Chapter 4 : 4.4 Chain Rule.

Chain Rule :

If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at x, and the derivative is given by

$$\boxed{(f \circ g)'(x) = f'[g(x)] \cdot g'(x)}$$

Example : Find the derivative of

$$h(x) = (3x^2 - 1)^2$$

solution:

$$h'(x) = 3(3x^2 - 1)^2 \cdot (6x - 0)$$

$$= 18x(3x^2 - 1)^2$$

Example : $h(x) = (2x+1)^3$

$$h'(x) = 3(2x+1)^2 \cdot 2$$

$$= 6(2x+1)^2$$

Example : $h(x) = \sqrt{x^2+1}$

$$h(x) = (x^2+1)^{\frac{1}{2}}$$

$$h'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{x}{(x^2+1)^{\frac{1}{2}}} = \frac{x}{\sqrt{x^2+1}}$$

Example: $h(x) = \sqrt[7]{2x^2+3x} = \left(2x^2+3x\right)^{\frac{1}{7}}$

$$h'(x) = \frac{1}{7}\left(2x^2+3x\right)^{\frac{1}{7}-1} \cdot (4x+3)$$

$$= \frac{4x+3}{7\left(2x^2+3x\right)^{\frac{6}{7}}} = \frac{4x+3}{7 \cdot \sqrt[7]{\left(2x^2+3x\right)^6}}$$

so $\boxed{h'(x) = \dfrac{4x+3}{7\sqrt[7]{\left(2x^2+3x\right)^6}}}$

Example: $h(x) = \left(\frac{x}{x+1}\right)^2$

$$h'(x) = 2 \cdot \left(\frac{x}{x+1}\right) \cdot \left[\frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}\right]$$

$$= 2\left(\frac{x}{x+1}\right) \cdot \frac{+1}{(x+1)^2}$$

$$\boxed{h'(x) = \frac{+2x}{(x+1)^3}}$$

Example: Find the derivative of

$$h(x) = (ax^2 - 2)^n$$

where $a > 0$ and n is a positive integer.

$$h'(x) = n(ax^2-2)^{n-1} \cdot 2ax$$

$$\boxed{h'(x) = 2nax(ax^2-2)^{n-1}}$$

Example: $\frac{d}{dx} \frac{1}{\sqrt{f(x)}}$

$$y = \frac{1}{\sqrt{f(x)}} = f(x)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2} f(x)^{-\frac{3}{2}} \cdot f'(x)$$

$$= \frac{-f'(x)}{2\sqrt{f^3(x)}}$$

Example: Find $\frac{d}{dx}[f(x)]^r$, r is a real number

$$\frac{d}{dx}[f(x)]^r = r[f(x)]^{r-1} f'(x)$$

Examples:

Suppose that $f'(x) = 3x - 1$; Find $\frac{d}{dx}\left(f(x^2)\right)$ at $x = 3$

$$\frac{d}{dx}\left[f(x^2)\right] = f'(x^2)\cdot 2x \quad , \quad \text{at } x = 3$$

$$= f'(9)\cdot 6$$

Now $f'(x) = 3x - 1$

$f'(9) = 3(9) - 1 = 26$

So $\frac{d}{dx}\left[f(x^2)\right] = (26)\cdot 6 = 156$

Example: $h(x) = \left(\sqrt{x^2+1} + 1\right)^2$

$$h'(x) = 2\left[\sqrt{x^2+1} + 1\right]^1 \left[\sqrt{x^2+1} + 1\right]' \quad , \quad \sqrt{x^2+1} = \left(x^2+1\right)^{\frac{1}{2}}$$

$$= 2\left[\sqrt{x^2+1} + 1\right]\left[\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\cdot 2x\right]$$

$$= \frac{2x\left[\sqrt{x^2+1} + 1\right]}{\sqrt{x^2+1}}$$

Example:

$$h(x) = \left(2x^3 - \sqrt{3x^4-2}\right)^3$$

$$h(x) = 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \cdot \left[2x^3 - \left(3x^4-2\right)^{\frac{1}{2}}\right]$$

$$= 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \cdot \left[6x^2 - \frac{1}{2}\left(3x^4-2\right)^{-\frac{1}{2}} \cdot 12x^3\right]$$

$$= 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \left[6x^2 - \frac{6x^3}{\sqrt{3x^4-2}}\right]$$

$$= 18\left(2x^3 - \sqrt{3x^4-2}\right)^2 \left[x^2 - \frac{x^3}{\sqrt{3x^4-2}}\right]$$

4.4.2 Implicit Differentiation الاشتقاق الضمني

Example

Find $\frac{dy}{dx}$ if $x^2+y^2=1$

$$2x+2y\,y'=0$$

$$y'=\frac{-2x}{2y}=-\frac{x}{y}$$

So $\boxed{\frac{dy}{dx}=\frac{-x}{y}}$

Example: Find $\frac{dy}{dx}$ if $y^3x^2-yx+2y^2=x$

$$y^3[2x]+x^2[3y^2y']-[y\cdot 1+xy']+4yy'=1$$

$$2xy^3+3x^2y^2y'-y-xy'+4yy'=1$$

$$[3x^2y^2-x+4y]\,y'=1+y-2xy^3$$

$$\boxed{y'=\frac{1+y-2xy^3}{3x^2y^2-x+4y}}$$

Example: Find $\frac{dy}{dx}$ when $y^2 = x^3$, Assume that $x>0, y>0$.

$$2y\bar{y} = 3x^2$$

$$\bar{y} = \frac{3x^2}{2y}$$

$$y = x^{\frac{3}{2}}$$

$$\text{So } \bar{y} = \frac{3x^2}{2 \cdot x^{\frac{3}{2}}} = \frac{3}{2}\sqrt{x}$$

4.4.4 Higher Derivatives

Example: Find the nth derivative of $f(x) = x^5$ for $n = 1, 2, 3, \ldots$

$$f(x) = x^5$$

$$\bar{f}(x) = 5x^4$$

$$\ddot{f}(x) = f^{(2)}(x) = 20x^3$$

$$\dddot{f}(x) = f^{(3)}(x) = 60x^2$$

$$\ddddot{f}(x) = f^{(4)}(x) = 120x$$

$$\overset{\equiv}{f}(x) = f^{(5)}(x) = 120$$

$$f^{(6)}(x) = 0$$

$$f^{(7)}(x) = 0$$

$$\vdots$$

$$f^{(n)}(x) = 0 \quad \forall n \geq 6$$

Example: Find the second derivative of $f(x) = \sqrt{x}$, $x \geq 0$

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f''(x) = \frac{-1}{4} x^{-\frac{3}{2}} = \frac{-1}{4\sqrt{x^3}}$$

Example: Find $\frac{d^2y}{dx^2}$, when $x^2+y^2=1$

$$2x + 2yy' = 0$$

$$y' = -\frac{2x}{2y} = -\frac{x}{y}$$

$$y'' = \frac{y \cdot -1 + x \cdot y'}{y^2} = \frac{-y + xy'}{y^2}$$

$$y'' = \frac{-y - x\left[\frac{+x}{y}\right]}{y^2} = \frac{-y + -\frac{x^2}{y}}{y^2} = \frac{-y^2 + -x^2}{y^3}$$

$$= \frac{-(y^2+x^2)}{y^3} = \frac{-1}{y^3}$$