Review of Chapter 7

Part 1: Find $\frac{dy}{dx}$ for the following:

1) $y = \sqrt{\cosh^2 x - \cos^2 x}$.

Sol:

$$y' = \frac{1}{2\sqrt{\cosh^2 x - \cos^2 x}} * \left(2\cosh x \sinh x + 2\cos x \sin x\right)$$

2) $y = 2^{\operatorname{csch}^{-1} x^2}$

Sol: $y' = \left(2^{\operatorname{csch}^{-1} x^2} \ln 2\right) * \frac{-1}{|x^2| \sqrt{1 + (x^2)^2}} * 2x$.

3) $y = \sqrt{(\tan^{-1} x)^2 + x} - \sec^{-1}\sqrt{e^x - 1}$

Sol: $$y' = \frac{1}{2\sqrt{(\tan^{-1} x)^2 + 1}} * \left(2\tan^{-1} x * \frac{1}{1 + x^2} + 1\right) -$$

$$\frac{1}{|\sqrt{e^x - 1}| \sqrt{(e^x - 1) - 1}} * \frac{1}{2\sqrt{e^x - 1}} * e^x.$$

4) $y = \cosh x \ln(1 - \ln \cosh x)$

$$y' = \cosh x * \frac{1}{1 - \ln \cosh x} * \frac{-\sinh x}{\cosh x} + \ln(1 - \ln \cosh x) * \sinh x$$

5) $y = x^{\sin^{-1} x}$

Sol: $\ln y = (\sin^{-1} x) \ln x \Longrightarrow$

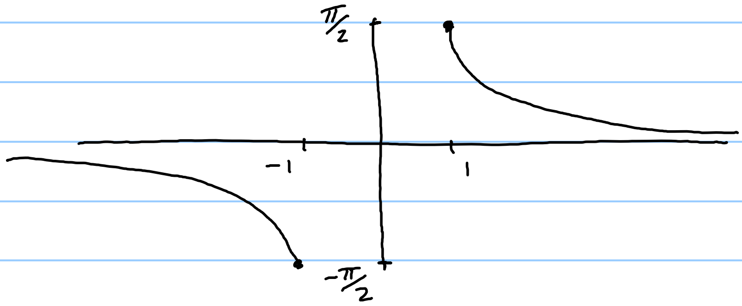
$\frac{1}{y} y' = \frac{\sin^{-1} x}{x} + \ln x * \frac{1}{\sqrt{1 - x^2}} \Longrightarrow$

$$y' = y\left(\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1 - x^2}}\right) = x^{\sin^{-1} x}\left(\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1 - x^2}}\right)$$

Part 2: Evaluate the following limits:

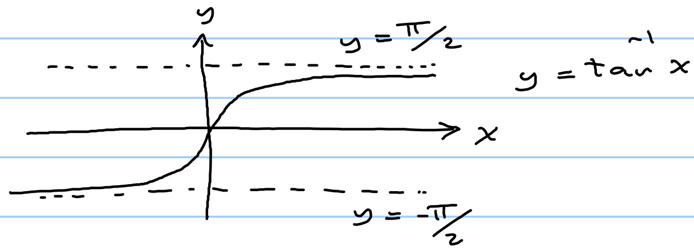
6) $\lim\limits_{x \to -\infty} \csc^{-1} x$

<u>sol:</u> Note that the graph of the fun $y = \csc^{-1} x$ is



so $\lim\limits_{x \to -\infty} \csc^{-1} x = 0$.

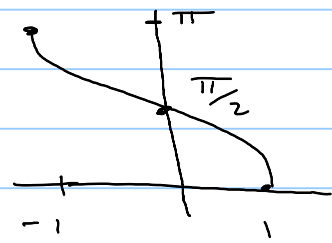
7) $\lim\limits_{x \to -\infty} \tan^{-1} x$



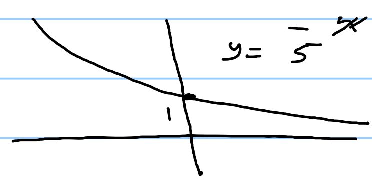
<u>sol:</u> From the graph,

$\lim\limits_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

8) $\lim\limits_{x \to -1^+} \cos^{-1} x = \pi$



9) $\lim\limits_{x \to \infty} \left(\frac{1}{5}\right)^x = \lim\limits_{x \to \infty} 5^{-x} = 0$



10) $\lim\limits_{x \to 0} \dfrac{(\tan^{-1} x)^2}{x\sqrt{x+1}} \quad \left(\frac{0}{0}\right)$

$$\overset{L.R}{=} \lim_{x \to 0} \frac{2\tan^{-1} x * \frac{1}{1+x^2}}{x \frac{1}{2\sqrt{x+1}} + \sqrt{x+1}} = \frac{0}{1} = \boxed{0}$$

11) $\lim\limits_{x \to 3^+} (x-2)^{\frac{1}{x-3}}$ $\quad (1^\infty)$

Sol: Consider $\lim\limits_{x \to 3^+} \ln (x-2)^{\left(\frac{1}{x-3}\right)} = \lim\limits_{x \to 3^+} \dfrac{\ln(x-2)}{x-3}$ $\left(\frac{0}{0}\right)$

$$\overset{L.R}{=} \lim_{x \to 3^+} \frac{\left(\frac{1}{x-2}\right)}{1} = 1$$

$\therefore$

$$\lim_{x \to 3^+} (x-2)^{\left(\frac{1}{x-3}\right)} = e^1 = \boxed{e}$$

12) $\lim\limits_{x \to \infty} x \tan^{-1}\left(\frac{1}{x}\right)$ $\quad (\infty \cdot 0)$

$$= \lim_{x \to \infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \left(\frac{0}{0}\right) \overset{L.R.}{=} \lim_{x \to \infty} \frac{\frac{1}{1+\left(\frac{1}{x}\right)^2} * \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$= \boxed{1}$$

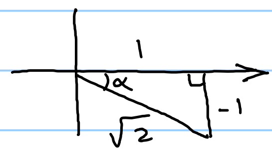
Part 3: Solve the following:

13) Find (a) $\csc^{-1}(-\sqrt{2})$ $\qquad$ (b) $\cos^{-1}\left(-\frac{1}{2}\right)$

(c) $\sec^{-1}(-1)$ $\qquad$ (d) $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Sol. (a) Set $\alpha = \csc^{-1}(-\sqrt{2}) = \sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

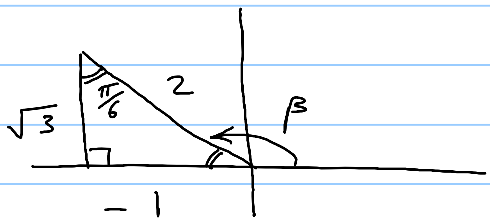
so $\sin\alpha = \frac{-1}{\sqrt{2}}$ and $\alpha \in [-\frac{\pi}{2}, 0]$



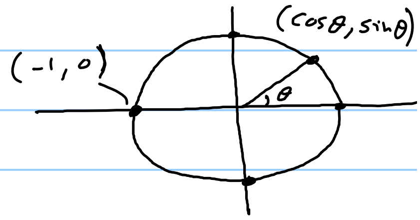
$$\therefore \alpha = \csc^{-1}(-\sqrt{2}) = \boxed{\frac{-\pi}{4}}$$

(b) Set $\beta = \cos^{-1}\left(\frac{-1}{2}\right) \Rightarrow \beta \in [\frac{\pi}{2}, \pi]$ and $\cos\beta = \frac{-1}{2}$

$$\therefore \beta = \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$$



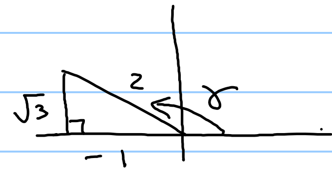
(c) $\sec^{-1}(-1) = \cos^{-1}\left(\frac{1}{-1}\right) = \cos^{-1}(-1) = \boxed{\pi}$



d) Set $\gamma = \cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) \quad (\neq \tan^{-1} -\sqrt{3})$

so

$\cot\gamma = \frac{-1}{\sqrt{3}}$ and $\gamma \in [\pi/2, \pi)$



$\therefore \boxed{\gamma = \frac{2\pi}{3}}$

14) Suppose that $\sinh x = -3$. Find

(a) the value of x

(b) $\cosh x$, $\tanh x$, $\operatorname{sech} x$, ...

Sol: $-3 = \sinh x = \frac{e^{x} - e^{-x}}{2} \Rightarrow e^{x} - e^{-x} = -6$

$\therefore e^{2x} - 1 = -6e^{x} \Rightarrow (e^{x})^2 + 6e^{x} - 1 = 0$

$\therefore e^{x} = \frac{-6 \mp \sqrt{36 + 4}}{2 * 1} = \frac{-6 \mp 6.325}{2}$

$e^{x} > 0$ لذلك السالب مرفوض)

$\therefore e^{x} = \frac{-6 + 6.325}{2} = 0.1625$

$\therefore x = \ln(0.1625) = \boxed{-1.8171}$

b) $\cosh^2 x - \sinh^2 x = 1$

$\therefore \cosh^2 x = 1 + \sinh^2 x = 1 + (-3)^2 = 10$

$\therefore$

$|\cosh x| = \sqrt{10} \qquad (\cosh x > 0) \Rightarrow$

$\cosh x = \sqrt{10}$

لاحظ: يمكن حساب $\cosh x$ باستخدام قيمة $x = -1.8171$.

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-3}{\sqrt{10}}$,

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\sqrt{10}}, \quad \operatorname{csch} x = \frac{-1}{3} \quad \text{and}$$

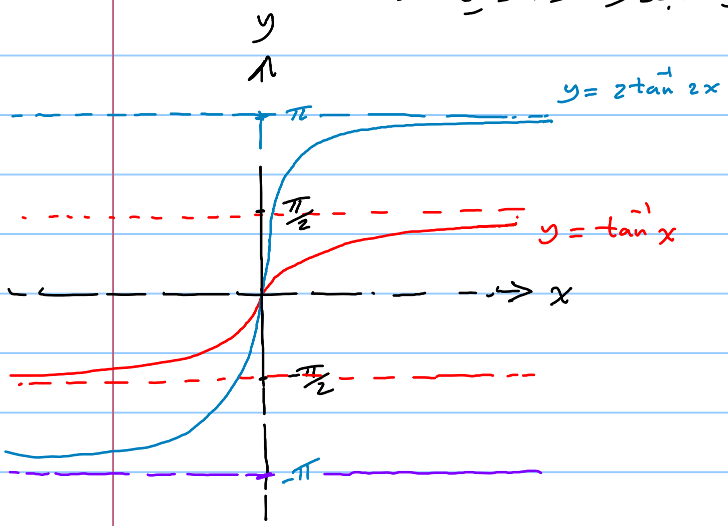
$$\tanh x = \frac{-\sqrt{10}}{3}$$

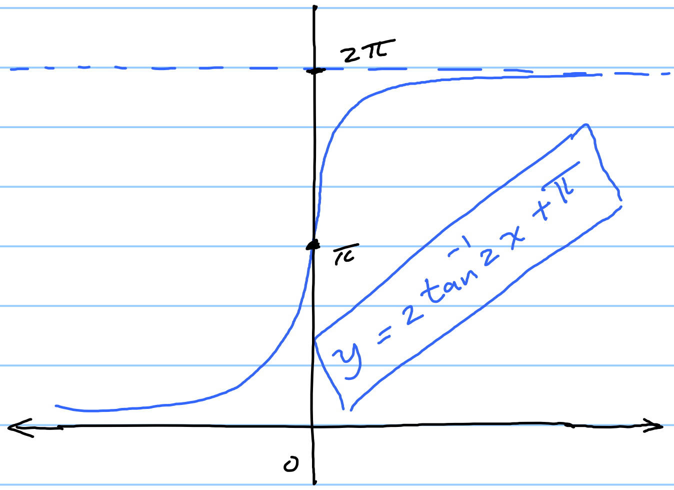
Part 4: Graph

15) $y = 2\tan^{-1}(2x) + \pi$

Sol: الرسمة المطلوبة هي نفس رسمة $y = 2\tan^{-1} 2x$ مزاحة للأعلى بمقدار π .

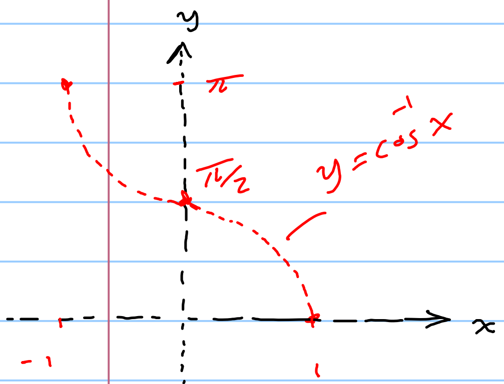
ورسمة $y = 2\tan^{-1} 2x$ هي نفس رسمة $y = \tan^{-1} x$ مضغوطة في إتجاه x بمقدار وحدتين وممددة في إتجاه y بمقدار وحدتين .

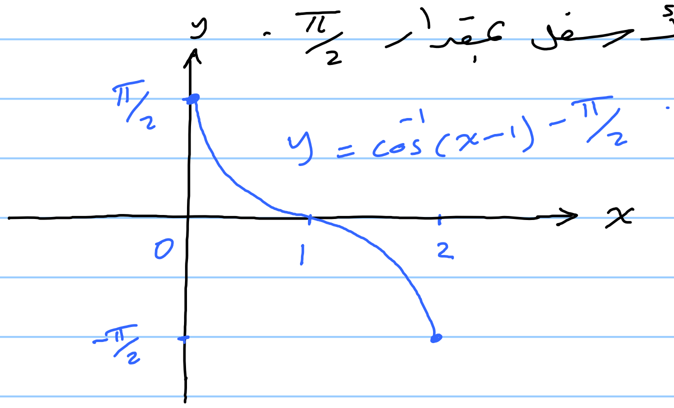




16) $y = \cos^{-1}(x-1) - \pi/2$

Sol: الرسمة المطلوبة نفس رسمة الدالة $y = \cos^{-1}x$ مزاحة لليمين بمقدار 1 وأسفل بمقدار $\pi/2$.





Part 5: Evaluate the following Integrals:

17) $\displaystyle\int_0^{\pi/2} \frac{\sin 2x \, dx}{\sqrt{4 - \cos^4 x}}$

$u = \cos^2 x$

$du = 2\cos x \cdot - \sin x \, dx$

$= - \sin 2x \, dx$

$x = 0 \longrightarrow u = 1$

$x = \pi/2 \longrightarrow u = 0$

$$= - \int_1^0 \frac{du}{\sqrt{4 - u^2}}$$

$$= - \sin^{-1}\left(\frac{u}{2}\right)\Big]_1^0$$

$$= - \sin^{-1} 0 + \sin^{-1} \tfrac{1}{2} = \boxed{\frac{\pi}{6}}$$

18) $\displaystyle\int \frac{e^r \, dr}{1 + e^r}$

$u = 1 + e^r$

$du = e^r \, dr$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|1 + e^r| + C$$

$$= \ln(1 + e^r) + C$$

19) $\displaystyle\int \frac{dr}{1 + e^r}$

حل ١

$$= \int \frac{e^{-r} \, dr}{e^{-r} + 1}$$

(الضرب بسطاً ومقاماً في e^{-r})

$u = e^{-r} + 1$

$du = - e^{-r} \, dr$

$$= - \int \frac{du}{u}$$

$$= - \ln|u| + C = \boxed{- \ln|1 + e^{-r}| + C}$$

يمكن اكمال التبسيط كما يأتي :

$$= - \ln\left|\frac{e^r + 1}{e^r}\right| + C = - \ln(e^r + 1) + \ln e^r + C$$

$$= \boxed{r - \ln(e^r + 1) + C}$$

حل 2

$$\int \frac{dr}{1+e^r} = \int \frac{1+e^r-e^r}{1+e^r} = \int 1 - \frac{e^r}{1+e^r}\, dr$$

18

$$= \boxed{r - \ln(1+e^r) + C}$$

20) $\displaystyle\int \frac{dx}{(2x-1)\sqrt{4x^2-4x-3}}$

sol: Complete a square

$$4x^2-4x-3 = 4\left[x^2 - x + \tfrac{1}{4} - \tfrac{1}{4}\right] - 3$$

$$= 4\left[\left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right] - 3 = (2x-1)^2 - 4$$

$$\therefore \int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$$

$u = 2x-1$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

$$= \frac{1}{2}\int \frac{du}{u\sqrt{u^2-4}} = \frac{1}{2} * \frac{1}{2} \sec^{-1}\left|\frac{u}{2}\right| + C$$

$$= \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + C$$

21) $\displaystyle\int \frac{x+4}{x^2+4} = \int \frac{x\,dx}{x^2+4} + 4\int \frac{dx}{x^2+4}$

في التكامل الأول / خد $u = x^2+4 \Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{u} + 4\int \frac{dx}{x^2+4} = \frac{1}{2}\ln|u| + 4 * \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

موجبة

$$= \boxed{\frac{1}{2}\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C}$$

22) $$\int \frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx$$

$$u=\cos^{-1}x$$
$$du=\frac{-1}{\sqrt{1-x^2}}\,dx$$

$$=-\int e^{u}\,du=-e^{u}+C$$
$$=\boxed{-e^{\cos^{-1}x}+C}$$

23) $$\int \frac{e^{x}\sin^{-1}e^{x}}{\sqrt{1-e^{2x}}}\,dx$$

$$u=\sin^{-1}e^{x}$$
$$du=\frac{1}{\sqrt{1-e^{2x}}}\,e^{x}\,dx$$

$$=\int u\,du=\frac{u^2}{2}+C=\boxed{\frac{(\sin^{-1}e^{x})^2}{2}+C}$$

24) $$\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$$

$$u=\ln x$$
$$du=\frac{1}{x}\,dx$$

$$=\int_0^1 \frac{du}{\sqrt{1+u^2}}$$

$$x=1\longrightarrow u=0$$
$$x=e\longrightarrow u=1$$

$$=\sinh^{-1}u\Big]_0^1=\sinh^{-1}1-\cancel{\sinh^{-1}0}^{\,0}=\sinh^{-1}1=\boxed{0.8814}$$

25) $$\int \frac{\operatorname{csch}(\cos\sqrt{x})\coth(\cos\sqrt{x}) * \sin\sqrt{x}\,dx}{\sqrt{x}}$$

sol:

$$u=\operatorname{csch}(\cos\sqrt{x})$$
$$\therefore du=-\operatorname{csch}(\cos\sqrt{x})\coth(\cos\sqrt{x}) * \frac{-\sin\sqrt{x}}{2\sqrt{x}}\,dx$$

$$=2\int du=2u+C$$
$$=\boxed{2\operatorname{csch}(\cos\sqrt{x})+C}$$