4.7 Derivatives of Inverse Functions, Logarithmic Functions and the inverse Tangent functions

4.7.1 Derivative of inverse Functions

$f(x) = x^2, \quad x \geq 0$

$y = x^2$

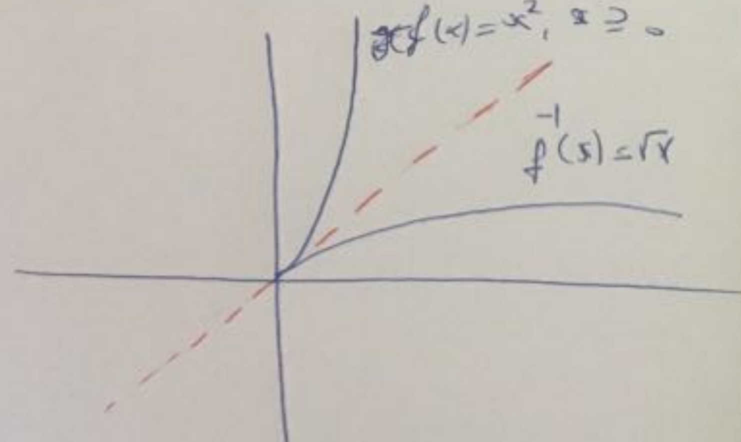
$x = \sqrt{y}$

$f^{-1}(x) = \sqrt{x}$

So $f(x) = x^2$, $f^{-1}(x) = \sqrt{x}$ and remember that $f \circ f^{-1}(x) = x$
and $f^{-1} \circ f(x) = x$

In general : take $f(x)$ and let $f^{-1}(x) = g(x)$

$$(f \circ g)(x) = f[g(x)] = f(\sqrt{x}) = (\sqrt{x})^2 = x, \quad x \geq 0$$



If $f(x)$ is given and $g(x) = f^{-1}(x)$, then

$$f[g(x)] = x$$

$$\frac{d}{dx}\left(f[g(x)]\right) = \frac{d}{dx}\cdot x$$

$$f'[g(x)]\, g'(x) = 1$$

$$g'(x) = \frac{1}{f'[g(x)]}$$

Derivative of an inverse Function

If $f(x)$ is one to one and differentiable with inverse functions $f^{-1}(x)$ and $f'(f^{-1}(x)) \neq 0$, then $f^{-1}(x)$ is differentiable and

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$$

also $\boxed{\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}}$

Example:

Let $f(x) = \frac{x}{x+1}, \ x > 0$

Find $\left. \frac{d f^{-1}(x)}{dx} \right|_{x=\frac{1}{3}}$

This $x = \frac{1}{3}$ is x wrt $f^{-1}(x)$ so it's y wrt $f(x)$

so $\frac{x}{x+1} = \frac{1}{3}$

$3x = x+1$

$2x = 1$

$\boxed{x = \frac{1}{2}}$ so on $f(x)$ the point is $\left(\frac{1}{2}, \frac{1}{3}\right)$

on $f^{-1}(x)$ the point is $\left(\frac{1}{3}, \frac{1}{2}\right)$

$$\left. \frac{d f^{-1}(x)}{dx} \right|_{x=\frac{1}{3}} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'\left(f^{-1}\left(\frac{1}{3}\right)\right)} = \frac{1}{f'\left(\frac{1}{2}\right)} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$$

$f(x) = \frac{x}{1+x}$ so $f'(x) = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$

so $f'\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}+1\right)^2} = \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{4}{9}$

Ex

$f(x) = 2x + e^x \quad , \quad x \in R$ 4

Find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=1}$

$x=1 \longrightarrow 1 = 2x + e^x$

$\boxed{x=0}$

on $f(x)$ $(0,1)$

on $f^{-1}(x)$: $(1,0)$

$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=1} = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2+1} = \frac{1}{3}$

$f(x) = 2x + e^x$

$f'(x) = 2 + e^x$