7.5 Indeterminate forms and L'Hôpital's Rule

Note Title ٢٣/٠٣/٢٥

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \left(\frac{0}{0}\right)$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Other meaningless expressions often occur, such as ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0, and 1^∞, which cannot be evaluated in a consistent way; these are called indeterminate forms as well. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2.

THEOREM 5— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Examples: Evaluate the following limits:

1) $\lim_{x \to 0} \frac{3x - \sin x}{x} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{x \to 0} \frac{3 - \cos x}{1} = 2$

2) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} \left(\frac{0}{0}\right)$

$\overset{L.R}{=} \lim_{x \to 0} \frac{\left(\frac{1}{2\sqrt{1+x}} - 0\right)}{1} = \boxed{\frac{1}{2}}$

3) $\lim_{x \to 0} \frac{x - \sin x}{x^3} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \left(\frac{0}{0}\right)$

$\overset{L.R}{=} \lim_{x \to 0} \frac{\sin x}{6x} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{x \to 0} \frac{\cos x}{6} = \boxed{\frac{1}{6}}$

4) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x-x^2} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{1-2x} = \dfrac{0}{1} = \boxed{0}$

ملاحظة: عند عدم الانتباه، وتطبيق نظرية لوبيتال مرة أخرى من المثال (4) مع الصيغة $\frac{0}{1}$ نحصل على نتيجة خاطئة.

5) Find the value of $a > 0$ such that

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{1-\cos(ax)} = \frac{1}{4}.$$

sol:

$$\frac{1}{4} = \lim_{x\to 0} \frac{x-\ln(1+x)}{1-\cos(ax)} \left(\frac{0}{0}\right)$$

$$\overset{L.R}{=} \lim_{x\to 0} \frac{\left(1-\frac{1}{1+x}\right)}{a\sin(ax)} \left(\frac{0}{0}\right)$$

$$\overset{L.R}{=} \lim_{x\to 0} \frac{\left(\frac{1}{(1+x)^2}\right)}{a^2\cos ax} = \frac{1}{a^2}$$

$$\Rightarrow a^2 = 4 \Rightarrow \boxed{a=2} > 0$$

ملاحظة: نظرية لوبيتال لا تزال صحيحة مع النهاية من جهة واحدة (كما أنه يمكن تطبيقها عندما $x \to \mp\infty$.

Example:

$$\lim_{x\to 0^+} \frac{\sin x}{x^2} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{x\to 0^+} \frac{\cos x}{2x} \left(\frac{1}{0^+}\right) = \infty$$

$$\lim_{x\to 0^-} \frac{\sin x}{x^2} \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x\to 0^-} \frac{\cos x}{2x} \left(\frac{1}{0^-}\right) = -\infty$$

$$\therefore \lim_{x\to 0} \frac{\sin x}{x^2} \text{ d.n.e.}$$

Indeterminate Forms $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$:

لا تزال نظرية لوبيتال صحيحة مع الصيغة $\frac{\infty}{\infty}$ بل إن بعض المراجع تبرهن النظرية للصيغة $\frac{\infty}{\infty}$. انظر الأمثلة التالية :

EXAMPLE 4 Find the limits of these ∞/∞ forms:

(a) $\displaystyle \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$ (b) $\displaystyle \lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^2}$.

Sol: $\displaystyle \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x} \left(\frac{\infty}{\infty}\right) \overset{L.R.}{=} \lim_{x \to \pi/2} \frac{\sec x \tan x}{\sec^2 x}$

$\displaystyle = \lim_{x \to \pi/2} \frac{\tan x}{\sec x} = \lim_{x \to \pi/2} \left(\frac{\sin x}{\cos x}\right) \times \cos x = 1$

b) $\displaystyle \lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} \left(\frac{\infty}{\infty}\right) \overset{L.R}{=} \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{\sqrt{x}}\right)}$

$\displaystyle = \lim_{x \to \infty} \frac{\sqrt{x}}{x} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$

c) $\displaystyle \lim_{x \to \infty} \frac{e^x}{x^2} \left(\frac{\infty}{\infty}\right) \overset{L.R}{=} \lim_{x \to \infty} \frac{e^x}{2x} \left(\frac{\infty}{\infty}\right)$

$\displaystyle \overset{L.R.}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty$

Indeterminate Products and Difference

إذا أدى التعويض المباشر في النهاية لأي من الصيغتين الغير محددتين $\infty - \infty$ أو $0 \cdot \infty$ فإنه يجب تحويلها أولاً للصيغة $\frac{0}{0}$ أو $\frac{\infty}{\infty}$ ومن ثم تطبيق نظرية لوبيتال. كما توضح الأمثلة التالية:

Examples:

1) $\displaystyle \lim_{x \to 0^+} x \cot x \quad (0 \cdot \infty)$

$\displaystyle = \lim_{x \to 0^+} \frac{x}{\tan x} \left(\frac{0}{0}\right) \overset{L.R}{=} \lim_{x \to 0^+} \frac{1}{\sec^2 x} = 1$

2) $\displaystyle \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \quad (\infty - \infty)$

$\displaystyle = \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x \to 0^+} \frac{\cos x - 1}{x \cos x + \sin x} \left(\frac{0}{0}\right)$

$\displaystyle \overset{L.R}{=} \lim_{x \to 0^+} \frac{-\sin x}{-x \sin x + \cos x + \cos x} = \frac{0}{2} = 0$

3) $\lim\limits_{x\to\infty}\left(x e^{\frac{1}{x}} - x\right)$ $(\infty-\infty)$

$= \lim\limits_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)$ $(0\cdot\infty)$ $= \lim\limits_{x\to\infty} \dfrac{\left(e^{\frac{1}{x}}-1\right)}{\left(\frac{1}{x}\right)}$ $\left(\frac{0}{0}\right)$

$\overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{\left(e^{\frac{1}{x}}\right)\cdot\cancel{\left(\frac{-1}{x^2}\right)} - 0}{\cancel{\left(\frac{-1}{x^2}\right)}} = \boxed{1}$

Indeterminate Powers:

النهايات التي تنتج صيغ غير محددة بالصورة 1^∞ ، 0^0 ، ∞^0 ، يتم التعامل معها بأخذ اللوغاريتم أولاً ليتم تحويلها إلى صيغة غير محددة من نوع آخر،

ثم بعدها يتم أخذ exp للحصول على نتيجة النهائية حسب توضيح النظرية التالية:

Thrm:

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Examples:

1) $\lim\limits_{x\to\infty} (1+x)^{\frac{1}{x}}$ (∞^0)

consider $\lim\limits_{x\to\infty} \ln(1+x)^{\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{\ln(1+x)}{x}$ $\left(\frac{\infty}{\infty}\right)$

$\overset{L.R.}{=} \lim\limits_{x\to\infty} \dfrac{\left(\frac{1}{1+x}\right)}{1} = 0$

$\therefore \lim\limits_{x\to\infty} (1+x)^{\frac{1}{x}} = e^0 = \boxed{1}$

2) $\lim\limits_{x\to 0^+} (1+x)^{\frac{1}{x}}$ (1^∞)

consider $\lim\limits_{x\to 0^+} \dfrac{\ln(1+x)}{x}$ $\left(\frac{0}{0}\right) = \lim\limits_{x\to 0^+} \dfrac{\left(\frac{1}{1+x}\right)}{1} = 1$

$\therefore \lim\limits_{x\to\infty} (1+x)^{\frac{1}{x}} = e^1 = \boxed{e}$

3) $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^{\sqrt{x}}$ $\quad (1^{\infty})$

Consider $\lim\limits_{x\to\infty} \sqrt{x}\,\ln\left(1+\frac{1}{x}\right)$ $\quad (\infty\cdot 0)$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{\sqrt{x}}\right)} \quad \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x\to\infty}\left[\frac{\frac{1}{\left(1+\frac{1}{x}\right)} * \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{2}\right) x^{-\frac{3}{2}}}\right]$$

$$= \lim_{x\to\infty} -2x^{3/2} * \frac{x}{x+1} * \frac{-1}{x^2} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x+1} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.R}{=} 2\lim_{x\to\infty} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{1} = 0$$

So $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}} = e^{0} = \boxed{1}$

ملحوظة:

1) $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}$ $\quad (1^{\infty})$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{a}{x}\right)}{\left(\frac{1}{x}\right)} \quad \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}} * a * \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}} = a$$

$\therefore \lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = \boxed{e^{a}}$

2) $\lim\limits_{x\to\infty}\left(x+e^{x}\right)^{\frac{1}{x}}$ $\quad (\infty^{0})$

Consider $\lim\limits_{x\to\infty} \frac{\ln(x+e^{x})}{x}$ $\quad \left(\frac{\infty}{\infty}\right)$

$$\overset{L.R}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{x+e^{x}}\right) * (1+e^{x})}{1} = \lim_{x\to\infty} \frac{1+e^{x}}{x+e^{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$\overset{L.R.}{=} \lim_{x\to\infty} \frac{e^{x}}{1+e^{x}} \quad \left(\frac{\infty}{\infty}\right) \overset{L.R}{=} \lim_{x\to\infty} \frac{e^{x}}{e^{x}} = 1$$

$\therefore \lim\limits_{x\to\infty}\left(x+e^{x}\right)^{\frac{1}{x}} = \boxed{e}$

3) $$\lim_{x\to 0} \frac{2^{x}-1}{x\sin x + x} \quad \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x\to 0} \frac{2^{x}\ln 2 - 0}{x\cos x + \sin x + 1}$$

$$= \frac{\ln 2}{1} = \boxed{\ln 2}$$