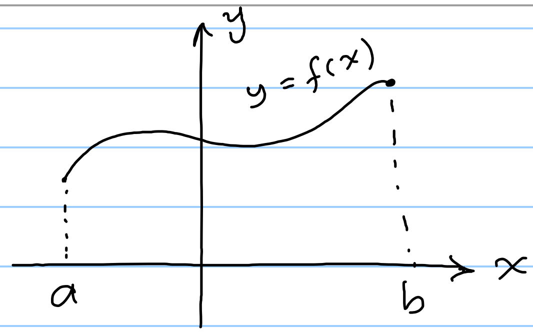
6.3 Arc Length



DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

Remark: A fun with continuous derivative on $[a,b]$ is called smooth, and its curve is called smooth curve on $[a,b]$.

EXAMPLE 1 Find the length of the curve (Figure 6.24)

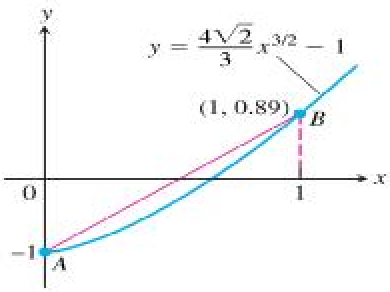
$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Sol: $f'(x) = \frac{4}{3}\sqrt{2} \cdot \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2}\cdot\sqrt{x}$ which is continuous on $[0,1]$. So

$$L = \int_0^1 \sqrt{1 + (f')^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{1}{8}\int_1^9 \sqrt{u}\, du = \boxed{\frac{13}{6}}$$

$u = 1 + 8x$
$du = 8\,dx$
$\frac{du}{8} = dx$
$x = 0 \longrightarrow u = 1$
$x = 1 \longrightarrow u = 9$



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المنحنى للدالة المرسومة

EXAMPLE 2 Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4.$$

Sol: $f' = \frac{3x^2}{12} - \frac{1}{x^2}$ which is continuous on $[1,4]$

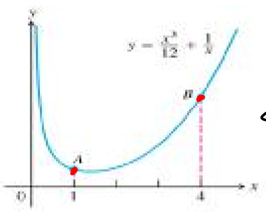
Note that $1 + (f')^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$\therefore \quad L = \int_1^4 \sqrt{1 + (f')^2}\, dx = \int_1^4 \sqrt{\left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2}\, dx$$

$$= \int_1^4 \frac{x^2}{4} + \frac{1}{x^2}\, dx \qquad \left(\frac{x^2}{4} + \frac{1}{x^2} > 0 \quad \forall x\right)$$

$$= \frac{x^3}{12} - \frac{1}{x}\Big]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = \boxed{6}$$



المنحنى الذي أوجدنا طوله موضح بالرسمة

ملحوظة: إذا لم تكن المشتقة f' متصلة على الفترة $[a,b]$ ، أو
كان التكامل باتجاه محور x صعب ، فإنه قد يكون من الأنسب التكامل
مع طول المنحنى نكامله باتجاه y حسب النص التالي :

Formula for the Length of $x = g(y),\ c \le y \le d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (4)$$

EXAMPLE 3 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Sol: $y' = \frac{2}{3}\left(\frac{x}{2}\right)^{-\frac{1}{3}} \cdot \frac{1}{2} = \left(\frac{\sqrt[3]{2}}{3}\right) \cdot \frac{1}{\sqrt[3]{x}}$

Note that y' is not continuous at $x = 0 \in [0, 2]$, so f is not smooth curve. In this case, we can't use formula (3). So, we try to use the other formula in (4) above as follows:

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}} \Longrightarrow x = 2y^{3/2}.$$

When $x \in [0, 2] \longrightarrow y \in [0, 1]$

($[0,1]$ is the range of $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ when $[0,2]$ is the domain)

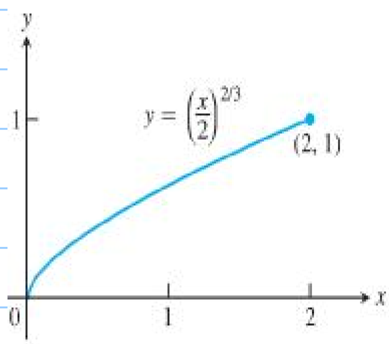
Now $\frac{dx}{dy} = 3\sqrt{y}$ which is continuous on $[0,1]$

Therefore, using formula (4), we get,

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \simeq \boxed{2.27}$$

انظر الرسمة



ملحوظة: الرسم ليس شرطاً للحل، وإنما يعطى للتوضيح.