Example 3: $f(x) = \tan x \quad , -\frac{\pi}{2} < x < \frac{\pi}{2}$ ①

$$\left.\frac{d f^{-1}(x)}{dx}\right|_{x=1}$$

$x=1$ " x wrt $f^{-1}(x)$ "

So $\tan x = 1 \rightarrow x = \frac{\pi}{4}$

$f : \left(\frac{\pi}{4}, 1\right)$

$f^{-1} : \left(1, \frac{\pi}{4}\right)$

$$\frac{d f^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'\left(\frac{\pi}{4}\right)} = \frac{1}{2}$$

$f(x) = \tan x \rightarrow f'(x) = \sec^2 x$

$f'\left(\frac{\pi}{4}\right) = \sec^2 \frac{\pi}{4} = 2$

so $\boxed{\left.\frac{d f^{-1}(x)}{dx}\right|_{x=1} = \frac{1}{2}}$

Example 4: $f(x) = \tan x, \ \frac{-\pi}{2} < x < \frac{\pi}{2}$ ②

Find $\frac{d}{dx} f^{-1}(x)$

$$\frac{d}{dx} f^{-1} = \frac{1}{f'(f^{-1}(x))} =$$

$f(x) = \tan x, \ y = f^{-1}(x) = \tan^{-1}(x) = \arctan x \Rightarrow \boxed{\tan y = x}$

Note that $\tan^{-1}(x)$ is different from $\frac{1}{\tan x}$

Let $\boxed{y = \arctan x}$

$$\frac{dy}{dx} = \frac{d}{dy} \arctan x = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1}$$

$$= \frac{1}{1 + x^2}$$

So $\boxed{\frac{dy}{dx} = \frac{d \tan^{-1} x}{dx} = \frac{1}{1 + x^2}}$

$$\boxed{\frac{d}{dx}\arcsin x = \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$ (3)

4.7.2 The Derivative of the Logarithmic Function

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$ (1)

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x(\ln a)}}$$ (2)

$$\boxed{\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}}$$ (3)

$$\boxed{\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}}$$ (4)

Example: $y = \ln(\sin x)$

$$y' = \frac{\cos x}{\sin x}$$

$$\boxed{y' = \cot x}$$

Ex: $y = \ln(\tan x + x)$

$$\boxed{y' = \frac{\sec^2 x + 1}{\tan x + x}}$$

Example: $y = \ln(x^2+1)$

$$\boxed{y' = \frac{2x}{x^2+1}}$$

Example: $y = \log(2x^3-1)$

$$\boxed{y' = \frac{6x^2}{(2x^3-1)\ln 10}}$$

4.7.3 Logarithmic Differentiation

Example: $y = x^x$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$\cancel{\ln y}$

$$\frac{y'}{y} = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = y\,[1 + \ln x]$$

$$y' = x^x\,[1 + \ln x]$$

If x^x looks strange write it as $x^x = e^{\ln x^x} = e^{x \ln x}$

So $y' = e^{x \ln x}\,[1 + \ln x]$

Ex: $y = (\sin x)^x$

$$\ln y = \ln (\sin x)^x$$

$$\ln y = x(\ln \sin x)$$

$$\frac{y'}{y} = x \frac{\cos x}{\sin x} + \ln \sin x$$

$$y' = y\left[x \cot x + \ln \sin x\right]$$

$$\boxed{y' = (\sin x)^x \left[x \cot x + \ln \sin x\right]}$$

Example:

$$y = \frac{e^x x^{\frac{3}{2}} \sqrt{1+x}}{(x^2+3)^4 (3x-2)^3}$$

$$\ln y = \ln\left[\frac{e^x x^{3/2} \sqrt{1+x}}{(x^2+3)^4 (3x-2)^3}\right]$$

$$= \ln\left[e^x x^{\frac{3}{2}} \sqrt{1+x}\right] - \ln\left[\cancel{e^x x^3} (x^2+3)^4 \cdot (3x-2)^3\right]$$

$$= \ln e^x + \ln x^{\frac{3}{2}} + \ln (1+x)^{\frac{1}{2}} - \ln (x^2+3)^4 - \ln (3x-2)^3$$

$$= x + \frac{3}{2} \ln x + \frac{1}{2} \ln (x+1) - 4 \ln (x^2+3) - 3 \ln (3x-2)$$

$$\frac{y'}{y} = 1 + \frac{3}{2x} + \frac{1}{2} \cdot \frac{1}{(x+1)} - 4 \cdot \frac{2x}{x^2+3} - 3 \cdot \frac{3}{3x-2}$$

$$y' = y\left[1 + \frac{3}{2x} + \frac{1}{2(x+1)} - \frac{8x}{x^2+3} - \frac{9}{3x-2}\right]$$