3.4 The Sandwich Theorem and Some Trigonometric Limits

$\boxed{\text{Sandwich Theorem}}$

If $f(x) \le g(x) \le h(x)$ for all x in an open interval that contains c (except possibly at c) and

$$\boxed{\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L}$$

then $\boxed{\lim_{x \to c} g(x) = L}$

Example: Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

solution:

$$-1 \le \sin \frac{1}{x} \le 1$$

multiply by x^2

$$x^2 \cdot -1 \le x^2 \cdot \sin \frac{1}{x} \le x^2 \cdot 1$$

$$-x^2 \le x^2 \sin \frac{1}{x} \le x^2$$

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$$

So by Sandwich Th. $\boxed{\lim_{x \to 0} x^2 \cdot \sin \frac{1}{x} = 0}$

Example: $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

solution:

$$-1 \leq \sin \frac{1}{x} \leq 1$$

multiply by x

1. If $x > 0$

$$-x \leq x \sin \frac{1}{x} \leq x$$

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0 \quad \text{since} \quad \lim_{x \to 0} -x = \lim_{x \to 0} x = 0$$

2. If $x < 0$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$-x \geq x \sin \frac{1}{x} \geq x$$

$$\lim_{x \to 0} -x = \lim_{x \to 0} x = 0$$

so by Sandwich th. $\lim_{x \to 0} \sin \frac{1}{x} = 0$

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$ and $$\boxed{\lim_{x \to 0} \frac{1-\cos x}{x} = 0}$$

Example Find the following Limits

a) $\lim_{x \to 0} \frac{\sin 3x}{5x}$

Let $u = 3x$

$\to x = \frac{u}{3}$

as $x \to 0$. $u \to 0$

$$\text{so } \lim_{x \to 0} \frac{\sin 3x}{5x} = \lim_{u \to 0} \frac{\sin u}{5 \cdot \frac{u}{3}} = \frac{3}{5} \lim_{u \to 0} \frac{\sin u}{u} = \frac{3}{5} \cdot 1$$

$$= \frac{3}{5}$$

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

b) $$\lim_{x \to 0} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{x}$$

$$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = 1^2 = 1$$

(e) $\lim\limits_{x\to 0} \dfrac{\sec x - 1}{x \sec x}$

$= \lim\limits_{x\to 0} \dfrac{\frac{1}{\cos x} - 1}{\frac{x}{\cos x}}$

Rule

$\overset{?}{=} 0$

$= \lim\limits_{x\to 0} \dfrac{\frac{1-\cos x}{\cos x}}{\frac{x}{\cos x}} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$

Outline exercises

$1+4+6+10+14+16+18+19+20$

(14) $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2} = 1$

(16) $\lim\limits_{x\to 0} \dfrac{1-\cos 2x}{3x}$

Let $u = 2x$

$x = \dfrac{u}{2}$

$x \to 0$ then $u \to 0$

$\lim\limits_{u\to 0} \dfrac{1-\cos u}{3 \cdot \frac{u}{2}} = \dfrac{2}{3} \lim\limits_{u\to 0} \dfrac{1-\cos u}{u} = \dfrac{2}{3} \cdot 0 = 0$

18. $\lim_{x\to 0} \frac{1-\cos\frac{x}{2}}{x}$

$\frac{x}{2} = u$

$x = 2u$

$\lim_{u\to 0} \frac{1-\cos u}{2u} = \frac{1}{2} \lim_{u\to 0} \frac{1-\cos u}{u} = \frac{1}{2} \cdot 0 = 0$

19. $\lim_{x\to 0} \frac{\sin x\,(1-\cos x)}{x^2}$

$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{1-\cos x}{x} \right)$

$= \left(\lim_{x\to 0} \frac{\sin x}{x} \right) \cdot \left(\lim_{x\to 0} \frac{1-\cos x}{x} \right)$

$= 1 \cdot 0 = 0$

20. $\lim_{x\to 0} \frac{\csc x - \cot x}{x \csc x} = \lim_{x\to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{\frac{x}{\sin x}}$

$= \lim_{x\to 0} \frac{1-\cos x}{x} = 0$