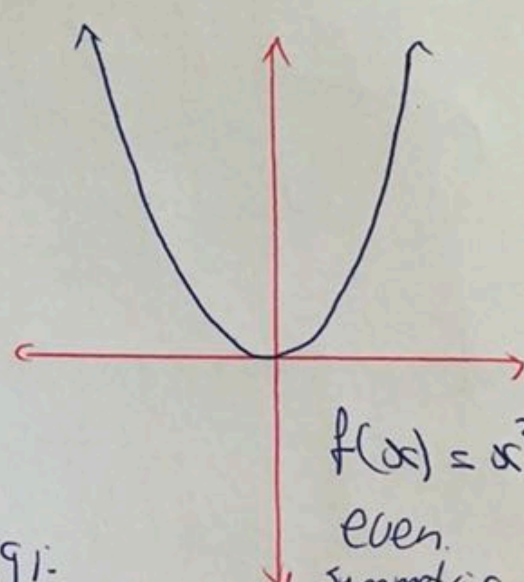
2. $f: A \longrightarrow B$ is called odd if

$$f(-x) = -f(x)$$

"graphically: f is symmetric about the origin"



$f(x) = x^3$
Odd
symmetric about origin.



$f(x) = x^2$
even.
symmetric about y-axis

$\sin x$	odd	$(\sin -x = -\sin x)$
$\cos x$	even	$(\cos -x = \cos x)$
$\tan x$	odd	$(\tan -x = -\tan x)$

Example: Is $y = \frac{x}{x^2+1}$ odd?

yes

$$f(x) = \frac{x}{x^2+1}$$

$$f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -\left(\frac{x}{x^2+1}\right) = -f(x)$$

so f is odd

Example: Is $y = \frac{x^3+1}{x^2+1}$ odd?

$$f(x) = \frac{x^3+1}{x^2+1} \quad \& \quad -f(x) = \frac{-(x^3+1)}{x^2+1}$$

$$f(-x) = \frac{(-x)^3+1}{(-x)^2+1} = \frac{-x^3+1}{x^2+1} \neq -f(x)$$

so $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

so $f(x)$ is neither even nor odd

Definition:

The composite function $f \circ g$ (also called the composition of f and g) is defined as

$$(f \circ g)(x) = f[g(x)]$$

for each x in the domain of g for which $g(x)$ is in the domain of f

$$x \xrightarrow{g} g(x) \xrightarrow{f} f[g(x)]$$

Example:

If $f(x) = \sqrt{x}$, $x \geq 0$ and $g(x) = x^2+1$, $x \in R$

Find

(a) $(f \circ g)(x) = f(g(x)) = f(g) = \sqrt{g} = \sqrt{x^2+1}$

(b) $(g \circ f)(x) = g(f(x)) = g(f) = g^2+1 = (\sqrt{x})^2+1$

$= x+1$

Note that $\sqrt{x^2} = |x|$

$(\sqrt{x})^2 = x$

Example:

If $f(x) = 2x^2$, $x \geq 2$ and $g(x) = \sqrt{x}$, $x \geq 0$

Find $(f \circ g)(x)$

Solution:

$$(f \circ g)(x) = f(g(x))$$
$$= f(g)$$
$$= 2g^2 = 2(\sqrt{x})^2 = 2x$$

1.2.2 Polynomial Functions

Def: A polynomial function is a function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where n is nonnegative integer and $a_0, a_1, a_2, \dots, a_n$ are real valued constant, with $a \neq 0$

$a_n \equiv$ leading coefficient

معامل أكبر قوة

$n =$ Degree. أعلى درجة للمتغير

Ex $f(x) = 2(x+1)^2 + 3$

$$f(x) = 2[x^2 + 1 + 2x] + 3$$
$$= 2x^2 + 2 + 4x + 3$$
$$f(x) = 2x^2 + 4x + 5$$

Deg $= 2$

L.C $= 2$

Ex $f(x) = x^2 + x^{-3} + 4$

Not a polynomial

Constant function $f(x) = c$ Degree $= 0$

Linear function $f(x) = mx+b$ Degree $= 1$

Quadratic $f(x) = ax^2+bx+c$ Degree $= 2$

Example: $y = x^n$, n odd odd function	Ex $f(x) = 5x^2 + x + 4$ Deg $= 2$ L.C $= 2$ C.T $= 4$
Examp $y = x^n$, n even even functi	Ex $f(x) = 5^3 * x^2 + x$ Deg $= 2$ L.c $= 1$ C.T $= 5^3$

1.2.3 Rational Functions

Ex

Def: A rational function is the quotient of two polynomi
functions $P(x)$ and $q(x)$

$$f(x) = \frac{P(x)}{q(x)} \quad \text{for } q(x) \neq 0$$

Ex $y = \frac{1}{x}$, $x \neq 0$

Ex $y = \frac{x^2+2x-1}{x-3}$, $x \neq 3$

1.2.4 Power Functions

Definition: A power function is of the form

$$f(x) = x^r$$

where r is a real number

Example:

$y = x^{\frac{1}{3}}$, $x \in \mathbb{R}$

$y = x^{\frac{5}{2}}$, $x \geq 0$

$y = x^{-\frac{1}{2}}$, $x > 0$

1.2.5 Exponential Functions

Definition: The function f is an exponential function with base a if

$$f(x) = a^x$$

where a is a positive constant other than 1.

The largest possible domain of f is $\mathbb{R}$

Recall

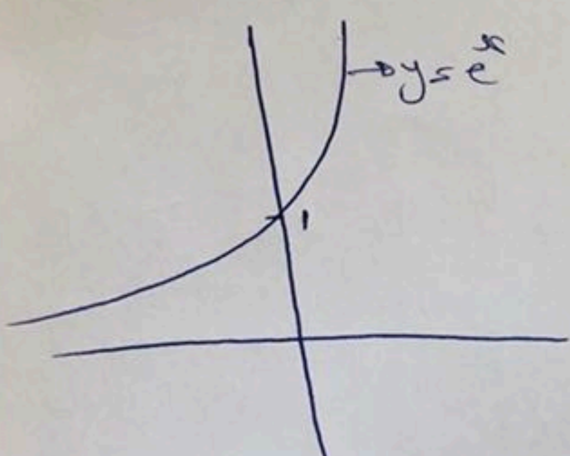
$$a^r \cdot a^s = a^{r+s}$$

$$\frac{a^r}{a^s} = a^{r-s}$$

$$a^{-r} = \frac{1}{a^r}$$

$$(a^r)^s = a^{rs}.$$

Example: $y = e^x$



1.2.6 Inverse Functions:

$$\begin{tikzcd} x \arrow[rr, bend left, "f(x)"] & & y \arrow[ll, bend left, "f^{-1}(x)"] \end{tikzcd}$$

One to One function

$f(x)$ is 1-1 iff $\forall\ x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

Example:

$f(x) = x^2$

is <u>not</u> one to one

$f(2) = f(-2)$ and $2 \neq -2$

Example:

$f(x) = x^3$

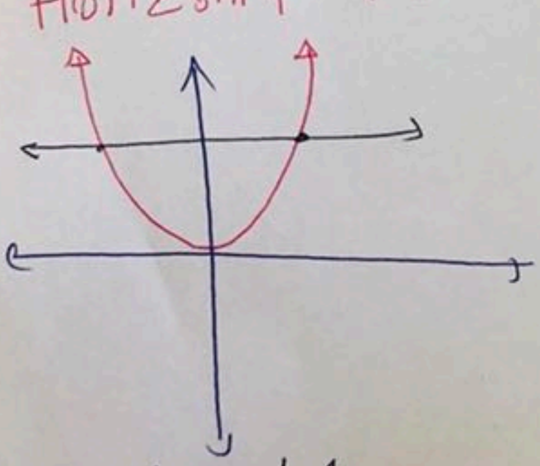
is one to one

since $\forall\ x_1 \neq x_2$

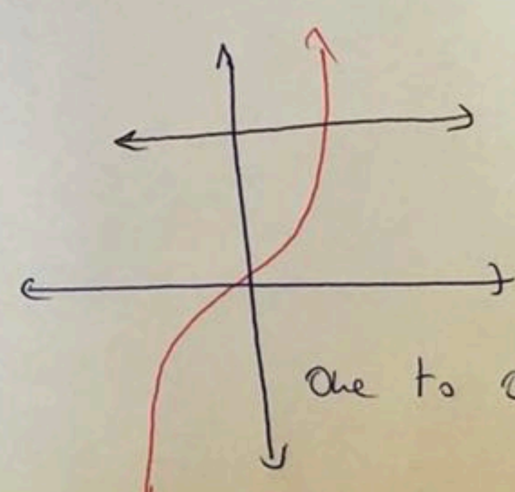
$$f(x_1) = x_1^3 \neq x_2^3 = f(x_2)$$

$f(x) = x^3$ is 1-1

Horizontal line test



Not 1-1



One to one

$f(x) = x^2$ is not one to one.
since for example $4 = f(2) = f(-2)$

$f(x) = x^3$ is one to one.
you can't find two different numbers with same image

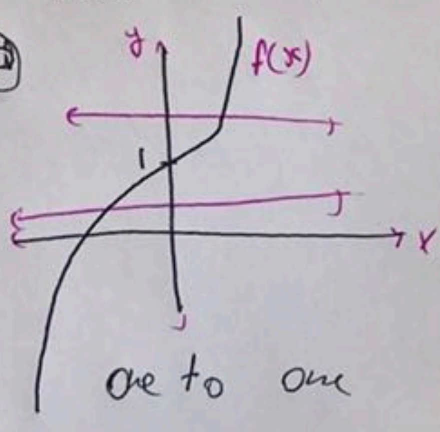
Definition:
Let $f: A \to B$ be one to one function. with range $f(A)$.
The inverse function, f^{-1} has domain $f(A)$ and range A
and is defined by. $f^{-1}(y) = x$ iff $y = f(x)$ $\forall y \in f(A)$

Example: Find the inverse function of $f(x) = x^3 + 1$, $x \geq 0$

Solution:

1. Is $f(x) = x^3 + 1, x \geq 0$ one to one?

(a)



one to one

Or
$f(x_1) = f(x_2)$
$x_1^3 + 1 = x_2^3 + 1$
$x_1^3 = x_2^3$
$x_1 = x_2$
so f is one to one

2. $y = x^3 + 1$
3. $x^3 = y - 1 \Rightarrow x = \sqrt[3]{y-1}$
4. $y = f^{-1}(x) = \sqrt[3]{x-1}$

Note: Graph of f and graph of f^{-1} are symmetric about the line $y = x$

Note that : f^{-1} doesn't mean the reciprocal of f i.e $\frac{1}{f}$

$f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(x) = x$

1.2.7 Logarithmic Functions:

① Definition: The inverse function of $f(x) = a^x$ is called the logarithm to base a and is written $f^{-1}(x) = \log_a x$.

* e^x is the inverse of $\ln x$

Note that $\ln e^x = x$ and $e^{\ln x} = x$

* 10^x is the inverse of $\log_{10} x = \log x$

$10^{\log x} = x$ and $\log 10^x = x$

In general a^x is the inverse of $\log_a x$

[1] $a^{\log_a x} = x$ for $x > 0$

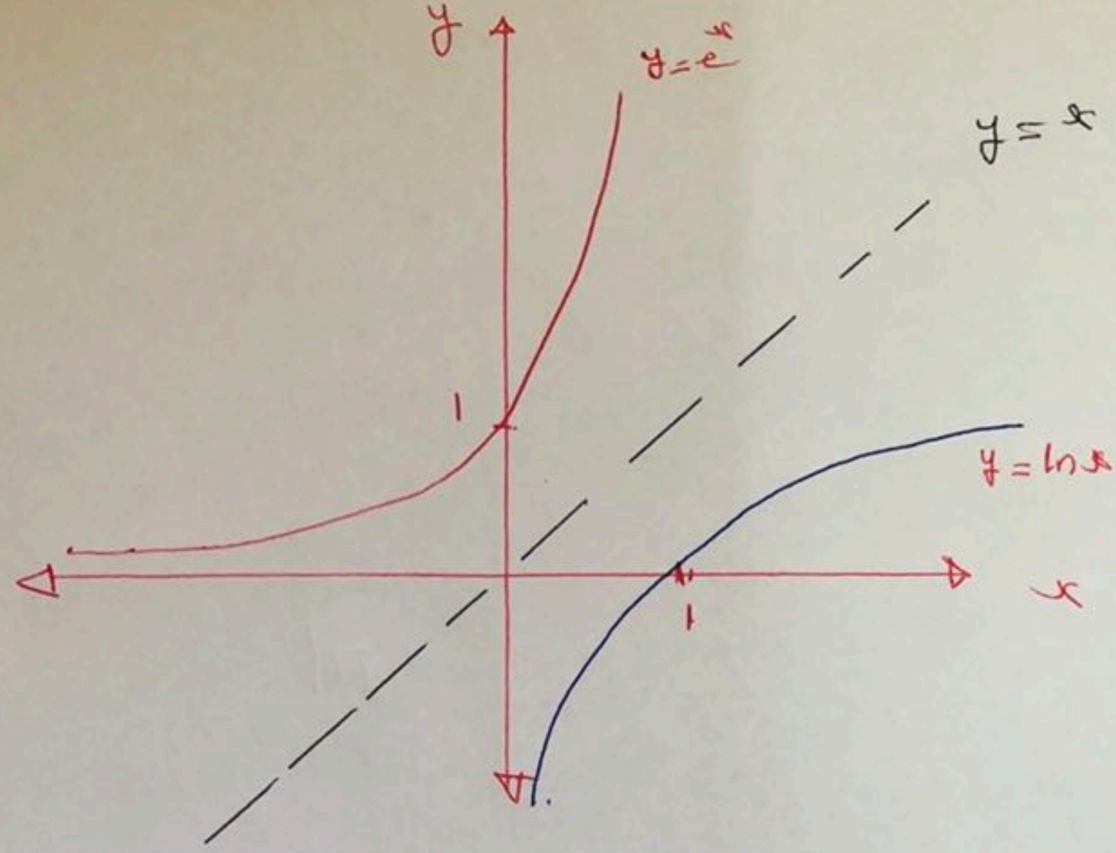
[2] $\log_a a^x = x$ for $x \in R$

also, recall the following properties

[1] $\log_a st = \log_a s + \log_a t$

[2] $\log_a \left(\frac{s}{t}\right) = \log_a s - \log_a t$

[3] $\log_a s^r = r \log_a s$



Example: Simplify the following expressions

(1) $\log_2 [8(x-2)]$

$= \log_2 8 + \log_2 x-2$

$= \log_2 2^3 + \log_2 x-2$

$= 3 + \log_2 x-2$

$\log_2 2^3 = 3$ Directly

or

$\log_2 2^3 = 3\log_2 2 = 3 \cdot 1$
$= 3$

Remember that

$\log_a a = 1$

(2) $\log_3 9^x$

$= \log_3 (3^2)^x = \log_3 3^{2x} = 2x$

(c) $\ln e^{3x^2+1} = 3x^2+1$

Identities:

1. $\boxed{a^x = e^{x\ln a}}$

$e^{x\ln a} = e^{\ln a^x} = a^x$

2. $\boxed{\log_a x = \dfrac{\ln x}{\ln a}}$

Example:

Write the following expressions in terms of base e.

(a) 2^x

$2^x = e^{\ln 2^x} = e^{x\ln 2}$

(b) 10^{x^2+1}

$10^{x^2+1} = e^{\ln(10^{x^2+1})} = e^{(x^2+1)\ln 10}$.

(c) $\log_3 x$

$\log_3 x = \dfrac{\ln x}{\ln 3}$

(d) $\log_2 3x-1$

$\log_2 3x-1 = \dfrac{\ln 3x-1}{\ln 2}$

12.8 Trigonometric Functions: الاقترانات مثلثية

Trigonometric functions are examples of periodic function

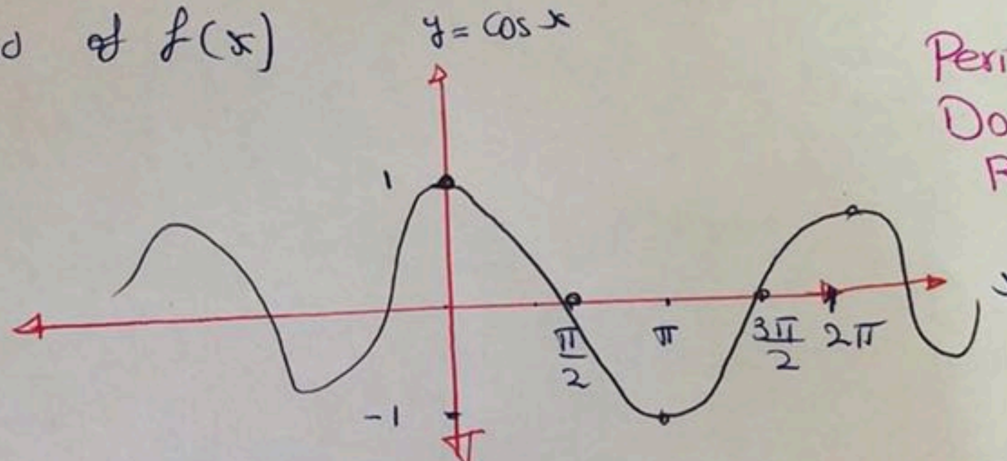
Definition:

A function $f(x)$ is periodic if there is a positive constant a such that

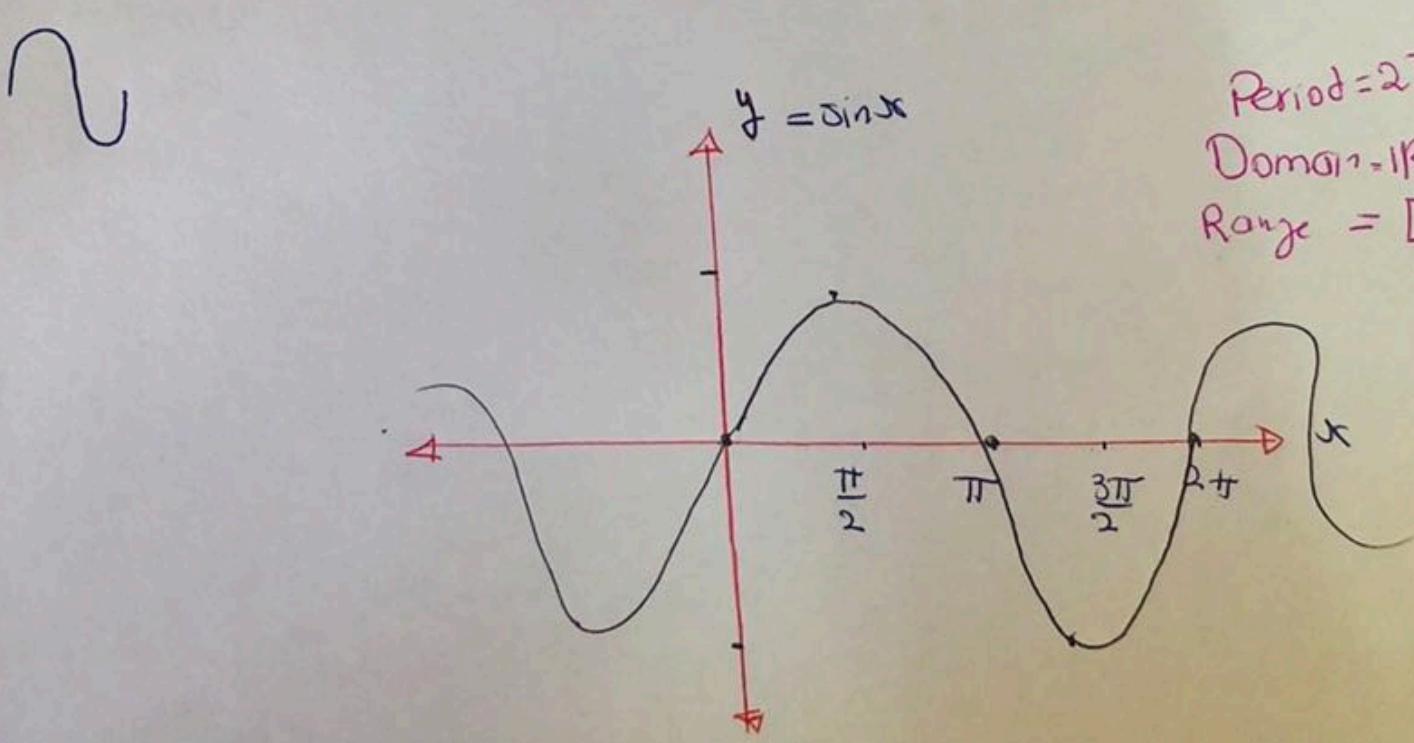
$$f(x+a)=f(x)$$

for all x in the domain

If a is the smallest number with this property, we call it the period of $f(x)$



Period $= 2\pi$
Domain $= \mathbb{R}$
Range $= [-1, 1]$



Period $= 2\pi$
Domain $= \mathbb{R}$
Range $= [-1, 1]$

$y = \tan x$

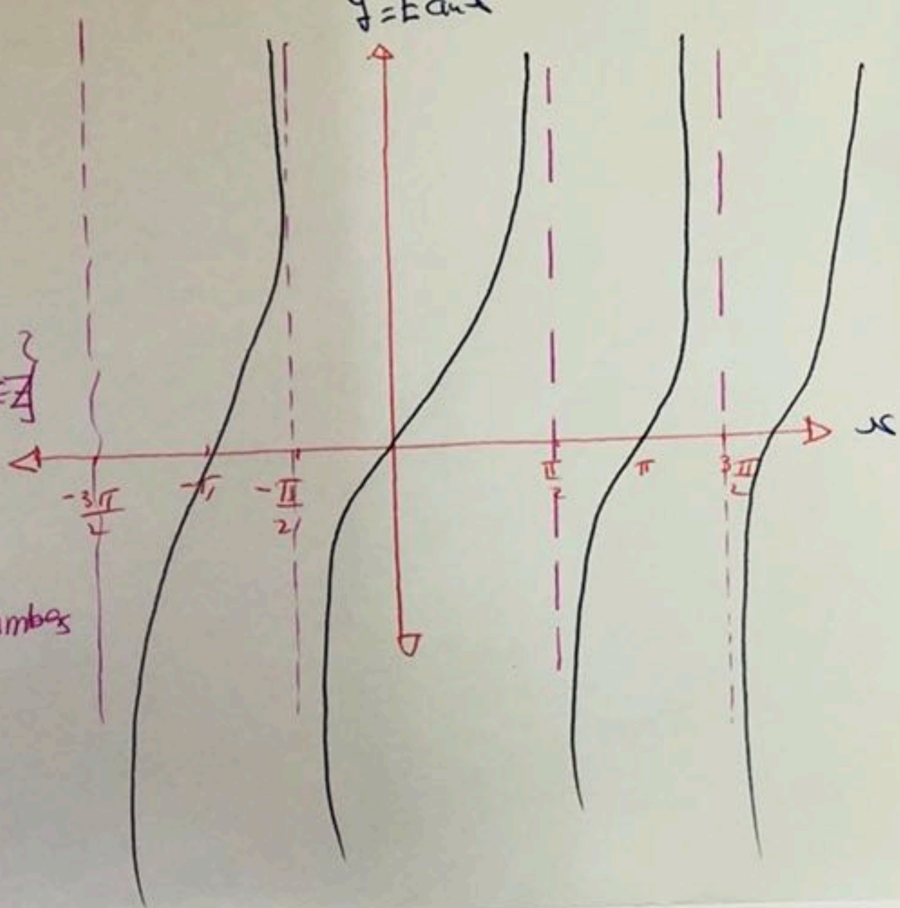
P

period $= \pi$

Domain $= \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\right\}$

Range $= (-\infty, \infty)$

Domain contains all real numbers except the odd integer multiples of $\frac{\pi}{2}$



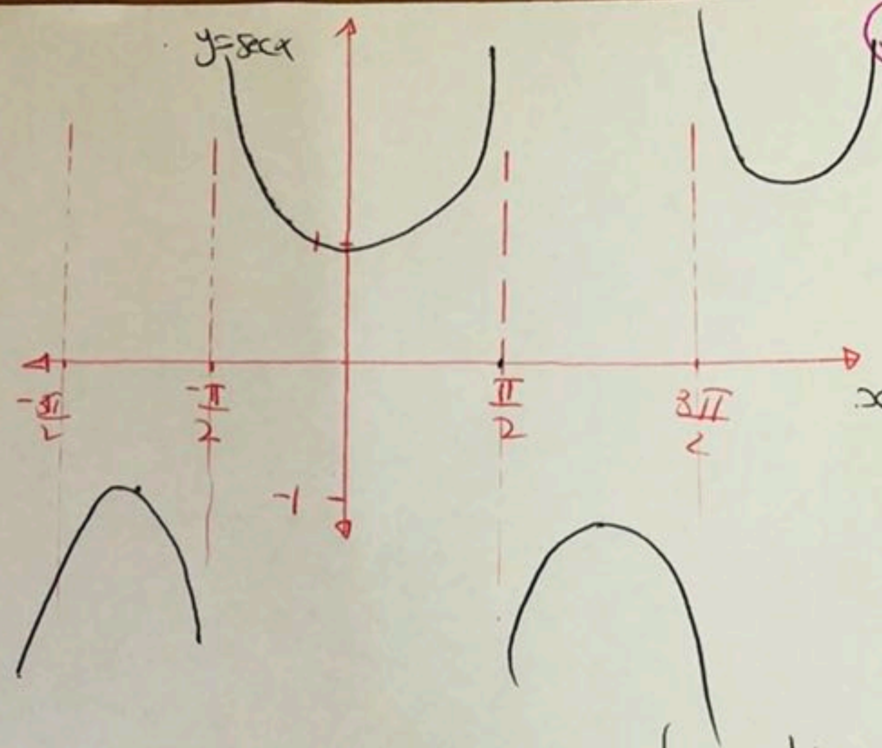
$y = \sec x$

period 2π

Domain $= \mathbb{R}$ except odd multiples of $\frac{\pi}{2}$

$= \mathbb{R} - \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$

Range $= (-\infty, -1] \cup [1, \infty)$

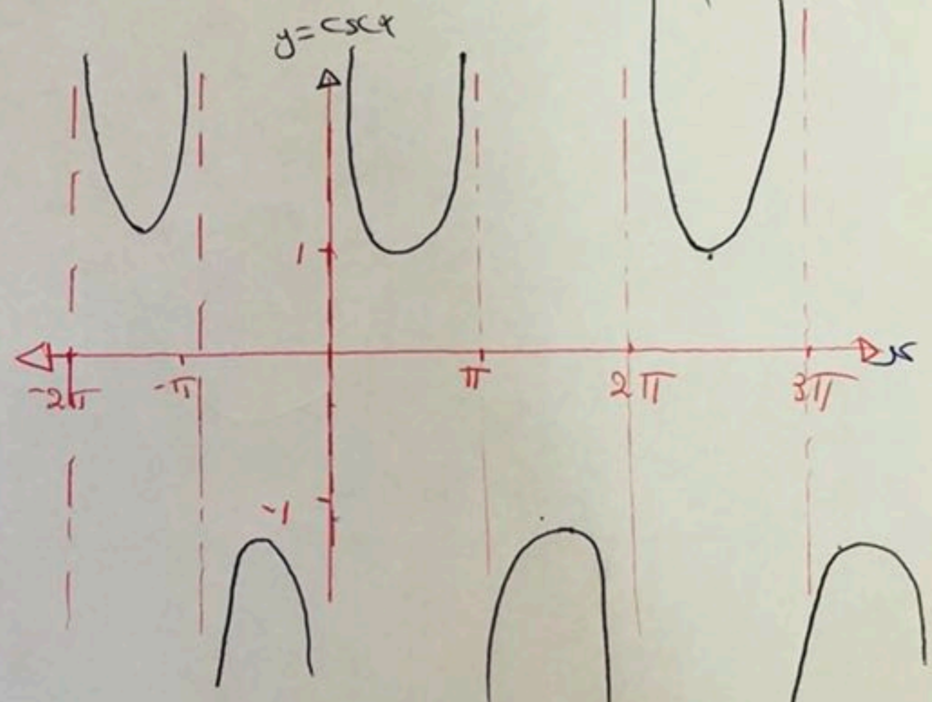


$y = \csc x$

Period $= 2\pi$

Domain $= \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$

Range $= (-\infty, -1] \cup [1, \infty)$

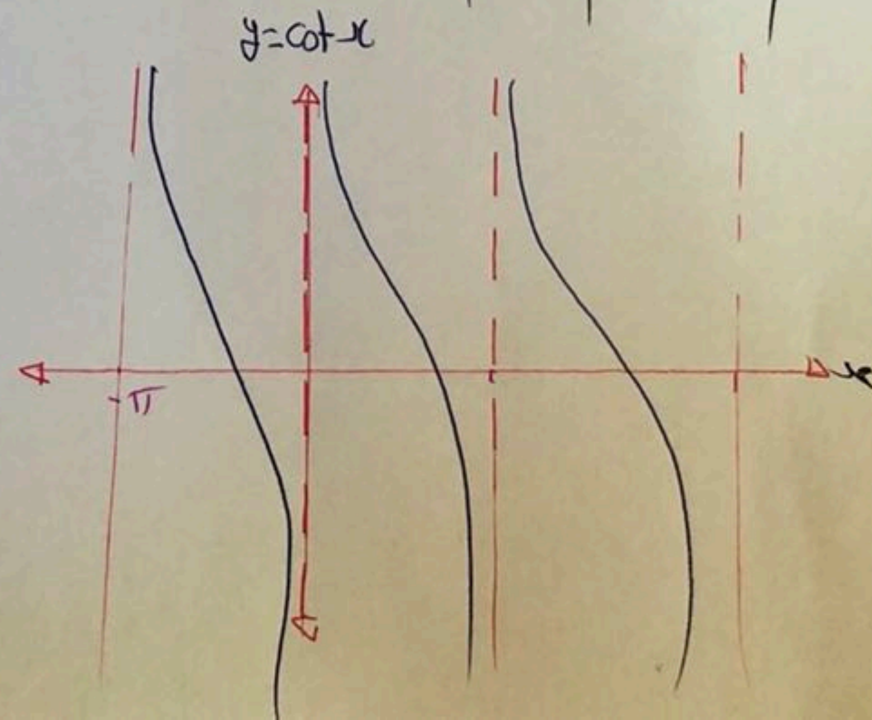


$y = \cot x$

Period $= \pi$

Domain $= \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$

Range $= (-\infty, \infty)$



How to compute period

$\boxed{y = \sin ax}$ or $\boxed{y = \cos ax}$, $\boxed{y = \csc ax}$ or. $\boxed{y = \sec ax}$

$$\text{period} = \frac{2\pi}{\text{coefficient of } x} = \frac{2\pi}{a}.$$

$\boxed{y = \tan ax}$ or $\boxed{y = \cot ax}$

$$\text{Period} = \frac{\pi}{\text{coefficient of } x} = \frac{\pi}{a}.$$

Example: Period of $\sin 4x$ is $\frac{2\pi}{4} = \frac{\pi}{2}$

period of $\tan \frac{x}{2}$ is $\frac{\pi}{\frac{1}{2}} = \pi * \frac{2}{1} = 2\pi$