5.3 The Definite Integral

Note Title ٢٣/٠١/١٠

Def: Let f be a fun on $[a,b]$. The definite integral of f from a to b is the unique number J – if exists – satisfies

$$L(P) \leq J \leq U(P)$$

for any partition P of $[a,b]$, where $L(P)$ is the lower sum and $U(P)$ is the upper sum of this partition. This number is denoted by

$$\int_a^b f(x)\,dx$$

ملحوظة١: التعريف السابق يكافئ تعريف أن التكامل المحدد يساوي نهاية مجموع ريمان لأي تجزئة عندما طول الفترات الجزئية يؤول للصفر وهو يؤدي أن تكون $n \to \infty$ حيث n هو عدد الفترات الجزئية للفترة $[a,b]$ و التعريف التالي هو تعريف التكامل باستخدام تعريف النهاية.

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\,\Delta x_k - J\right| < \epsilon.$$

طول أطول فترة في التجزئة

Integrable and Nonintegrable funs

إذا وجد رقم وحيد بين المجموع الأدنى والأعلى لكل التجزيئات على الفترة $[a,b]$ فإن التكامل يكون معرفاً و يكافئ أن نهاية مجموع ريمان تكون موجودة في هذه الحالة يقال للدالة f أنها قابلة للتكامل على الفترة $[a,b]$، (f is integrable on $[a,b]$.)

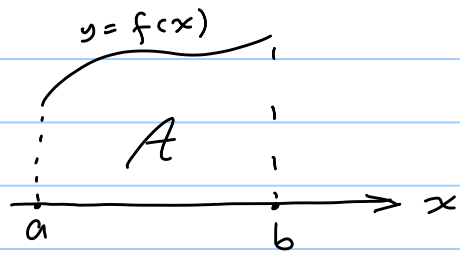
وإذا لم يكن هناك رقم وحيد / فإنه التكامل المحدود غير موجود /

والدالة تكون غير قابلة للتكامل (non integrable on [a,b])

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

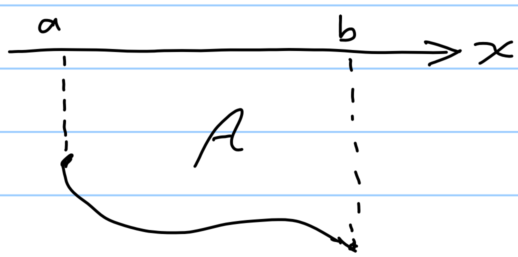
Defs. a) If f is integrable fun on $[a,b]$, and if $f(x) \geq 0$, then the area A under the curve and over the x-axis from a to b is equal

$$A = \int_a^b f(x)\,dx$$



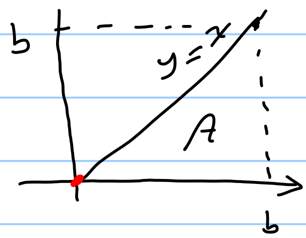
b) If $f(x) \leq 0$, then

$$A = -\int_a^b f(x)\,dx$$



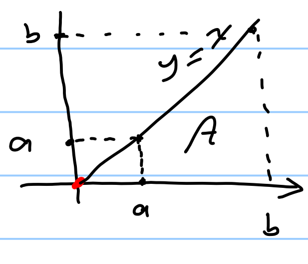
Examples:

1) $\displaystyle\int_0^b x\,dx = \frac{1}{2}\, b \cdot b = \frac{b^2}{2}$. (مساحة المثلث)



2) $\displaystyle\int_a^b x\,dx = \frac{1}{2}(a+b)(b-a)$ (مساحة شبه المنحرف)

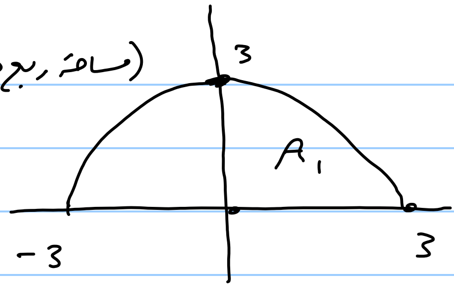
$$= \frac{b^2}{2} - \frac{a^2}{2}$$



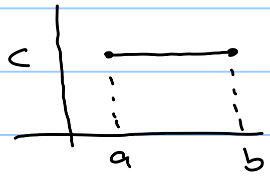
3) a) $\displaystyle\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi r^2$ (مساحة ربع دائرة)

$= \boxed{\frac{9\pi}{4}}$

b) $\displaystyle\int_{-3}^3 \sqrt{9-x^2}\,dx = \frac{1}{2}\pi r^2 = \boxed{\frac{9\pi}{2}}$



4) $\displaystyle\int_a^b c\,dx = c\,(b-a)$



Properties of the Definite Integrals

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition

2. *Zero Width Interval:* $\displaystyle\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

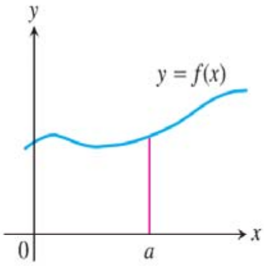
6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $\displaystyle f(x) \ge g(x) \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

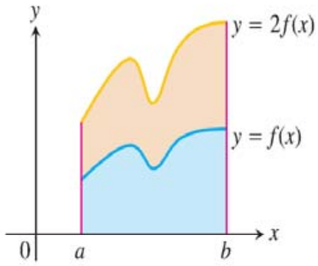
$$f(x) \ge 0 \text{ on } [a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$$ (Special Case)

ملحوظة: كحالة خاصة وعندما تكون الدوال موجبة فإن التكامل المحدد يمثل هندسياً بالمساحة تحت المنحنى، وعليه، يمكن توضيح القوانين السابقة هندسياً من خلال هذه الحالة الخاصة مع الأخذ بعين الاعتبار أن القوانين أشمل من هذه الحالة فقط.



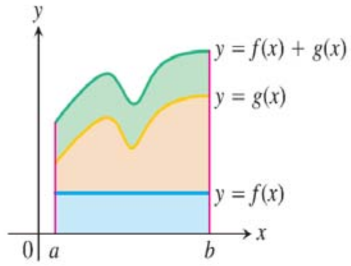
(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0$$



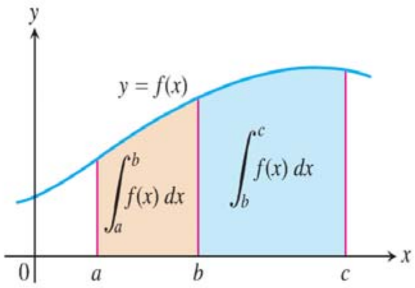
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



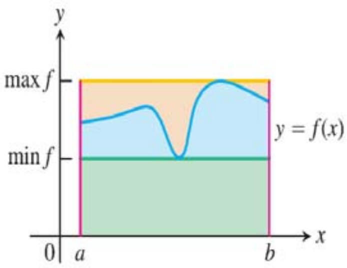
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



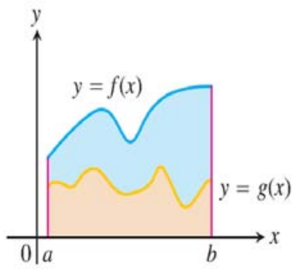
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Examples:

1) Suppose that $\int_0^2 f(x)\,dx = 2$, $\int_0^5 f(x)\,dx = 8$ and $\int_2^5 g(x)\,dx = 1$. Evaluate $\int_5^2 \frac{1}{2} f(x) + 3g(x) - 2\,dx$

Sol:

$$\int_5^2 \frac{1}{2} f(x) + 3g(x) - 2\,dx = \frac{1}{2}\int_5^2 f(x)\,dx + 3\int_5^2 g(x)\,dx - \int_5^2 2\,dx$$

$$= \frac{1}{2}\left(\int_5^0 f(x)\,dx + \int_0^2 f(x)\,dx\right) + 3\left(-\int_2^5 g(x)\,dx\right) - 2(2-5)$$

$$= \frac{1}{2}(-8+2) + 3(-1) + 6 = 0$$

ملحوظة: يمكنك حل السؤال بأكثر من طريقة وباستخدام خصائص مختلفة.

2) Evaluate $\int_1^3 f(x)\,dx$ if $f(x) = \begin{cases} x, & 1 \le x < 2 \\ 2, & 2 \le x \le 3 \end{cases}$

sol:

$$\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_1^2 x\,dx + \int_2^3 2\,dx = \left(\frac{2^2}{2} - \frac{1^2}{2}\right) + 2(3-2)$$

$$= \frac{3}{2} + 2 = \boxed{\frac{7}{2}}$$

3) Find upper and lower bounds for the definite integral $\int_0^1 \sqrt{1+x^4}\,dx$

sol: Let $f(x) = \sqrt{1+x^4}$ on $[0,1]$.

clearly $m = \sqrt{1} = 1$ is abs. min of f and

$M = \sqrt{2}$ is abs. max of f (افحص ذلك)

so $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \Longrightarrow$

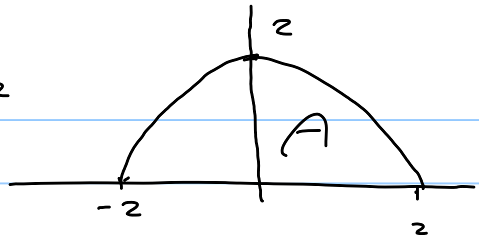
$$1 \le \int_0^1 \sqrt{1+x^4}\,dx \le \sqrt{2}.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$.

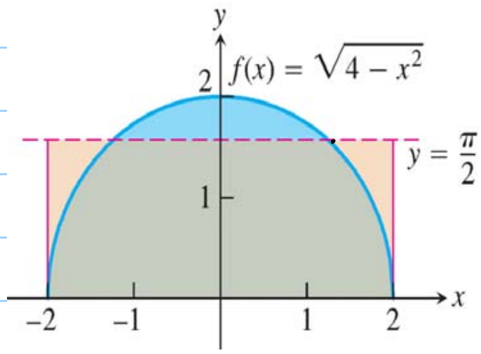
Sol: $\int_{-2}^{2} \sqrt{4-x^2}\, dx = A = \frac{1}{2}\pi r^2$



$= 2\pi$

so $av(f) = \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2}\, dx = \frac{2\pi}{4} = \frac{\pi}{2}$

التمثيل الهندسي لـ av(f) موضح بالرسمة التالية .



Mean Value Thrm for Definite Integral

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx. \quad (= av(f))$$

Example: Apply MVT for definite integral on previous example.

Sol: In previous example, f is continuous on $[-2,2]$

and we find $av(f) = \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2}\, dx = \frac{\pi}{2}$

so $\exists\, c \in (-2, 2)$ s.t. $f(c) = \pi/2 \Longrightarrow$.

$\sqrt{4-c^2} = \frac{\pi}{2} \Longrightarrow 4 - c^2 = \frac{\pi^2}{4}$

$\therefore\ c^2 = 4 - \frac{\pi^2}{4} = 1.533 \Longrightarrow c = \mp 1.238 \in (-2, 2)$