Note Title ٢٣/٠٤/١٩

Review of Ch8

Evaluate the following integrals

1) $\int (x+1)^2 e^x \, dx$

sol: By parts:

$f(x)$		$g(x)$
$(x+1)^2$	$+$	e^x
$2(x+1)$	$-$	e^x
2	$+$	e^x
0		e^x

$$= \left[(x+1)^2 - 2(x+1) + 2\right] e^x + C$$

2) $\int \frac{x^3}{\sqrt{x^2-4}} \, dx$

$x = 2\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$

$dx = 2\sec\theta \tan\theta \, d\theta$

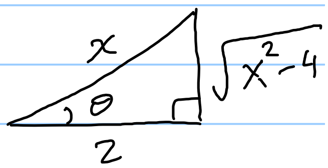
$\sqrt{x^2-4} = 2\tan\theta$

$$= \int \frac{8\sec^3\theta \cdot \cancel{2} \sec\theta \cancel{\tan\theta} \, d\theta}{\cancel{2} \cancel{\tan\theta}}$$

$$= 8\int \sec^4\theta \, d\theta = 8\int \sec^2\theta \cdot \sec^2\theta \, d\theta$$

$$= 8\int (\tan^2\theta + 1) \cdot \sec^2\theta \, d\theta = 8\left[\frac{\tan^3\theta}{3} + \tan\theta\right] + C$$

$$= 8\left[\frac{1}{3}\left(\frac{\sqrt{x^2-4}}{2}\right)^3 + \frac{\sqrt{x^2-4}}{2}\right] + C$$



3) $\int \frac{2\,dx}{x^2\sqrt{x-1}}$

$u = \sqrt{x-1}$

$u^2 = x-1$

$x = u^2+1$

$dx = 2u\,du$

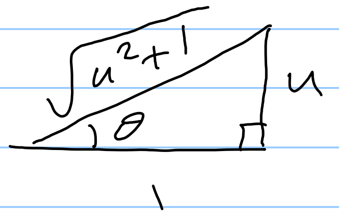
$$= \int \frac{2 \cdot 2u\,du}{(u^2+1)^2 \cdot u}$$

$$= 4\int \frac{du}{(u^2+1)^2} \qquad u = \tan\theta,\ -\pi/2 < \theta < \pi/2$$
$$du = \sec^2\theta\, d\theta$$

$$= 4\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} \qquad u^2+1 = \sec^2\theta$$

$$= 4\int \cos^2\theta\, d\theta = 2\int 1 + \cos 2\theta\, d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$



$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\cdot\tan^{-1}u + 2\cdot\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} + C$$

$$= 2\tan^{-1}\left(\sqrt{x-1}\right) + \frac{2\sqrt{x-1}}{x} + C$$

4) $\displaystyle\int \frac{\sqrt{1-\ln^2 x}}{x\ln x}\, dx$ $\qquad u = \ln x$
$$du = \frac{1}{x}\, dx$$

$$= \int \frac{\sqrt{1-u^2}}{u}\, du$$

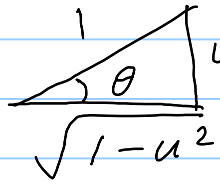
$$u = \sin\theta,\ -\pi/2 \le \theta \le \pi/2$$
$$du = \cos\theta\, d\theta$$
$$\sqrt{1-u^2} = \cos\theta$$

$$= \int \frac{\cos\theta\cdot\cos\theta\, d\theta}{\sin\theta}$$

$$= \int \cot\theta\cdot\cos\theta\, d\theta$$

By parts: $u = \cot\theta \qquad dv = \cos\theta\, d\theta$
$$du = -\csc^2\theta\, d\theta \qquad v = \sin\theta$$

$$= \sin\theta \cdot \cot\theta + \int \sin\theta \cdot \csc^2\theta \, d\theta$$



$$= \cos\theta + \int \csc\theta \, d\theta$$

$$= \cos\theta + \ln\left|\csc\theta - \cot\theta\right| + C$$

$$= \sqrt{1-u^2} + \ln\left|\frac{1}{u} - \frac{\sqrt{1-u^2}}{u}\right| + C$$

$$= \sqrt{1-\ln^2 x} + \ln\left|\frac{1-\sqrt{1-\ln^2 x}}{\ln x}\right| + C$$

5) $\int \sqrt{x^2+x+2}\, dx$

sol: إكمل المربع

$$x^2+x+2 = x^2 + x + \frac{1}{4} + \frac{7}{4} = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4}$$

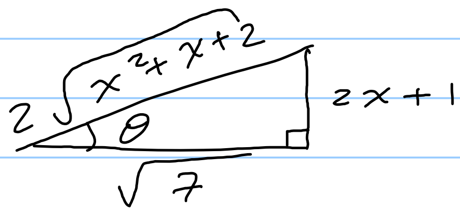
$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}}\, dx \qquad x + \tfrac{1}{2} = \frac{\sqrt{7}}{2}\tan\theta$$

$$dx = \frac{\sqrt{7}}{2}\sec^2\theta \, d\theta$$

$$= \int \frac{\sqrt{7}}{2}\sec\theta \cdot \frac{\sqrt{7}}{2}\sec^2\theta \, d\theta \qquad \sqrt{\left(x+\tfrac{1}{2}\right)^2 + \frac{7}{4}} = \frac{\sqrt{7}}{2}\sec\theta$$

محلول من الأمثلة

$$= \frac{7}{4}\int \sec^3\theta \, d\theta = \frac{7}{4}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right] + C$$



$$= \frac{7}{4}\left[\frac{1}{2} \cdot \frac{2}{\sqrt{7}} \cdot \sqrt{x^2+x+2} \cdot \frac{(2x+1)}{\sqrt{7}} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+x+2}}{\sqrt{7}} + \frac{2x+1}{\sqrt{7}}\right|\right] + C$$

6) $\int e^t \sqrt{\tan^2 e^t + 1}\, dt = \int \sqrt{\tan^2 u + 1}\, du$ $\qquad u = e^t$, $du = e^t\, dt$

$= \int \sqrt{\sec^2 u}\, du = \int |\sec u|\, du$

$= \mp \ln|\sec u + \tan u| + C = \boxed{\mp \ln|\sec e^t + \tan e^t| + C}$

7) $\int \frac{x+1}{x^2(x-1)}\, dx$

$\frac{x+1}{x^2(x-1)} \overset{P.F}{=} \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$\Rightarrow \quad x+1 = A\,x(x-1) + B(x-1) + C x^2.$

at $x=0$: $\quad 1 = -B \Rightarrow \boxed{B = -1}$

at $x=1$: $\quad \boxed{2 = C}$

لايجاد A خذ معامل x^2 في الطرفين مساوياً للصفر

$A + C = 0 \Rightarrow \boxed{A = -2}$

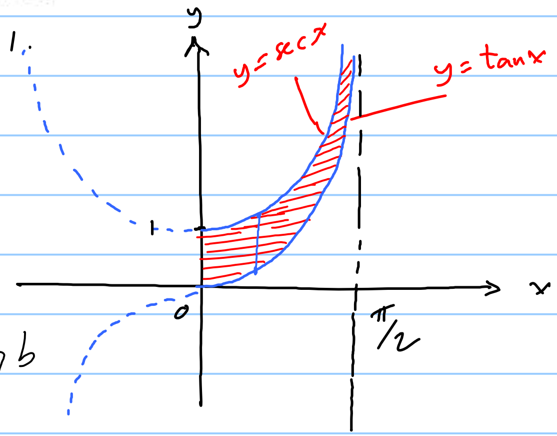
$\therefore \int \frac{x+1}{x^2(x-1)}\, dx = -2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 2\int \frac{dx}{x-1} = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

$= \frac{1}{x} + 2\ln\left|\frac{x-1}{x}\right| + C$

8) Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

Sol:

$\text{Area} = \int_0^{\pi/2} \sec x - \tan x\, dx$



$= \lim_{b \to \frac{\pi}{2}^-} \int_0^b \sec x - \tan x\, dx$

$= \lim_{b \to \frac{\pi}{2}^-} \Big[\ln|\sec x + \tan x| - \ln|\sec x|\Big]_0^b$

$= \lim_{b \to \frac{\pi}{2}^-} \left[\ln\left|\frac{\sec x + \tan x}{\sec x}\right|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \ln|1 + \sin x|\Big]_0^b$

$= \lim_{b \to \frac{\pi}{2}^-} \ln|1 + \sin b| = \boxed{\ln 2}$

9) $\displaystyle\int_0^{\infty} \frac{dx}{(x-1)(x^2+1)}$ (تكامل معتل من النوعين ، حيث
هناك نقطة عدم اتصال لانهائية عند $x=1$)

$$= \int_0^1 \frac{dx}{(x-1)(x^2+1)} + \int_1^2 \frac{dx}{(x-1)(x^2+1)} + \int_2^{\infty} \frac{dx}{(x-1)(x^2+1)}$$

يكون التكامل المعتل الأساسي تقاربياً إذا كان جميع التكاملات الثلاثة تقاربية . بداية ، أوجد التكامل العادي .

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x-1)$$

at $\underline{x=1:}$ $\quad 1 = 2A \Rightarrow \boxed{A=\frac{1}{2}}$ (انعزل)

$$-\tfrac{1}{2}x^2+\tfrac{1}{2} = -\tfrac{1}{2}(x-1)(x+1) = (Bx+C)(x-1)$$

$$\Rightarrow -\tfrac{1}{2}x-\tfrac{1}{2} = Bx+C \Rightarrow B=-\tfrac{1}{2}, \quad C=-\tfrac{1}{2}$$

$$\therefore \int \frac{dx}{(x-1)(x^2+1)} = \int \frac{\frac{1}{2}\,dx}{x-1} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \tfrac{1}{2}\ln|x-1| - \tfrac{1}{4}\ln|x^2+1| - \tfrac{1}{2}\tan^{-1}x + C$$

Now consider

$$\int_0^1 \frac{dx}{(x-1)(x^2+1)} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)(x^2+1)}$$

$$= \lim_{b\to 1^-} \left[\tfrac{1}{2}\ln|x-1| - \tfrac{1}{4}\ln|x^2+1| - \tfrac{1}{2}\tan^{-1}x\right]_0^b$$

$$= \lim_{b\to 1^-} \left[\ln|b-1| - \tfrac{1}{4}\ln|b^2+1| - \tfrac{1}{2}\tan^{-1}b\right] = -\infty \quad \text{which is div.}$$

لذلك ومن دون إكمال بقية التكاملات بأن التكامل الأصلي سيكون تباعدي

$\displaystyle\int_0^{\infty} \frac{dx}{(x-1)(x^2+1)}$ is divergent .

Test for Convergence:

10) $$\int_1^{\infty} \frac{e^{-x}}{1+x}\, dx$$

واضح أن التكامل صعب / ولتكوين انطباع عن كون التكامل تقاربي أم تباعدي لاحظ ما يلي:

$$\frac{e^{-x}}{1+x} = \frac{1}{e^{x}(1+x)}$$

ولأن الدالة e^{x} أسرع في النمو من الدالة x^2 يمكن تكوين انطباع أن التكامل تقاربي

$$\frac{e^{-x}}{1+x} < ??$$

Note that for $x \in [1,\infty)$, $1+x > 1 \Longrightarrow$

$$\frac{1}{1+x} < 1 \Longrightarrow \frac{e^{-x}}{1+x} < e^{-x} = \frac{1}{e^{x}}.$$

Consider $\displaystyle\int_1^{\infty} e^{-x}\, dx = \lim_{b\to\infty} \frac{e^{-x}}{-1}\Big]_1^b$

$$= \lim_{b\to\infty}\left[-e^{-b} + e^{-1}\right] = \frac{1}{e} \quad \text{converges.}$$

So by DCT, $\displaystyle\int_1^{\infty} \frac{e^{-x}}{1+x}\, dx$ converges

11) $$\int_1^{\infty} \frac{dx}{x^2(1+e^{-x})}$$

Sol: Let $f(x) = \dfrac{1}{x^2(1+e^{-x})}$, $g(x) = \dfrac{1}{x^2}$.

$\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ is conv. (p-integral, $p=2$)

$\displaystyle\lim_{x\to\infty} \frac{f}{g} = \lim_{x\to\infty} \frac{x^2}{x^2(1+e^{-x})} = 1 \Longrightarrow$ By LCT,

$\displaystyle\int_1^{\infty} \frac{dx}{x^2(1+e^{-x})}$ is convergent.

12) Evaluate $\int \frac{dx}{1+\sqrt{x}}$

Sol: $u = \sqrt{x} + 1 \Rightarrow x = (u-1)^2 = u^2 - 2u + 1$

$dx = (2u-2)\,du$

$$\therefore \int \frac{dx}{1+\sqrt{x}} = \int \frac{2u-2}{u}\,du = \int 2 - \frac{2}{u}\,du = 2u - 2\ln|u| + C$$

$$= 2\sqrt{x} + 2 - 2\ln(\sqrt{x}+1) + C = 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$$

13) $$\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$$

$$= \int \frac{\cancel{(x-2)^2}\tan^{-1}(2x)}{(4x^2+1)\cancel{(x-2)^2}}\,dx - 3\int \frac{x\,\cancel{(4x^2+1)}}{\cancel{(4x^2+1)}(x-1)^2}\,dx$$

$$= \int \frac{\tan^{-1}(2x)}{4x^2+1}\,dx - 3\int \frac{x}{(x-1)^2}\,dx$$ P.F.

$\int u \cdot du$

أكمل الحل

End of Ch. 8