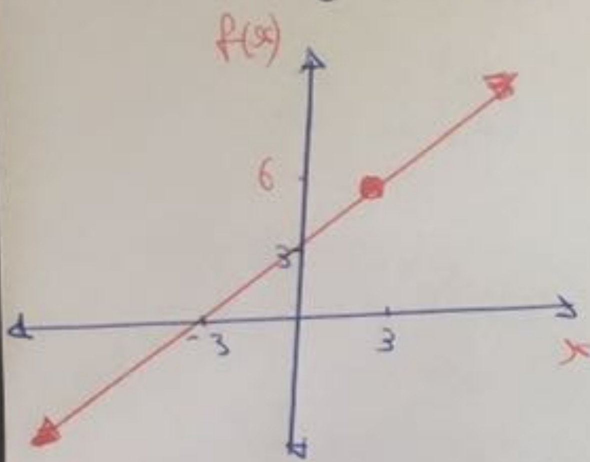
3.2 Continuity

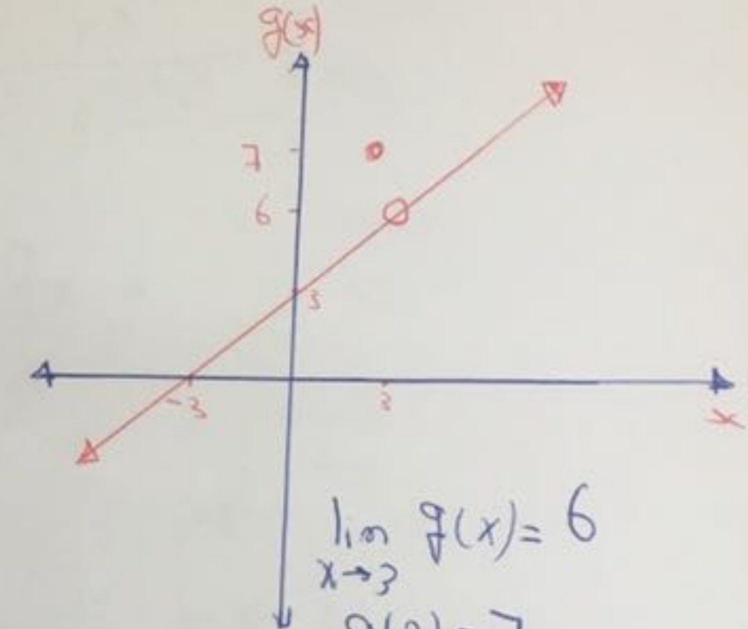
3.2.1 What is continuity?

Consider the two functions

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$$

$$g(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ 7, & x = 3 \end{cases}$$





$\lim_{x \to 3} f(x) = 6$

$f(3) = 6$

Continuous

$\lim_{x \to 3} g(x) = 6$

$g(3) = 7$

discontinuous

Definition:
A function f is said to be Continuous at $x = c$
if $\lim_{x \to c} f(x) = f(c)$

To check whether a function is continuous at $x=c$ ②
we need to check the following three conditions:

1. $\lim_{x\to c} f(x)$ exists

2. $f(x)$ is defined at $x=c$

3. $\lim_{x\to c} f(x) = f(c)$

Ex) show that $f(x) = 2x-3$, $x \in \mathbb{R}$ is continuous at $x=1$

$f(1) = 2(1) - 3 = -1$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} 2x-3 = -1$

$\lim_{x\to 1} f(x) = f(1) = -1$

so $f(x)$ is continuous at $x=c$

Ex) Let $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x-3}, & x \neq 3 \\ a, & x = 3 \end{cases}$

be continuous. Find a

$\lim_{x\to 3} f(x) = f(3)$

$\lim_{x\to 3} \dfrac{x^2 - x - 6}{x-3} = a$

$\lim_{x\to 3} \dfrac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} x+2 = \boxed{5 = a}$

Definition:

A function f is said to be continuous from the right at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$

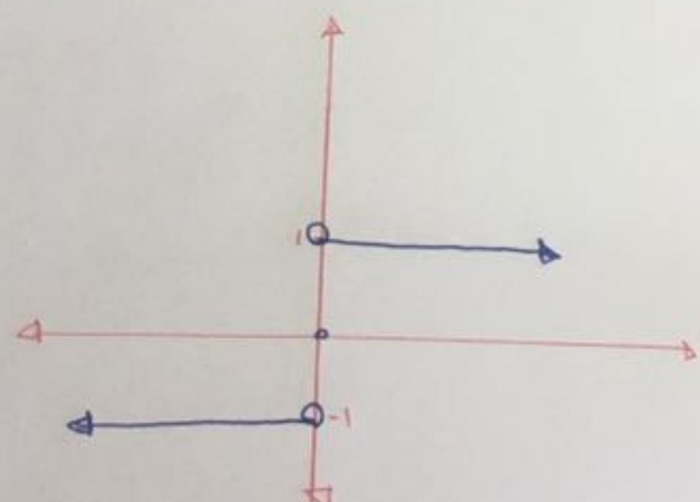
and continuous from Left at $x = c$ if

$$\lim_{x \to c^-} f(x) = f(c)$$

Ex Show that $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is discontinuous at $x = 0$, and that discontinuity can't be removed

$$f(x) = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{-x}{x}, & x < 0 \\ 0, & x = 0 \end{cases}$$



$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1$

So $\lim_{x \to 0} f(x)$ DNE (Jump)

Example At which point is the function

$$f(x) = \frac{1}{(x-4)^2}$$

discontinuous? Can the discontinuity be removed?

at $x=4$ the function $f(x)$ is discontinuous

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{1}{(x-4)^2} = \infty \text{ (limit DNE)}$$

3.2.2 Combination of Continuous Functions

Suppose that a is a constant and the functions f and g are continuous at $X = C$.

Then the following functions are continuous at $X = C$

1. $a.f$
2. $f+g$
3. $f \circ g$
4. $\frac{f}{g}$ provided that $g(c) \neq 0$

The following functions are continuous wherever they are defined

(5)

1. Polynomial functions
2. rational functions
3. Power functions
4. Trigonometric functions
5. exponential functions of the form a^x, $a > 0$, $a \neq 1$
6. logarithmic functions of the form $\log_a x$, $a > 0$, $a \neq 1$

Ex For which ~~of~~ values of $x \in R$ are the following functions continuous

(a) $f(x) = 2x^3 - 3x + 1$

Cont. $\forall x \in \mathbb{R}$

(b) $f(x) = \dfrac{x^2 + x + 1}{x - 2}$

Cont. $\forall x \neq 2$

(c) $f(x) = x^{\frac{1}{4}}$

Cont. $\forall x \geq 0$

(d) $f(x) = 3\sin x$

Cont. $\forall x$

(e) $f(x) = \tan x$

Cont. $\forall x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

(f) $f(x) = 3^x$

Cont. $\forall x \in R$

(g) $2\ln(x+1)$

Cont. $\forall$ $x + 1 > 0$

$x > -1$

Theorem:

If $g(x)$ is continuous at $x=c$ with $g(c)=L$ and $f(x)$ is continuous at $x=L$, then $(f\circ g)(x)$ is continuous at $x=c$. In particular

$$\lim (f\circ g)(x) = \lim_{x\to c} f[g(x)] = f[\lim g(x)] =$$

$$f[g(c)] = f(L)$$

$\boxed{Ex}$ Determine where the following functions are continuous

(a) $h(x) = e^{-x^2}$

$h(x) = \frac{1}{e^{x^2}}$

cont. $\forall x \in R$

(b) $h(x) = \sin\frac{\pi}{x}$

$\cancel{\frac{\pi}{x}}$ $x \neq 0$

Discontinuous at $x=0$

(c) $h(x) = \frac{1}{1+2x^{\frac{1}{3}}}$

$1+2x^{\frac{1}{3}} = 0$

$x^{\frac{1}{3}} = \frac{-1}{2} \to \boxed{x = \frac{-1}{8}}$ Discont. at $x = \frac{-1}{8}$

Ex $\lim_{x\to 3} \sin\left(\pi \frac{x^2-1}{4}\right)$

$= \lim \sin\left(\pi \cdot \frac{8}{4}\right) = \sin 2\pi = 0$

Ex Find $\lim_{x\to 1} \sqrt{2x^3-1}$

$\sqrt{\lim_{x\to 1} 2x^3-1} = \sqrt{2-1} = \sqrt{1} = 1$

Ex Find $\lim_{x\to 0} e^{x-1}$

$= \lim_{x\to 0} e^{x-1} = e^{-1} = \frac{1}{e}.$

Ex Find $\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} \quad \left(\frac{0}{0}\right)$

$\lim_{x\to 0} \frac{\left(\sqrt{x^2+16}-4\right)\left(\sqrt{x^2+16}+4\right)}{x^2 \cdot \left(\sqrt{x^2+16}+4\right)}$

$= \lim_{x\to 0} \frac{x^2+16-16}{x^2\left(\sqrt{x^2+16}+4\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+16}+4} = \frac{1}{8}.$