5.5 Indefinite Integrals and the Substitution Method

Indefinite Integrals

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Illustration:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int (2x + \cos x)\,dx = x^2 + \sin x + C.$$

Table of Antiderivatives

التكاملات التالية يمكن استنتاجها بإشتقاق الطرف الأيمن من خلال معرفتنا السابقة بالإشتقاقات.

1) $\int k\,dx = kx + C$

2) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad , \quad n \neq -1$

$\left[\int x^{10}\,dx = \dfrac{x^{11}}{11} + C \quad , \quad \int x^{-\frac{2}{3}}\,dx = 3x^{\frac{1}{3}} + C \right]$

3) $\int \sin kx\,dx = \dfrac{-\cos kx}{k} + C$

$\left[\int \sin 3x\,dx = \dfrac{-\cos 3x}{3} + C \right]$

4) $\int \cos kx\,dx = \dfrac{\sin kx}{k} + C$

$\left[\int \cos \dfrac{x}{2}\,dx = 2\sin \dfrac{x}{2} + C \right]$

5) $\int \sec^2 kx \, dx = \dfrac{\tan kx}{k} + C$

6) $\int \csc^2 kx \, dx = -\dfrac{\cot kx}{k} + C$

7) $\int \sec kx \tan kx \, dx = \dfrac{\sec kx}{k} + C$

8) $\int \csc kx \cot kx \, dx = -\dfrac{\csc kx}{k} + C$

Thrm: (Rules for Antiderivatives)

1) $\int k f(x) \, dx = k \int f(x) \, dx$

2) $\int f(x) \mp g(x) \, dx = \int f(x) \, dx \mp \int g(x) \, dx$.

Examples:

1) $\int 5x + \dfrac{1}{\sqrt{x}} - 3\cos 2x \, dx$

$$= \int 5x \, dx + \int x^{-1/2} \, dx - 3 \int \cos 2x \, dx$$

$$= \left(\frac{5}{2}x^2 + c_1\right) + \left(2\sqrt{x} + c_2\right) - \left(\frac{3}{2}\sin 2x + c_3\right)$$

$$= \frac{5}{2}x^2 + 2\sqrt{x} - \frac{3}{2}\sin 2x + (c_1 + c_2 - c_3)$$

لاحظ أنه جمع ثلاث ثوابت $c_1+c_2+c_3$ هو ثابت جديد ، فإذا

أخذنا $C = c_1 + c_2 + c_3$ نحصل على

$$\int 5x + \frac{1}{\sqrt{x}} - 3\cos 2x \, dx = \boxed{\frac{5}{2}x^2 + 2\sqrt{x} - \frac{3}{2}\sin 2x + C}$$

ملحوظة: عندما يتم توزيع التكامل على عملية الجمع ، فإنه يمكنه إيجاد ثابت مشترك يمثل جمع الثوابت كما في المثال السابق

2) $\int x^4 - 4x^3 + 3x - 2 \, dx$

$$= \frac{x^5}{5} - x^4 + \frac{3}{2}x^2 - 2x + C.$$

3) $\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C.$$

4) $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$

5) Is $\int x \cos x \, dx = x \sin x + \cos x + C$?

Sol:

$$\frac{d}{dx}\left(x \sin x + \cos x + C\right) = x \cos x + \sin x - \sin x + 0$$
$$= x \cos x \Longrightarrow$$

$$\int x \cos x \, dx = x \sin x + \cos x + C \qquad \square$$

Substitution Method

عملية التعويض البسيط هي عملية عكسية لقانون السلسلة / فإذا كانت

$F(x)$ هي أصل إشتقاق لـ $f(x)$ فإنه بإستخدام السلسلة

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x)$$
$$= f(g(x))\, g'(x)$$

وعليه فإنه

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

والسؤال كيف يمكن اجراء هذا التكامل بشكل مباشر ؟!

الإجابة ببساطة هو من خلال التعويض البسيط بأخذ

$u = g(x)$ فتكون $\frac{du}{dx} = g'(x)$ ومنه $du = g'(x)\,dx$

عوضه من التكامل نحصل على

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C$$
$$= F(g(x)) + C.$$

Examples:

1) $$\int 3x^2 (x^3+5)^9\,dx = \int u^9\,du$$
$$= \frac{u^{10}}{10} + C = \frac{(x^3+5)^{10}}{10} + C$$

$u = x^3+5$
$du = 3x^2\,dx$

2) $$\int 3\sec^2(3t+1)\,dt$$
$$= \int \sec^2 u\,du = \tan u + C$$
$$= \tan(3t+1) + C$$

$u = 3t+1$
$du = 3\,dt$

3) $$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2}$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \boxed{\frac{1}{3}(2x+1)^{3/2} + C}$$

$u = 2x+1$
$du = 2\,dx$
$\frac{du}{2} = dx$

4) $$\int x\sqrt{2x+1}\,dx$$
$$= \int \frac{1}{2}(u-1)\sqrt{u}\cdot\frac{du}{2}$$
$$= \frac{1}{4}\int u^{3/2} - u^{1/2}\,du = \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right) + C$$

$u = 2x+1$
$du = 2\,dx$
$\frac{du}{2} = dx$
$x = \frac{1}{2}(u-1)$

5) $$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})^2} = 2\int \frac{du}{u^2}$$
$$= 2\left(\frac{-1}{u}\right) + C = \frac{-2}{u} + C$$

$u = 1+\sqrt{x}$
$du = \frac{dx}{2\sqrt{x}}$

$$= \frac{-2}{1+\sqrt{x}} + C \qquad \therefore\ 2du = \frac{dx}{\sqrt{x}}$$

6) $\int x^5 \sqrt{x^2-1}\, dx$

$u = x^2 - 1$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

$x^2 = u + 1$

$$= \int (x^2)^2 \cdot x \sqrt{x^2-1}\, dx$$

$$= \int (u+1)^2 \cdot \sqrt{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int (u^2 + 2u + 1)\sqrt{u}\, du = \frac{1}{2}\int u^{5/2} + 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{1}{2}\left[\frac{2}{7}u^{7/2} + 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{1}{2}\left[\frac{2}{7}(x^2+1)^{7/2} + \frac{4}{5}(x^2-1)^{5/2} + \frac{2}{3}(x^2-1)^{3/2}\right] + C$$

7) $\int 2x^2 \sin x^3\, dx = \frac{2}{3}\int \sin u\, du$

$u = x^3$

$du = 3x^2\, dx$

$\frac{du}{3} = x^2\, dx$

$$= \frac{-2}{3}\cos u + C$$

$$= \frac{-2}{3}\cos x^3 + C$$

8) $\int \sin^3 x \cos^4 x\, dx$

(في مثل هذه الصيغ نفضل أخذ $u = \cos x$ وهي ذات الأس الزوجي ، ثم تحويل الدالة الأخرى بدلالة $\cos x$)

$u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$$\int \sin^2 x \cos^4 x \cdot \sin x\, dx$$

$$= \int (1-\cos^2 x)\cos^4 x \cdot \sin x\, dx$$

$$= \int (\cos^4 x - \cos^6 x)\sin x\, dx = -\int u^4 - u^6\, du$$

$$= -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C = \boxed{\frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C}$$

ملخص التكاملات

1) $\int \sin^4(7\theta+3)\cos(7\theta+3)\,d\theta$

$u = \sin(7\theta+3)$

$du = 7\cos(7\theta+3)d\theta$

$= \frac{1}{7}\int u^4 \cdot du = \frac{1}{7}\left(\frac{u^5}{5}\right) + C$

$= \frac{\sin^5(7\theta+3)}{35} + C$

2) $\int \frac{\sin^5\sqrt{x}\,\cos^3\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sin\sqrt{x}$

$du = \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$

$= \int \sin^5\sqrt{x}\,\cos^2\sqrt{x} \cdot \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$= \int \sin^5\sqrt{x}\,(1-\sin^2\sqrt{x})\,\frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int u^5(1-u^2)\cdot 2\,du$

$= 2\int u^5 - u^7\,du = 2\left(\frac{u^6}{6} - \frac{u^8}{8}\right) + C$

$= \frac{\sin^6\sqrt{x}}{3} - \frac{\sin^8\sqrt{x}}{4} + C$

3) $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

$u = 2 - \frac{1}{x}$

$du = \frac{1}{x^2}\,dx$

$= \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}\left(2-\frac{1}{x}\right)^{3/2} + C$

4) $\int 3x^5\sqrt{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$x^3 = u-1$

$= \int x^3\sqrt{x^3+1}\cdot 3x^2\,dx$

$= \int (u-1)\sqrt{u}\cdot du = \int u^{3/2} - u^{1/2}\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$