

Complex number

→ Counting numbers \equiv natural numbers
 $\{1, 2, 3, 4, \dots\}$ (\mathbb{N})

→ Integers $\equiv \{0, \pm 1, \pm 2, \pm 3\}$ (\mathbb{Z})

Close operation:
 العمليات المغلقة: الجمع، الطرح، الضرب، القسمة

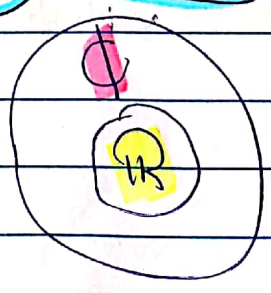
→ Rational numbers $\mathbb{Q} = \left\{ x = \frac{a}{b}, a, b \in \mathbb{Z} \right\}$
 Example: $\frac{2}{3} = \frac{6}{9}$

→ The Real numbers \mathbb{R} "Rational + Non Rational"
 Example: $\sqrt{2}, \pi$



$\sqrt{-4} \Rightarrow$ complex numbers \mathbb{C}

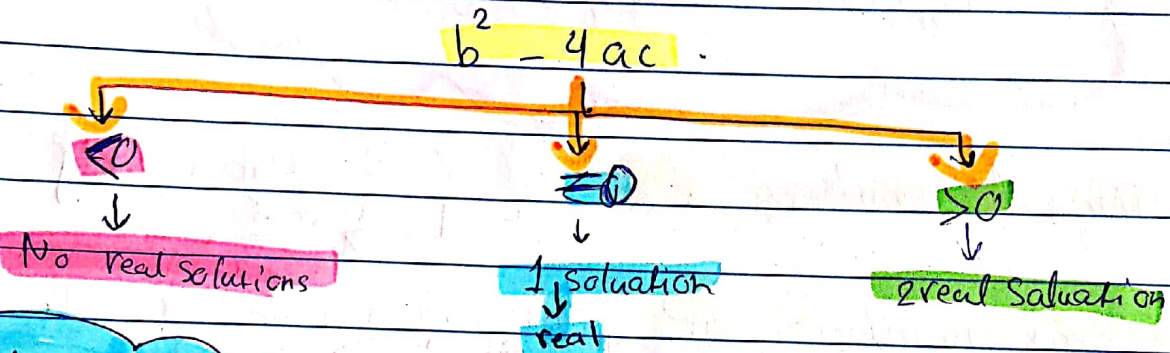
→ The Complex numbers is the set of all numbers of the form $z = a + ib$, a, b are real number
 $i^2 = -1$, $i = \sqrt{-1}$



Solving Equation

$x+a=0 \rightarrow \mathbb{Z}$ "the solution from \mathbb{Z} "
 $ax+b=0 \rightarrow \mathbb{Q}$ "the solution from \mathbb{Q} "
 $ax^2+bx+c=0 \Rightarrow$ "Quadratic"

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

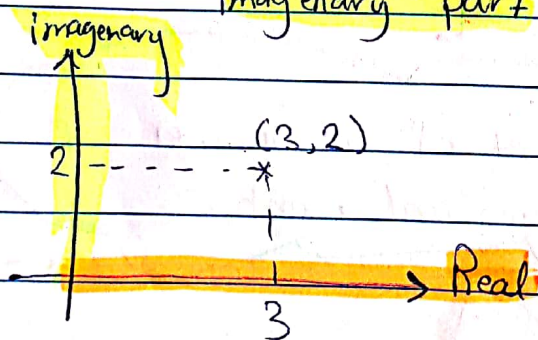


z: Complex number

$z = a + ib \rightarrow a = \text{the real part}$
 $b = \text{the imaginary part}$
 $i^2 = -1 \rightarrow \sqrt{-1} = i$

$$\mathbb{C} = \{ z : z = a + ib, a, b \in \mathbb{R} \}$$

real part $\rightarrow (a, b)$ order pair (a, b) \rightarrow imaginary part



Ex: $3 + 2i$

Notes:-

$$1 \quad a_1 + b_1 i = a_2 + b_2 i$$

$$(a_1, b_1) = (a_2, b_2)$$

$$\text{if } a_1 = a_2$$

$$b_1 = b_2$$

$$\text{Ex: } 2 + 3i = 2 + xi$$

$$\Rightarrow \boxed{x = 3}$$

$$2 \quad z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$(a_1 + b_1 i) + (a_2 + b_2 i)$$

$$(a_1 + a_2) + (b_1 + b_2)i$$

$$\text{Ex: } (2 - 3i) + (4 + i) =$$

$$\boxed{6 - 2i}$$

$$3 \quad z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$= a_1 a_2 + a_1 b_2 i + i b_1 a_2 + i^2 b_1 b_2$$

$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$$

$$\text{Ex: } (2 + 3i)(1 - 2i) =$$

$$2 - 4i + 3i + 6$$

$$= 8 - i$$

4. if $z = a + ib$, then the complex conjugate of z is $\bar{z} = a - ib$

$$\text{Ex: } z = 2 - 3i$$

$$\Rightarrow \bar{z} = 2 + 3i$$

$$\text{Ex: } \frac{2+i}{1-i} \times \frac{(1+i)}{(1+i)}$$

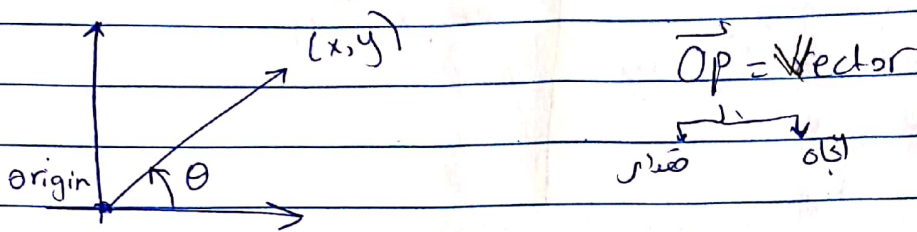
$$\frac{(2-1) + (2+1)i}{(1-1) + (1+1)} = \frac{1+3i}{2} = \frac{1}{2} + \frac{3i}{2}$$

$$\text{Ex: } \frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \times \frac{(a_2 - ib_2)}{(a_2 - ib_2)}$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{\bar{z}_2}{\bar{z}_1}$$

Argan Diagrams

$$Z = x + iy \equiv (x, y)$$



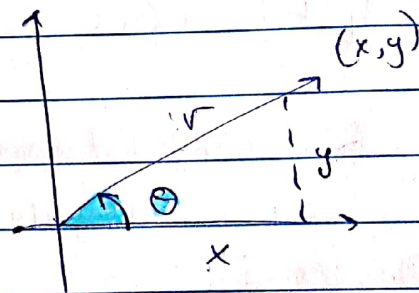
$\theta = \text{argument of } z \text{ (arg } z)$
Absolute Value = $|z| = |\vec{OP}|$

$$|z| = \sqrt{x^2 + y^2} \Rightarrow r$$

$$\tan \theta = \frac{y}{x} \Rightarrow \theta$$

$$\cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta$$

$$\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta$$



$$z = x + iy$$

$$z = (r \cos \theta + i r \sin \theta)$$

$$z = r (\cos \theta + i \sin \theta)$$

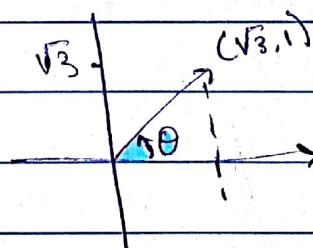
Ex. $z = 1 + \sqrt{3}i$

$$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\tan \theta = \frac{\sqrt{3}}{1} \rightarrow \theta = 60^\circ$$

$$z = 2 (\cos 60^\circ + i \sin 60^\circ)$$

$$\Rightarrow z = 2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$



Note: If $z = x + iy$

$$\bar{z} = x - iy$$

$$\text{then } z \cdot \bar{z} = (x + iy)(x - iy)$$

$$\rightarrow x^2 + y^2$$

$$\rightarrow |z|^2$$

$$\Rightarrow z \cdot \bar{z} = |z|^2$$

Euler's formula :-

$$\rightarrow e^{i\theta} = \cos\theta + i \sin\theta$$

$$\text{Ex: } e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4}$$

$$\text{If } z = x + iy \Rightarrow z = r [\cos\theta + i \sin\theta]$$

$$\Rightarrow z = re^{i\theta}$$

$$\text{Ex: } z = 1 - \sqrt{3}i$$

$$r = 2$$

$$\theta = -60 = -\frac{\pi}{3}$$

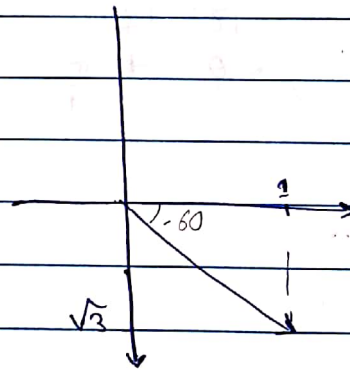
$$z = 2 \left(\cos\frac{-\pi}{3} + i \sin\frac{-\pi}{3} \right)$$

$$= 2 \left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3} \right)$$

$$\Rightarrow z = 2e^{i\frac{-\pi}{3}}$$

$$z = 2e^{i\frac{-\pi}{3}}$$

\downarrow	LP	US
LP	$\sin\theta(+)$ $\cos\theta(-)$ $\tan(-)$	$\sin\theta(+)$ $\cos\theta(+)$ $\tan\theta(+)$
US	$\sin\theta(-)$ $\cos\theta(-)$ $\tan\theta(+)$	$\sin\theta(-)$ $\cos\theta(+)$ $\tan\theta(-)$
\downarrow	LP	LP



* I f $z_1 = x_1 + iy_1 \rightarrow z_1 = r_1 e^{i\theta_1}$

$z_2 = x_2 + iy_2 \rightarrow z_2 = r_2 e^{i\theta_2}$

?

1 then $z_1 \cdot z_2 = (r_1 e^{i\theta_1}) \times (r_2 e^{i\theta_2})$

$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$

2 $\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$

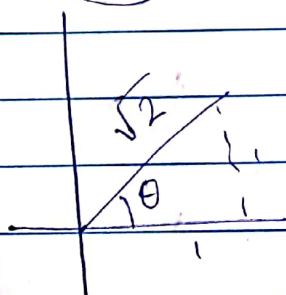
3 $z = x + iy = r e^{i\theta}$
 $z^n = (r e^{i\theta})^n$

$z^n = r^n e^{i n \theta}$

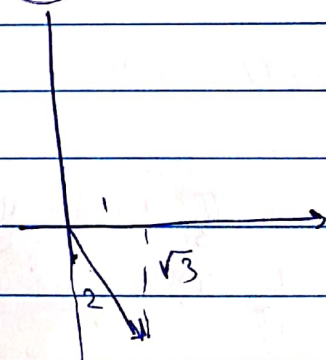
Ex: $z_1 = 1 + i$

$z_2 = \sqrt{3} - i$

$z_1 \rightarrow |z_1| = \sqrt{2}$
 $\theta_1 = 45 = \frac{\pi}{4}$



$z_2 \rightarrow r = 2$
 $\theta_2 = -\frac{\pi}{6}$



$\rightarrow z_1 \cdot z_2 = (\sqrt{2} e^{i\frac{\pi}{4}}) (2 e^{-i\frac{\pi}{6}})$
 $= 2\sqrt{2} e^{i(\frac{\pi}{4} - \frac{\pi}{6})} = 2\sqrt{2} e^{i\frac{\pi}{12}}$

$\rightarrow \frac{z_1}{z_2} = \frac{\sqrt{2} e^{i\frac{\pi}{4}}}{2 e^{i\frac{\pi}{6}}} = \frac{\sqrt{2}}{2} e^{i(\frac{5\pi}{12})}$

$\rightarrow z_2^6 = 2^6 (e^{-i\frac{\pi}{6}})^6 = 64 \times e^{-i\pi}$
 $= 64 [\cos -\pi + i \sin -\pi]$
 $= 64 [-1 + i(0)] = -64$

De Moivre's Theorem

$$(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$$

$$\text{Ex: } 1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$$

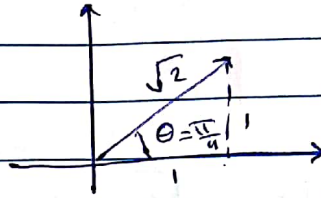
$$r = \sqrt{2}, \quad \theta = \frac{\pi}{4}$$

$$z = \sqrt{2} [\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}]$$

$$z^{10} = (\sqrt{2} e^{i\frac{\pi}{4}})^{10}$$

$$= 32 e^{i\frac{10\pi}{4}}$$

$$= 32 [\cos \frac{5\pi}{2} + i \sin \frac{5\pi}{2}]$$



* Note:

when you solve a problem in this section ($\sin \theta$ & $\cos \theta$) always prove together.

Complex Roots

$$\sqrt[n]{z} = w \Leftrightarrow w^n = z$$

$$\Rightarrow \rho^n e^{in\alpha} = r e^{i\theta}$$

$$\rho^n = r \Rightarrow \rho = \sqrt[n]{r}$$

$$\Rightarrow n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}$$

$k = 0, +1, \dots$

$$k=0 \Rightarrow w_0 \Rightarrow \alpha = \frac{\theta}{n}$$

$$k=0 \Rightarrow w_0 \Rightarrow \alpha = \frac{\theta}{n} \Rightarrow w_0 = \sqrt[n]{r} e^{i(\frac{\theta}{n})}$$

$$k=1 \Rightarrow w_1 \Rightarrow w_1 = \sqrt[n]{r} e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})}$$

$$w_{k-1} = \sqrt[n]{r} \exp(i(\frac{\theta}{n} + \frac{2k\pi}{n}))$$

* Note:

$$\exp = e$$

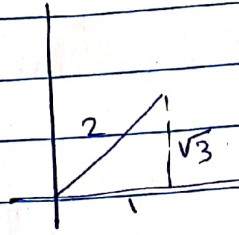
$$\exp(i\theta) = e^{i\theta}$$

$$k = 0, \dots, n-1$$

Ex. Find the 2 square roots of $z = 1 + \sqrt{3}i$

$$r = |z| = 2$$

$$\theta = \frac{\pi}{3}$$



$$\sqrt{z} = \sqrt{2e^{i\frac{\pi}{3}}}$$

$$w_k = \sqrt{r} \exp i \left(\frac{\theta}{n} + \frac{2k\pi}{n} \right)$$

$$w_k = \sqrt{r} \exp i \left(\frac{\theta}{n} + k\pi \right) \quad k = 0, 1$$

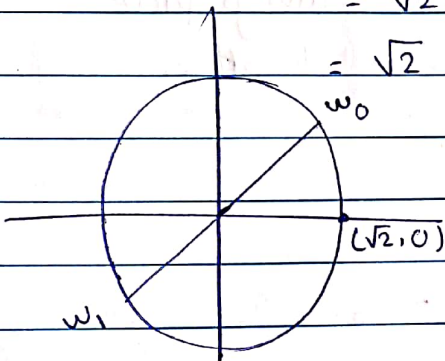
$$k=0 \Rightarrow w_0 = \sqrt{2} \exp i \left(\frac{\pi}{6} \right) = \sqrt{2} \left[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right] \\ = \sqrt{2} \left[\frac{\sqrt{3}}{2} + i \frac{1}{2} \right]$$

$$k=1 \Rightarrow w_1 = \sqrt{2} \exp i \left(\frac{\pi}{6} + \pi \right)$$

$$= \sqrt{2} \exp i \left(\frac{7\pi}{6} \right)$$

$$= \sqrt{2} \left[\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right]$$

$$= \sqrt{2} \left[-\frac{\sqrt{3}}{2} - i \frac{1}{2} \right]$$



The fundamental theorem

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0, \quad a_n \neq 0$$

$p(z)$ = Complex polynomial of degree

$p(z) = 0$ has exactly "n" roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* Note: the number of roots equal

(n)

Ch. 1: Functions

1.1 Def: A function from a set D to a set B is a rule that assigns a unique element $y \in B$ for each element $x \in D$.

$$y = f(x) \rightarrow f: D \rightarrow B$$

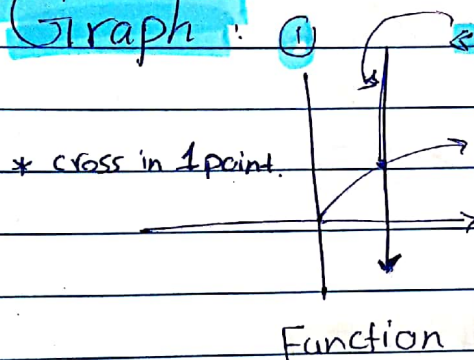
$$(x) \rightarrow y = f(x)$$

* inputs \rightarrow The domain $\rightarrow x \rightarrow$ independent

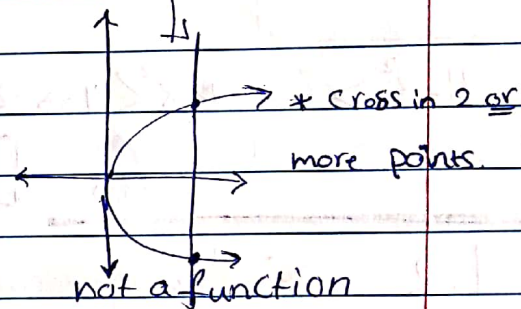
* outputs \rightarrow The range $\rightarrow y \rightarrow$ dependent

$$y = f(x)$$

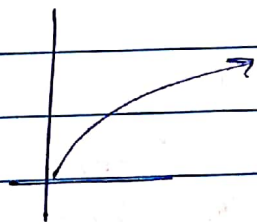
Graph:



vertical line test



①



$$f(x) = \sqrt{x}$$

$$D: x \geq 0$$

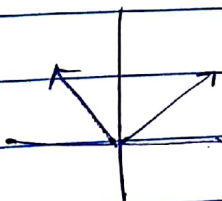
$$R: [0, \infty) : y \geq 0$$

②

$f(x) = |x|$ Absolute value

$$f(x) = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$$-x ; x < 0$$



$$D = \mathbb{R}$$

$$R = [0, \infty)$$

Note $\sqrt{x^2} = |x|$

1.2 \Rightarrow Trigonometric Functions

① $f(x) = \sin x$

② $f(x) = \cos x$

③ $f(x) = \tan x = \frac{\sin x}{\cos x}$

④ $\csc x = \frac{1}{\sin x}$

⑤ $\sec x = \frac{1}{\cos x}$

⑥ $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

Remark :

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

$$\sec(x + 2\pi) = \sec x$$

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periodic function

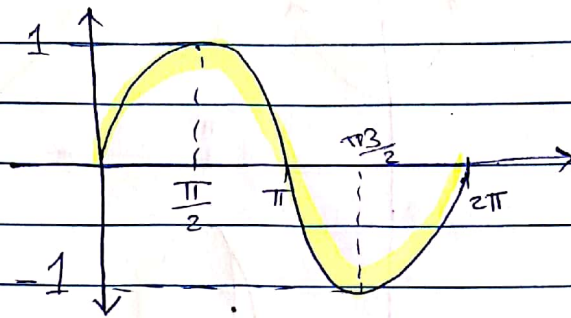
* Def: the function $f(x)$ is a periodic function with period p $f(x) + p$.

→ $f(x) = \sin x$

$D = (-\infty, \infty)$

$R = [-1, 1]$

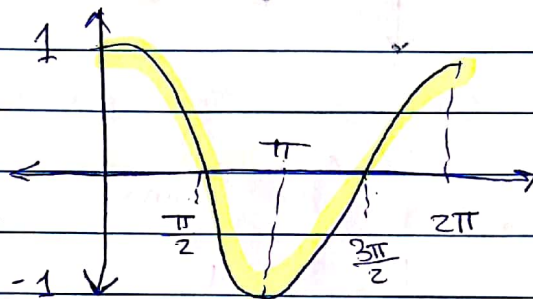
* $\sin x$ is periodic with $\{2\pi\}$



→ $f(x) = \cos x$

$D = (-\infty, \infty)$

$R = [-1, 1]$



→ $f(x) = \tan x$

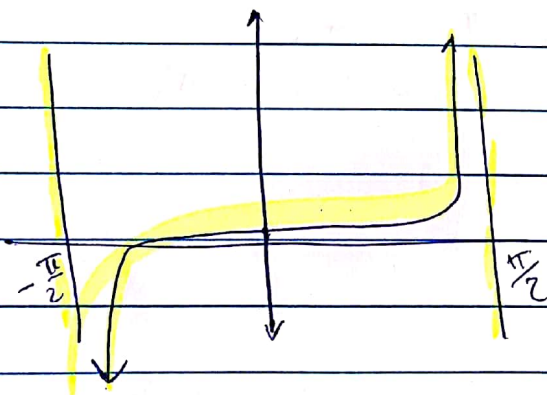
$\tan x = \frac{\sin x}{\cos x}$

⇒ $D: \cos \neq 0$

$x \notin \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

$R = (-\infty, \infty)$

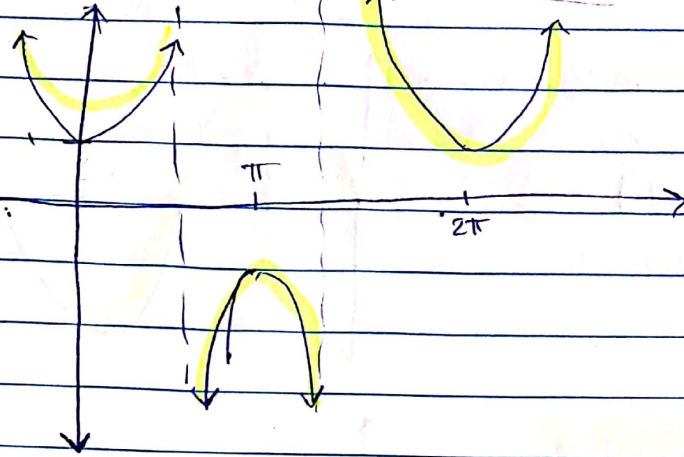
period = π



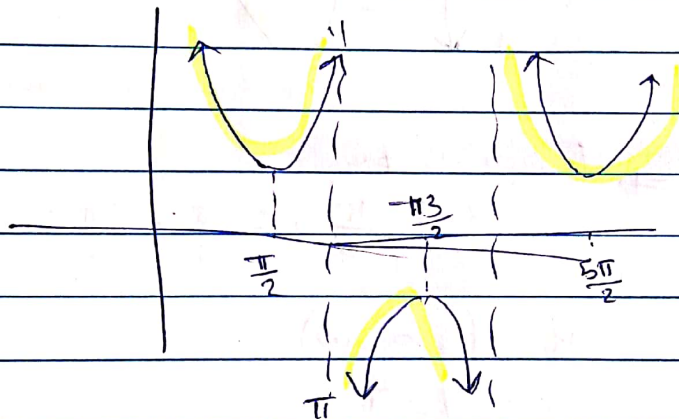
Nbts: the most ugly Graph I have drawn it

④ $y = \sec x = \frac{1}{\cos x}$

D: $\cos \theta \neq 0 \Rightarrow \theta \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}$
 R: $\mathbb{R} \setminus (-1, 1)$ $\theta = \frac{\pi}{2} + n\pi, n=0, +1, +2, \dots$



⑤ $y = \csc x = \frac{1}{\sin x}$

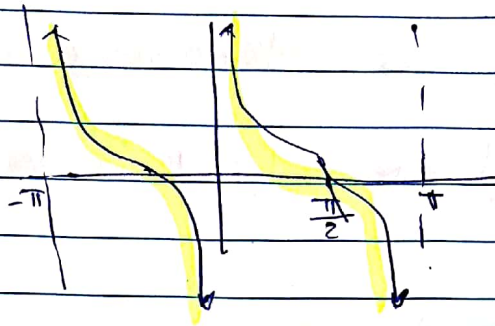


D: $\sin x \neq 0 \Rightarrow x \neq 0, \pm \pi, \pm 2\pi, \dots$
 R: $\mathbb{R} \setminus (-1, 1)$ $\mathbb{R} \setminus (-1, 1)$

$$y = \cot x = \frac{\cos x}{\sin x}$$

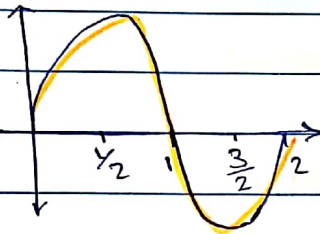
Domain: $\sin x \neq 0 \Rightarrow x \neq 0, \pm\pi, \pm 2\pi$

R: $(-\infty, \infty)$



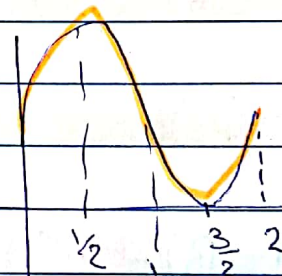
Ex: Graph $y = \sin \pi x$

$$\frac{2\pi}{\pi} = 2$$



* period = 2π
x-axis
Cos
Sin

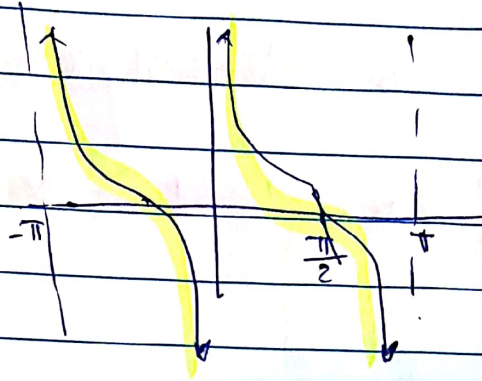
Ex: $y = \sin(\pi x) + 1$



* Period = π
x-axis
tan

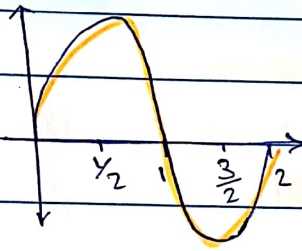
④ $y = \cot x = \frac{\cos x}{\sin x}$

Domain : $\sin x \neq 0 \Rightarrow x \neq 0, \pm\pi, \pm 2\pi$
 R : $(-\infty, \infty)$



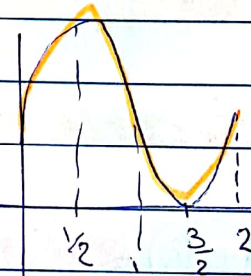
Ex: Graph $y = \sin \pi x$

$\frac{2\pi}{\pi} = 2$



* period = 2π
 x does Cos Sin

Ex: $y = \sin(\pi x) + 1$



* Period = π
 x does tan

Trigonometric Identities

$$(1) \sin^2 x + \cos^2 x = 1$$

$$\frac{s^2}{s^2} + \frac{c^2}{s^2} = \frac{1}{s^2} \Rightarrow 1 + \cot^2 x = \csc^2 x$$

$$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2} \Rightarrow 1 + \tan^2 x = \sec^2 x$$

$$(2) \sin 2x = 2 (\cos x) (\sin x)$$

$$(3) \cos 2x = \cos^2 x - \sin^2 x$$

$$(4) \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$(5) \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$(6) \sin (A+B) = \sin A \cos B + \sin B \cos A$$

$$(7) \cos (A+B) = \cos A \cos B - \sin A \sin B$$

$$(8) \sin x + \frac{\pi}{2} = \cos x$$

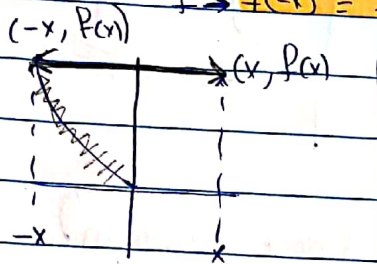
$$\text{Example: } \sin(x + \frac{\pi}{2}) = \sin x \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos x \\ = \cos x$$

$$\text{Prove } \sin(x + \frac{\pi}{2}) = \cos x$$

1.3 \Rightarrow Even and odd functions

\Rightarrow Def: $f(x)$ is an even function

if $f(-x) = f(x)$ symmetry about y-axis.



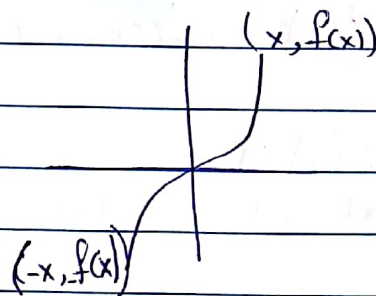
$$f(x) = x^2$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

an even function

$\Rightarrow f(x)$ is an odd function

if $f(-x) = -f(x)$



\Rightarrow If $f(-x) \neq f(x) \neq -f(x)$ then $f(x)$ is neither even nor odd.

Ex: $f(x) = \frac{x}{x^2+1} \Rightarrow f(-x) = \frac{-x}{x^2+1} = -f(x)$ odd

Ex: $f(x) = x^3 + x^2 + 1$
 $f(-x) = -x^3 + x^2 + 1 \neq f(x)$ neither even nor odd
 $\neq -f(x)$

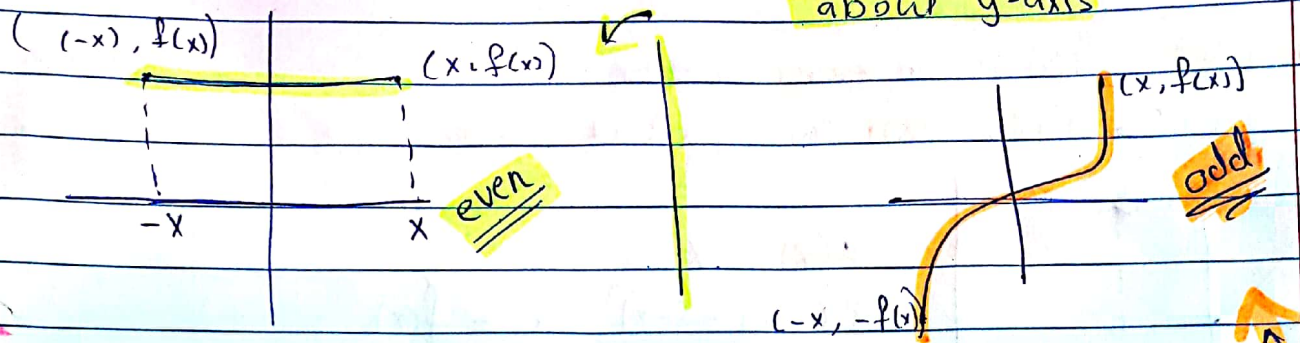
Ex: $y = \cos x \Rightarrow f(x) = f(-x)$ even

Ex: $y = \sin x \Rightarrow f(x) = -f(-x)$ odd

Ex: $\tan x = \frac{\sin x \rightarrow \text{odd}}{\cos x \rightarrow \text{even}}$ odd

Ex: $|x|$ even

* Note : $f(-x) = f(x) \Rightarrow$ even \Rightarrow Symmetry about y-axis



* $f(-x) = -f(x) \Rightarrow$ odd \Rightarrow Symmetry about the origin

* $f(-x) \neq f(x) \neq -f(x) \Rightarrow$ neither even nor odd.

Ex. $f(x) = |x|$
 $f(-x) = |-x| = |x| = f(x) \Rightarrow$ even

Ex. ① If $f(x)$ is even, $g(x)$ is odd?
 $(f \cdot g)$ is $f \cdot g \Rightarrow (f \cdot g)(-x) = f(-x) \cdot g(-x)$
 $= f(x) \cdot -g(x)$
 $= -f(x) \cdot g(x)$
 $= -f(x) \cdot g(x)$ odd

② $f \circ g(x)$?
 $f(g(-x)) = f(-g(x)) = -f(g(x)) = -f \circ g(x)$
even

Limits and Continuity

11 \Rightarrow Note: ① If $p(x)$ is a polynomial
then $\lim_{x \rightarrow x_0} p(x) = p(x_0)$

Ex: $\lim_{x \rightarrow -1} x^3 + 2x^2 + 10 = 11$

* If $R(x) = \frac{p(x)}{q(x)} \rightarrow q(x) \neq 0$
Rational function

then $\lim_{x \rightarrow x_0} R(x) = \frac{p(x_0)}{q(x_0)} \rightarrow q(x_0) \neq 0$

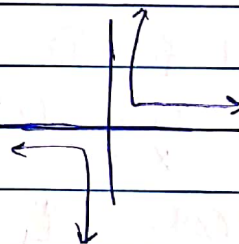
Ex: $\lim_{x \rightarrow -1} \frac{x+2}{x-1} = -\frac{1}{2}$

Ex: $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = 4$

Indeterminate form

Ex: $f(x) = \frac{1}{x}$

$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$



$\lim_{x \rightarrow 0^+} f(x) = \frac{1}{0^+} = +\infty$

$\lim_{x \rightarrow 0^-} f(x) = \frac{1}{0^-} = -\infty$

Ex: $\lim_{x \rightarrow -1} \frac{1}{1+x}$

$\lim_{x \rightarrow -1^+} \frac{1}{1+x} = \frac{1}{0^+} = +\infty$

$\lim_{x \rightarrow -1^-} \frac{1}{1+x} = \frac{1}{0^-} = -\infty$

Ex:
$$\lim_{x \rightarrow -1} \frac{(\sqrt{x^2 + 8} - 3) \times (\sqrt{x^2 + 8} + 3)}{x + 1}$$

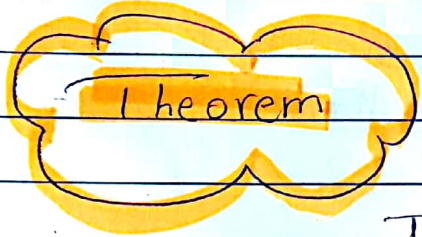
$$\lim_{x \rightarrow -1} \frac{x^2 + 8 - 9}{x + 1} = \lim_{x \rightarrow -1} \frac{(x-1)(x+1)}{(x-1)(\sqrt{x^2 + 8} + 3)}$$

$$= \frac{-2}{6} = -\frac{1}{3}$$



$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Ex:
$$\lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} = 2$$

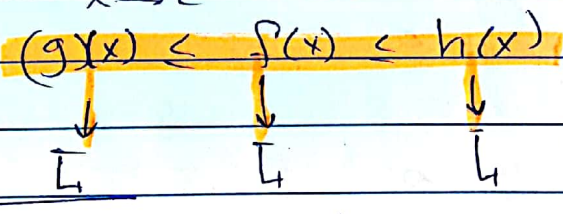


Theorem The sandwich theorem

If ① $g(x) \leq f(x) \leq h(x)$, for all x

$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$ in an open interval containing:

then $\lim_{x \rightarrow c} f(x) = L$



Ex: show that $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$

$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$

$x / (-1 \leq \sin x \leq 1)$

$$\lim_{x \rightarrow \infty} \left(\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x} \right)$$

↓ ↓ ↓ ⇒ $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$

zero zero zero

$$\text{Ex: } \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} \times \frac{1 + \sqrt{x}}{1 + \sqrt{x}} = \frac{1}{2}$$

$$\text{Ex: } \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta} \times \frac{(1 + \cos \theta)}{(1 + \cos \theta)}$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{2 \sin \theta \cos \theta} \times \frac{(1 + \cos \theta)}{(1 + \cos \theta)}$$

$$\lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{2 \sin \theta \cos \theta} = \frac{0}{4} = 0$$

$$(j) \lim_{t \rightarrow 3^+} \frac{\lfloor t \rfloor}{t}$$

$$\lim_{t \rightarrow 3^-} \frac{\lfloor t \rfloor}{t} = \frac{3}{3} = 1$$

$$\lim_{t \rightarrow 3^+} \frac{\lfloor t \rfloor}{t} = \frac{2}{3}$$

Example

$$\lfloor 1.7 \rfloor = 1$$

$$\lceil 1.7 \rceil = 2$$

2.2 \Rightarrow

Continuity

* **Def.** A function $f(x)$ is continuous at a point $x = x_0$ if:

1) $f(x_0)$ exists.

2) $\lim_{x \rightarrow x_0} f(x) = \text{exists}$

$$\lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

3) $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

* **Not continuous \equiv Discontinuous.**

Ex: $f(x) = \frac{|x|}{x}$ $x \neq 0$

$$\rightarrow \lim_{x \rightarrow 0^+} f(x) = 1$$

$$\rightarrow \lim_{x \rightarrow 0^-} f(x) = -1$$

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

* **Note:** $f(x)$ is a continuous function if it is call at each point of its domain.

* $f(x)$ is defined on $[a, b]$

$$a < c < b \Rightarrow \lim_{x \rightarrow c} f(x) = f(c)$$

$$x = b \Rightarrow \lim_{x \rightarrow b^-} f(x) = f(b)$$

$$x = a \Rightarrow \lim_{x \rightarrow a^+} f(x) = f(a)$$

* **Note:** (i) **Polynomial Continuous.**

$$\lim_{x \rightarrow c} p(x) = p(c)$$

② Rational function:

$$R(x) = \frac{p(x)}{q(x)} \text{ cont. for all such } q(x) \neq 0$$

③ $\sin x, \cos x, |x|$ cont for all x

$$\Rightarrow \tan x = \frac{\sin x}{\cos x}, \cos x \neq 0$$

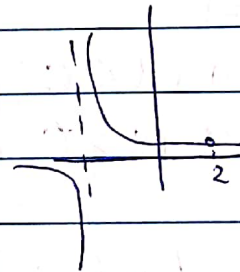
$$x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \quad \left(\frac{\pi}{2} + n\pi, n=0,1,2 \right)$$

Ex: $f(x) = \frac{x-2}{x^2-4} \quad x \neq 2, -2$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \frac{1}{4}$$

$$\frac{x-2}{x^2-4} + \frac{1}{x+2} \Rightarrow \begin{matrix} f(x) = g(x) \\ D(f) \neq D(g) \end{matrix}$$

$$F(x) = \begin{cases} \frac{x-2}{x^2-4}, & x \neq 2 \\ \frac{1}{4}, & x = 2 \end{cases}$$

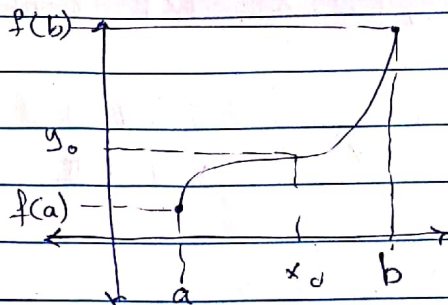


$F(x)$ continuous
at $x=2$

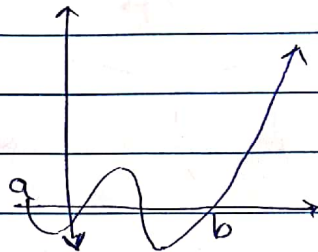
Removable discontinuity.

Theorem: The Intermediate value (I.V.P)

* If $f(x)$ is continuous on $[a, b]$, and y_0 is a value between $f(a)$ and $f(b)$, then there is $x_0 \in [a, b]$ such that $f(x_0) = y_0$.



* Note. If $f(x)$ is cont. on $[a, b]$ and $f(a) * f(b) < 0$, then $f(x) = 0$ has at least one solution on $[a, b]$.



* Ex: $f(x) = x^3 + x^2 + 1$ on $[-2, 2]$

$f(-2) = -8 + 4 + 1 < 0$

$f(2) = 8 + 4 + 1 > 0$

$f(-2) * f(2) < 0$

There is at least $c \in (-2, 2)$ such that $f(c) = 0$

* $R(x) = \frac{p(x)}{q(x)}$, $\lim_{x \rightarrow \pm\infty} R(x) = \pm\infty$

- \Rightarrow degree $p(x) <$ degree $q(x) \Rightarrow 0$
- degree $p(x) =$ degree $q(x) \Rightarrow$ Constant
- degree $p(x) >$ degree $q(x) \Rightarrow \pm\infty$

$$\lim_{x \rightarrow \infty} f(x) = 0 \quad \text{or} \quad \lim_{x \rightarrow a^+} f(x) = +\infty$$

isn't important

Ex: ① $\lim_{x \rightarrow \infty} \frac{2x+1}{x-1} = 2$

② $\lim_{x \rightarrow \infty} \frac{x}{x^2+5x+10} = \text{zero}$

③ $\lim_{x \rightarrow \infty} \frac{x^3+1}{x^2+5} = +\infty$

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Rational function \Rightarrow Graph

$$R(x) = \frac{p(x)}{q(x)}, \quad q(x) = 0$$

isn't important

$$\times \lim_{x \rightarrow a^+} R(x)$$

$$\times \lim_{x \rightarrow \infty} R(x)$$

Asymptotes Lines

1 → Def: the line $y=b$

is a horizontal asymptote of the curve $y=f(x)$ if either,

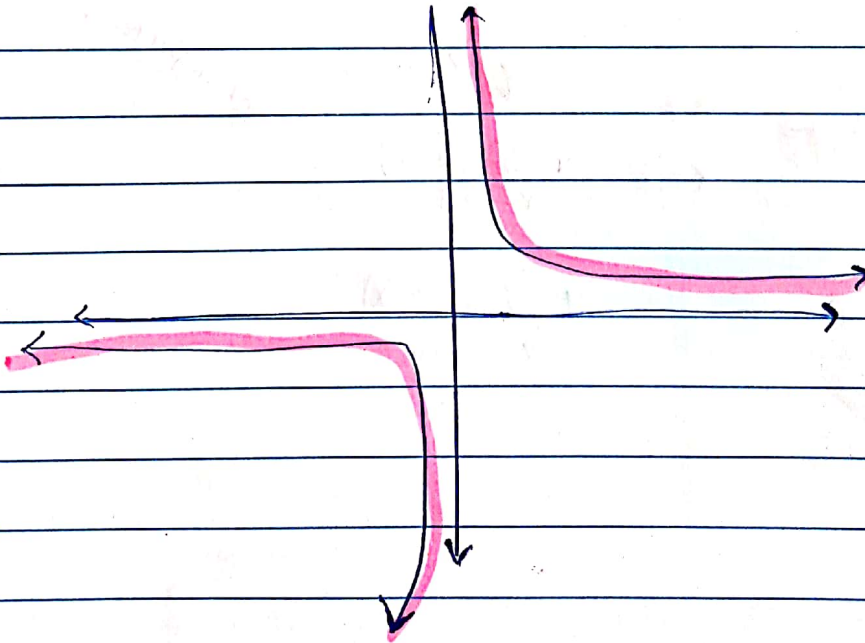
$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

Ex: $f(x) = \frac{1}{x} \Rightarrow \lim_{x \rightarrow \infty} f(x) = 0$

$$\lim_{x \rightarrow -\infty} f(x) = 0$$

x-axis

⇒ $y=0$ is a horizontal asymptote



2) Def: The line $x = a$, is

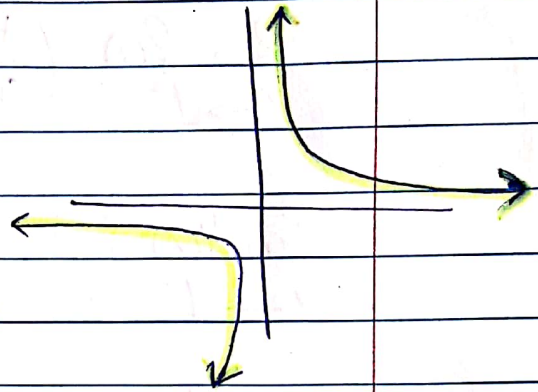
a vertical asymptote for $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm \infty$$

Ex: $f(x) = \frac{1}{x}$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \left(\frac{1}{0^+}\right) = +\infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = \left(\frac{1}{0^-}\right) = -\infty$$



$x = 0$ (y-axis) is a vertical asymptote

Ex: $f(x) = \frac{\sin x}{x}$

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 \Rightarrow y = 0 \text{ H. Asy}$$

Sandwich theorem

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \Rightarrow \text{No vertical asymptotes.}$$

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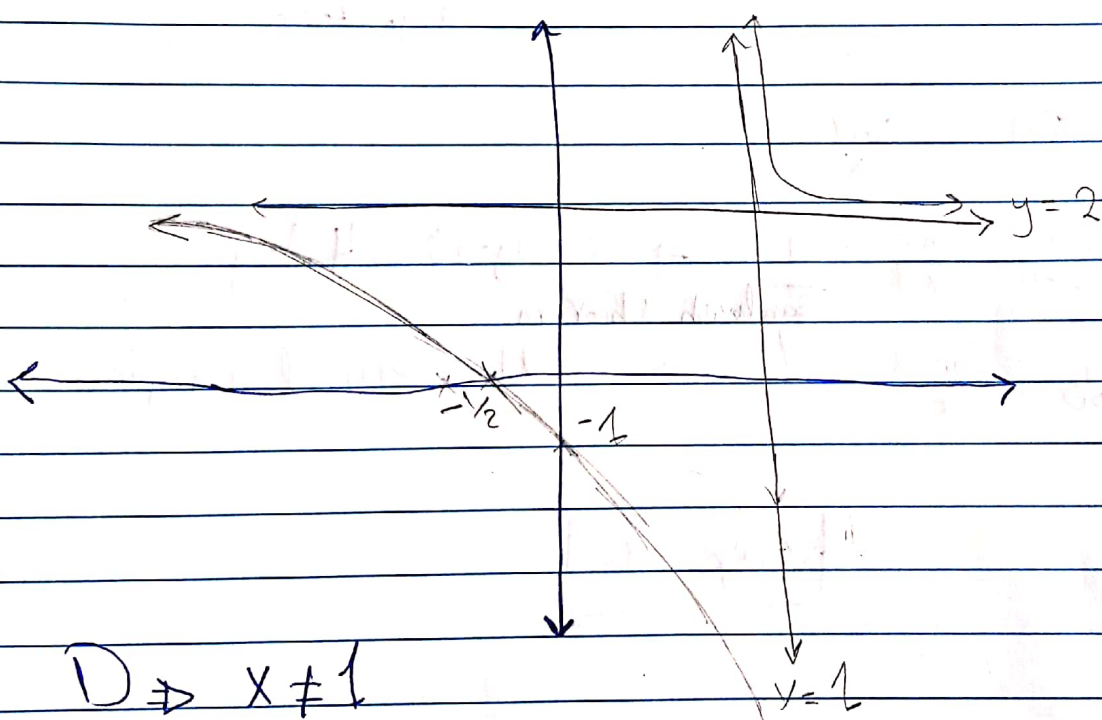
Ex: $f(x) = \frac{2x+1}{x-1}$

$$\lim_{x \rightarrow -\infty} f(x) = 2 \rightarrow y = 2 \text{ H. Asy}$$

$$\lim_{x \rightarrow 1^+} f(x) \left(\frac{+}{0^-} \right) = +\infty$$

$$\lim_{x \rightarrow 1^-} f(x) \left(\frac{+}{0^-} \right) = -\infty$$

$\Rightarrow x = 1$ Vertical esy



D $\Rightarrow x \neq 1$

R $\Rightarrow y \neq 2$

Ex: $f(x) = \frac{x^2 + x}{x^2 - 1} \quad y \neq -1, 1$

$$\frac{x(x+1)}{(x+1)(x-1)} = \frac{x}{x-1}$$

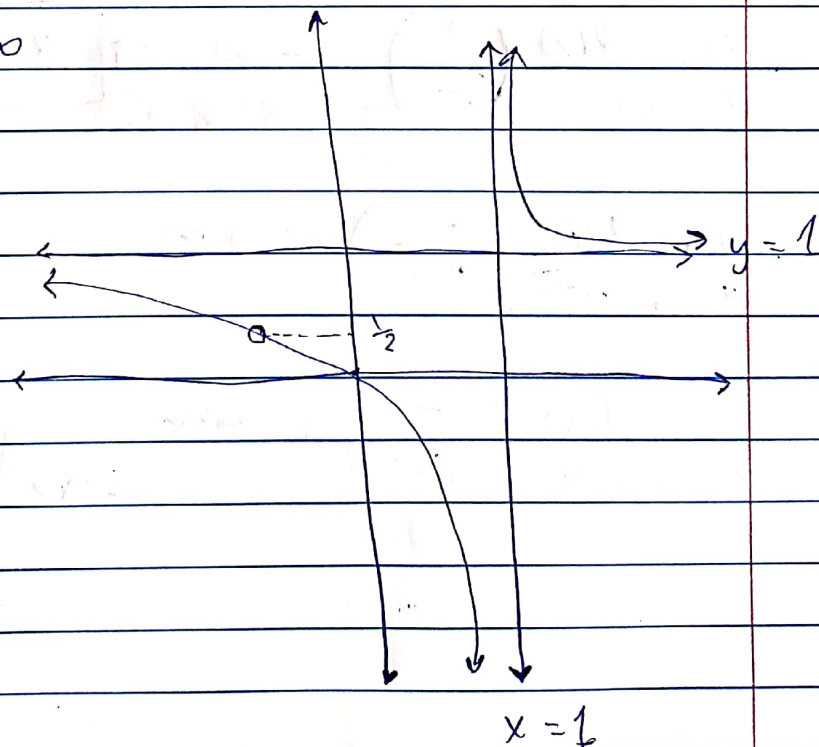
$$\lim_{x \rightarrow -1} f(x) = \frac{-1}{-2} = \frac{1}{2}$$

No vertical asy at $x = -1$

$$\lim_{x \rightarrow \pm\infty} f(x) = 1 \Rightarrow y = 1 \text{ H. Asy}$$

$$\lim_{x \rightarrow 1^+} f(x) = +\infty \Rightarrow x = 1 \text{ V. Asy}$$

$$\lim_{x \rightarrow 1^-} f(x) = -\infty$$



Ex: $f(x) = \frac{x}{x^2-4}$

$$f(-x) = \frac{-x}{x^2-4} = -f(x) \quad \underline{\text{odd}}$$

$$\left. \begin{array}{l} y\text{-int: } x=0 \Rightarrow y=0 \\ x\text{-int: } y=0 \Rightarrow x=0 \end{array} \right\} \Rightarrow (0,0)$$

$$\lim_{x \rightarrow \infty} f(x) = 0, \quad \lim_{x \rightarrow -\infty} f(x) = 0$$

$$\Rightarrow \boxed{y=0} \quad \text{H.A}$$

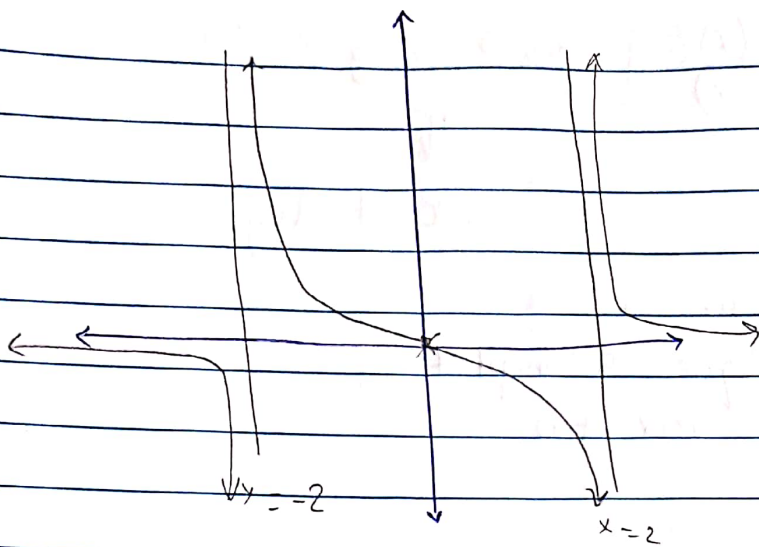
$$\lim_{x \rightarrow 2^+} f(x) \left(\frac{+}{0^+} \right) = +\infty$$

$$\lim_{x \rightarrow 2^-} f(x) \left(\frac{+}{0^-} \right) = -\infty \Rightarrow \boxed{x=2} \quad \text{V.A}$$

$$\lim_{x \rightarrow -2^+} f(x) \left(\frac{-}{0^-} \right) = +\infty$$

$$\lim_{x \rightarrow -2^-} f(x) \left(\frac{-}{0^+} \right) = -\infty \Rightarrow \boxed{x=-2} \quad \text{V.A}$$

V.A



Asymptotes:

$$R(x) = \frac{P(x)}{Q(x)}$$

$$\lim_{x \rightarrow \infty} R(x) = b \Rightarrow y = b \text{ H.A.}$$

$$\lim_{x \rightarrow a^+} R(x) = +\infty \Rightarrow x = a \text{ V.A.}$$

$$* R(x) = \frac{P(x)}{q(x)}$$

$$\deg P = \deg (q+1)$$

↓

No H. Asy

= linear + ...

oblique asymptote

$$y = mx + b$$

Ex. $f(x) = \frac{x^2}{x-1}, x \neq 1$

$$= \frac{x^2 - 1}{x-1} + \frac{1}{x-1}$$

$$= x+1 + \frac{1}{x-1}$$

oblique asymptote

$$f(x) = x+1 + \frac{1}{x-1}$$

$$x \rightarrow \begin{matrix} -\infty \\ +\infty \end{matrix} \Rightarrow f(x) \approx x+1$$

$$x \rightarrow 1 \Rightarrow f(x) \approx \frac{1}{x-1}$$

$$y = x+1 \text{ oblique}$$

$$\Rightarrow \frac{x^3}{x^2+1}$$

No H. Asy

No V. Asy

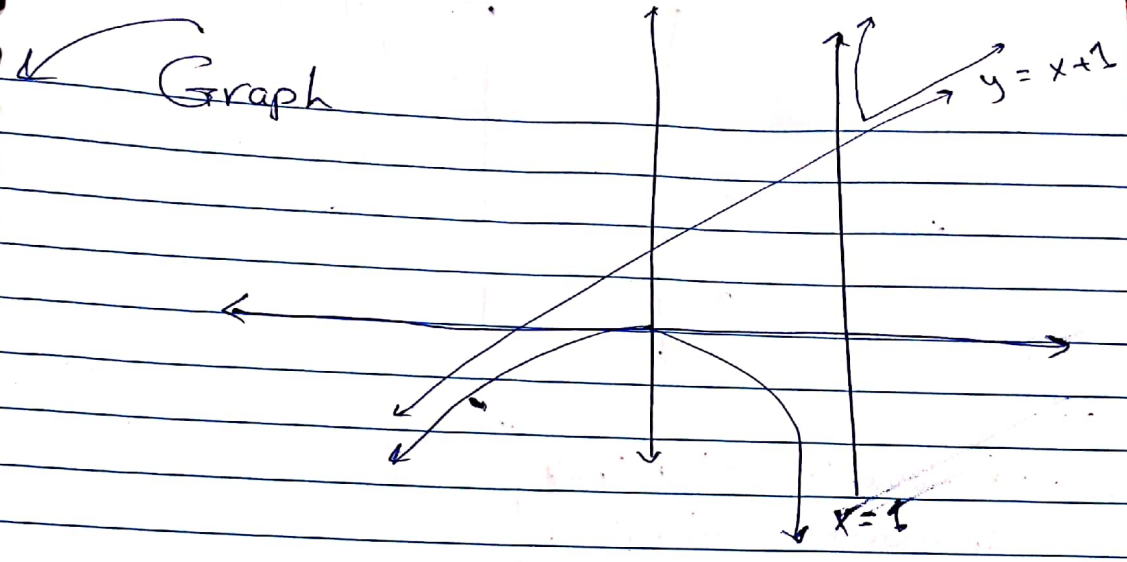
$$\lim_{x \rightarrow 1^+} f(x) = \left(\frac{+}{0^+} \right) = +\infty$$

$$\lim_{x \rightarrow 1^-} f(x) = \left(\frac{-}{0^-} \right) = -\infty$$

$$y = 1 \text{ vertical}$$

$$x=0 \Rightarrow f(x)=0$$

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Ex: $f(x) = \frac{x^3+1}{x^2-1}$, $x+1-1$ → may be (V.A.S) = degree x^3+1
 - No (H. Asy) - degree x^2-1
 - oblique

$\begin{array}{r} x \\ x^2-1 \overline{) x^3+1} \\ \underline{-x^3+x} \\ x+1 \end{array}$	$\Rightarrow f(x) = x + \frac{x+1}{x^2-1}$ $= x + \frac{1}{x-1}$
--	--

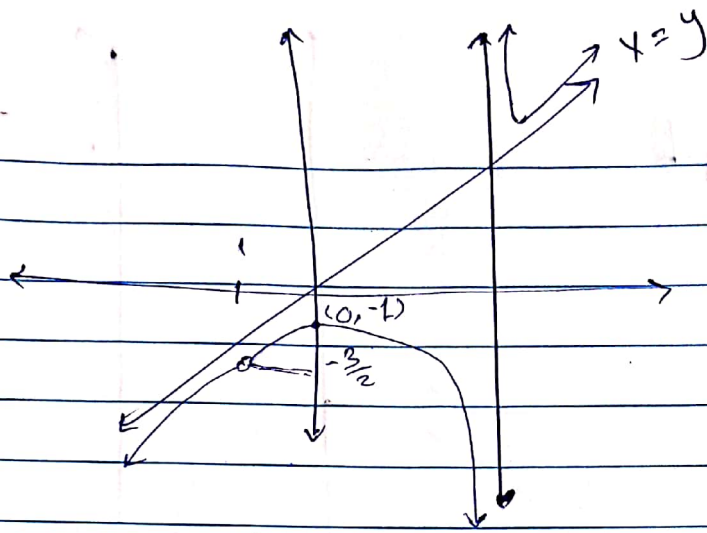
$\Rightarrow \lim_{x \rightarrow -1} f(x) = -\frac{3}{2} \Rightarrow$ No. V. Asy

$\Rightarrow y = x \Rightarrow$ oblique Asy

$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = +\infty$

$\lim_{x \rightarrow 1^-} f(x) = -\infty$

$x = 1$ V. Asy



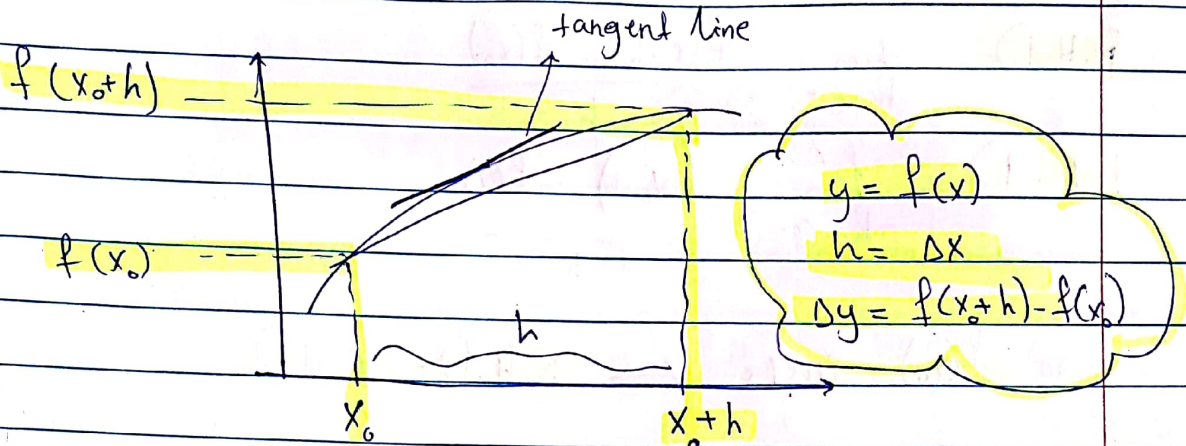
$x = -1 \Rightarrow$ removable discontinuity

$x = 1 \Rightarrow$ irremovable discontinuity

$$f(x) = \begin{cases} f(x) & , x \neq -1 \\ -\frac{1}{2} & , x = -1 \end{cases}$$

3.1

Differentiation



average rate of change

$\frac{\Delta y}{\Delta x}$ = slope of the secant line

3.1 Def: Given a function $f(x)$ the first derivative of $f(x)$, with respect to x at $x = x_0$, denoted by $f'(x)$ is given by the following limit

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$f(x)$ is a differentiable function at $x = x_0$

If $y = f(x)$, then $f(x)$ is diff if $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

exists for all x in the domain of $f(x)$.

$$f(x) = \frac{dy}{dx}$$

$$\text{R.H.D} \quad \lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\text{L.H.D} \quad \lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

$$f(x_0) \text{ exists} \iff \text{R.H.D} = \text{L.H.D}$$

$$\text{Ex: } f(x) = |x|$$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

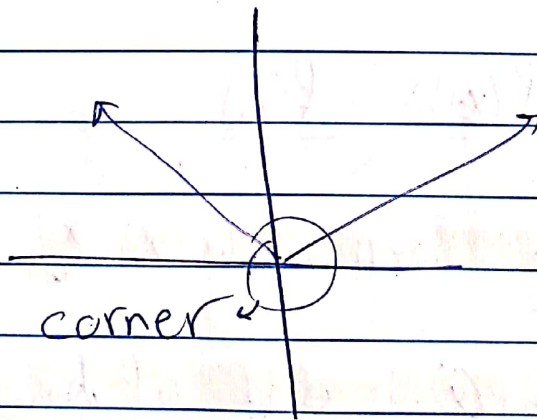
$$\text{R.H.D} = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = 1$$

$$\text{L.H.D} = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

$$\text{R.H.D} \neq \text{L.H.D}$$

$$1 \neq -1$$

SO $f(x)$ is not diff^l
at $x=0$



Derivatives

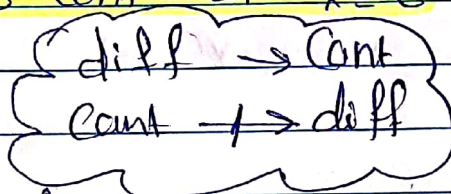
$$f'(a) = \frac{dy}{dx} \Big|_{x=a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$f'(x) = f'(a) \equiv$ slope of the tangent to the curve $f(x)$ at $x=a$.

$f(x) = |x| \Rightarrow$ not diff at $x=0$ cont at $x=0$

Note: If $f(x)$ is diff at $x=c$, then $f(x)$ is cont at $x=c$



3.2 Rules of differentiability:

1 $y = f(x) = x^n$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

$$f'(x) = n x^{n-1}$$

2 If f, g are differentiable then

$$(f \pm g)' = f' \pm g'$$

$$3 \quad (f \cdot g)' = f'g + g'f$$

$$4 \quad \left(\frac{f}{g}\right)' = \frac{g f' - f g'}{g^2}$$

$$5) (f \circ g)(x) = f(g(x))$$

The Chain Rule: If $y = f(u)$ and $u = g(x)$
 $y = f(g(x)) = (f \circ g)(x)$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= f'(u) \cdot g'(x) \\ &= f'(g(x)) \cdot g'(x) \end{aligned}$$

3.3 Trigonometric Functions:-

$$6) \frac{d}{dx} (\sin x) = \cos x$$

$$\frac{d}{dx} \sin u(x) = \cos(u(x)) \cdot u'(x)$$

$$7) \frac{d}{dx} (\cos x) = -\sin x$$

$$8) \frac{d}{dx} (\tan x) = \sec^2 x$$

$$9) \frac{d}{dx} (\csc x) = -\csc x \cot x$$

$$10) \frac{d}{dx} (\sec x) = \sec x \tan x$$

$$11) \frac{d}{dx} (\cot x) = -\csc^2 x$$

Ex: $y = (x^2 + 3)^{\frac{7}{3}}$
 $\dot{y} = \frac{7}{3} (x^2 + 3)^{\frac{4}{3}} \cdot 2x = \frac{14}{3} (x^2 + 3)^{\frac{4}{3}}$

12) $y = \sec \sqrt{x}$
 $\dot{y} = \sec \sqrt{x} \tan \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$

* to all trigonometric function.

$$[3] \quad y = \tan x^2 * \cos(2x+3)$$

$$(\sec^2 x^2 * 2x (\cos(2x+3))) + (\tan x^2 * -\sin(2x+3) * 2)$$

$$[4] \quad y = \sin^3(2x+4)$$

$$y' = 3 \sin^2(2x+4) * \cos(2x+4) * 2$$

3.4 Implicit Differentiation

$$y = f(x) \Rightarrow \frac{dy}{dx}$$

\downarrow Dep \downarrow Indep

$$[1] \quad x^3 + x y^2 + 10y = 0$$

$$3x^2 + x 2y y' + y^2 + x 10y' + 10y' = 0$$

$$3x^2 + y^2 + ~~10y~~ + y'(2xy + 10x) = 0$$

$$y'(2xy + 10x) = -3x^2 - y^2$$

$$y' = \frac{-3x^2 - y^2}{2xy + 10x}$$

$$[2] \quad xy = \cos(xy)$$

$$x y' + y = -\sin(xy) * (x y' + y)$$

$$x y' + y = -[\sin(xy) * x y' + y \sin(xy)]$$

$$(x + x \sin xy) y' = -[y \sin xy + y]$$

$y', y'', y''', y^{(4)}$

Ex: If $x^{2/3} + y^{2/3} = 1$ find y'' ?

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{1/3} y' = 0 \Rightarrow y' = -\left(\frac{y}{x}\right)^{1/3}$$

$$y'' = -\frac{1}{3} \left[\frac{x y' - y}{x^2} \right]^{-2/3}$$

$$= -\frac{1}{3} \left[\frac{-x \left(\frac{y}{x}\right)^{1/3} - y}{x^2} \right]^{-2/3}$$

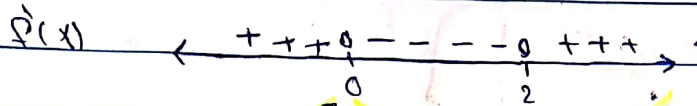
Exp: Draw this function $f(x) = x^3 - 3x^2 + 1$

1) $D(f) = \mathbb{R}$

2) $f'(x) = 3x^2 - 6x$

$0 = 3x(x-2)$

(critical point) $\Rightarrow x=0, x=2 \in D(f)$



$f(x)$ behave

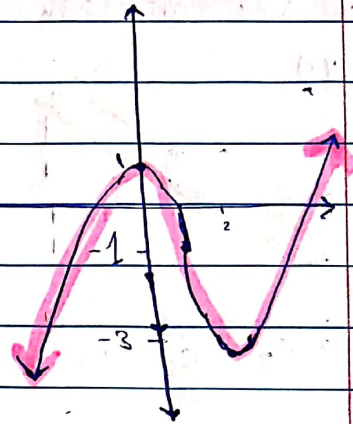
* f is \uparrow on $(-\infty, 0] \cup [2, \infty)$

* f is \downarrow on $[0, 2]$

3) key point $(0, 1)$

$f(0) = 1$ is local max

$f(2) = -3$ is local min

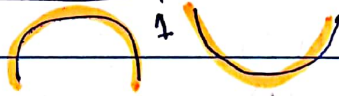


4) inflection point

$f''(x) = 6x - 6$

$0 = 6(x-1)$

$x = 1$



⇒ 3.5 Linearization and differentials

Linearization:

$$y - y_1 = m(x - x_1)$$

$$y = y_1 + m(x - x_1)$$

$$x_1 = a \Rightarrow L(x) = f(a) + f'(a)(x - a)$$

Ex: Find the linearization? ~~and differentials~~

$$f(x) = \tan x \quad \text{at } x = \frac{\pi}{4} \quad ?$$

$$f\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1 \Rightarrow \left(\frac{\pi}{4}, 1\right)$$

$$f'(x) = \sec^2 x = \sec^2 \frac{\pi}{4} = 2 = f'\left(\frac{\pi}{4}\right)$$

$$x = \frac{\pi}{4}, \quad y = 1, \quad m = 2$$

$$y - 1 = 2\left(x - \frac{\pi}{4}\right)$$

$$y = 1 + 2\left(x - \frac{\pi}{4}\right) = L(x)$$

Differentials:

$$y = f(x)$$
$$\frac{dy}{dx} = f'(x) \Rightarrow \text{derivative}$$

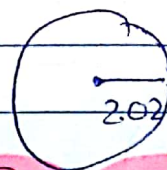
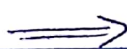
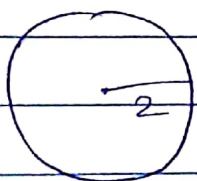
$$dy = f'(x) dx \Rightarrow \text{differential}$$

exp: $y = x^2 \Rightarrow y' = 2x$ derivative
 $dy = 2x dx$ differentials of y .

exp: Radius of a circle increased from 2 to 2.02.

(a) Estimate the resulting change in area

(b) express the estimate as a percentage of the circle's original area



$$A = \pi (2.02)^2$$

$$r = 2.02 = 2 + 0.02$$
$$= r + dr$$

(a) $A(r) = \pi r^2$

$$dA = 2\pi r dr = 2\pi (2) (0.02) = 0.08\pi$$

$$A(2) = \pi (2)^2 = 4\pi$$

estimate area = $A + dA$

$$= 4\pi + 0.08\pi = 4.08\pi$$

(b) True area = $\pi (2.02)^2 = (4.0804)\pi$

$$\text{error} = | \text{True} - \text{estimate} |$$

$$= | 4.0804\pi - 4.08\pi |$$

$$= 0.0004\pi$$

Applications of derivatives

4.1 → Increasing and decreasing functions:

Def: Let $f(x)$ be a function defined on an interval I . Then

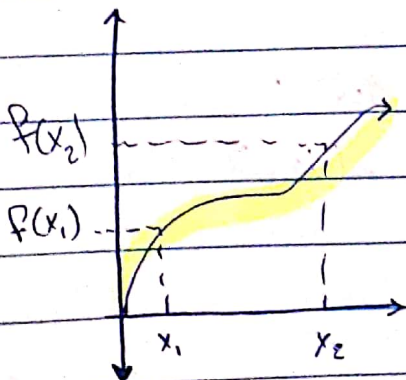
(a) f is increasing on I if whenever $x_2 > x_1$, then $f(x_2) > f(x_1)$,
for all x_1, x_2 in I .

(b) f is decreasing on I if whenever $x_2 > x_1$, then $f(x_2) < f(x_1)$,
for all x_1, x_2 in I .

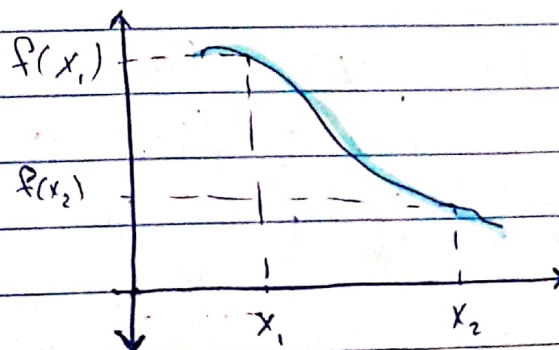
Theorem :- Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) then:-

(a) If $f'(x) > 0$, for all $x \in (a, b)$ then f is increasing on (a, b)

(b) If $f'(x) < 0$, for all $x \in (a, b)$ then f is decreasing on (a, b)

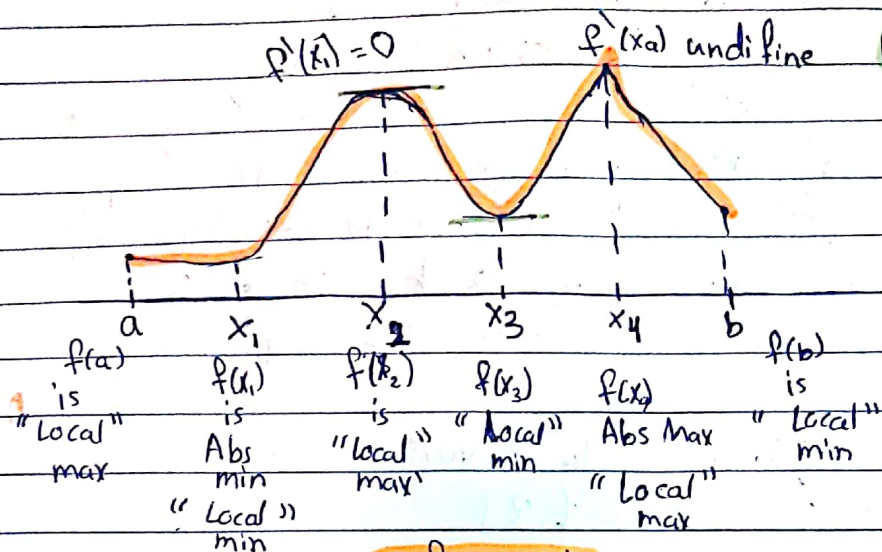


$x_2 > x_1$
 $f(x_2) > f(x_1)$
increasing



$x_2 > x_1$
 $f(x_2) < f(x_1)$
decreasing

4.2 → Extrem Values

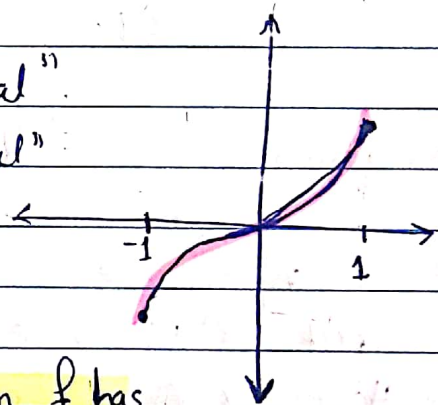


← Figure 4 →

Exp: $f(x) = x^3$ on $[-1, 1]$, find the extrem value?

$f(1) = 1^3 = 1$ is Abs Max "local"

$f(-1) = -1$ is Abs Min "local"

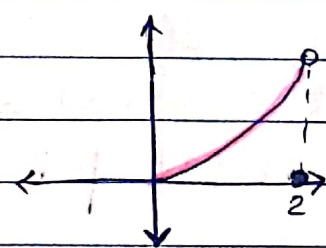


Th (Extrem Value theorem):

⇒ If f cont. on $[a, b]$ then f has Abs Max and Abs Min.

Exp: Consider this function.

$$f(x) = \begin{cases} x^2 & 0 \leq x < 2 \\ 0 & x = 2 \end{cases}$$



Find Extrem Values?

$f(0) = 0 = 0$ is Abs Min

$f(2) = 0$ is Abs Min

NO Abs Max.

* $f(x)$ is discont. at $x = 2$

Def. An ^{موجود في المجال} interior point c ($c \in D(f)$) is called **critical point** if
 (1) $f'(c) = 0$ or (2) $f'(c)$ undefined.

Remark: Note the Extrem. Values occur at either critical points or End points for a function defined on $[a, b]$ see Fig 1.

Exp: Find extrem values for $f(x) = x^{2/3}$ on $[-1, 8]$

End points: $f(-1) = (-1)^{2/3} = \sqrt[3]{(-1)^2} = 1$
 $f(8) = (8)^{2/3} = \sqrt[3]{(8)^2} = 4$

Critical points: $f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$

f' never zero.

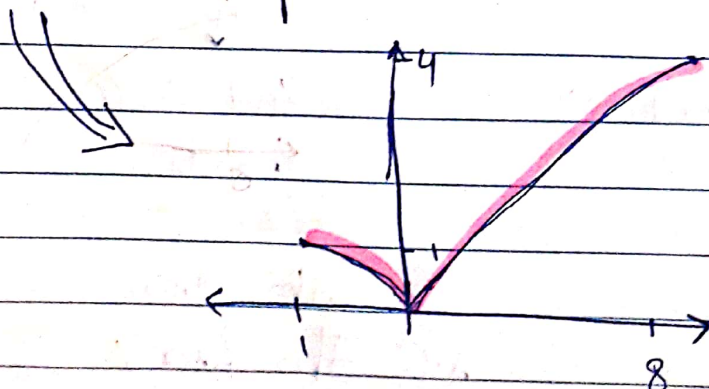
but f' is undefined at $x = 0 \in [-1, 8]$

Hence $x = 0$ is a critical point

$f(0) = 0$

So $f(0) = 0$ is Abs Min
 and $f(8) = 4$ is Abs Max

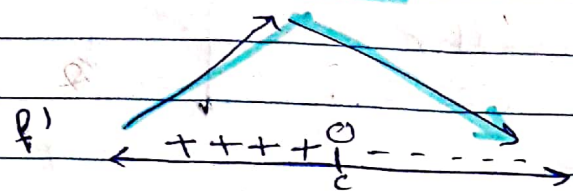
* f is cont. on $[-1, 8]$ | $f(1) = 1$ is local Max



The (1st) Derivative Test for Extrem Values

Assume $f'(c) = 0$ where c is interior point

✓ If f' changes signs from $+$ to $-$ around c then $f(c)$ is local Max



✓ If f' changes signs from $-$ to $+$ around c then $f(c)$ is local Min.



✓ If f' does not change signs around c then f has no extrem values

The (2nd) Derivative Test

Assume $f'(c) = 0$ where c is interior point

✓ If $f''(c) < 0$, then $f(c)$ is local Max

✓ If $f''(c) > 0$, then $f(c)$ is local Min

✓ If $f''(c) = 0$ the test doesn't work

Remark * An interior point c ($c \in D(f)$)

where ① f has tangent at c .

and ② f changes concavity at c is called inflection point.

* To find the inflections we start with $f''(x) = 0$ and find x .

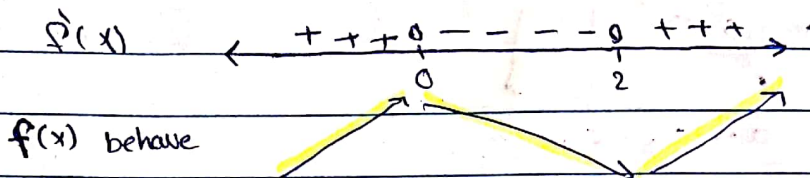
Exp: Draw this function $f(x) = x^3 - 3x^2 + 1$

1 $D(f) = \mathbb{R}$

2 $f'(x) = 3x^2 - 6x$

$0 = 3x(x-2)$

(critical point) $\Rightarrow x=0, x=2 \in D(f)$



* f is \uparrow on $(-\infty, 0] \cup [2, \infty)$

* f is \downarrow on $[0, 2]$

3 key point $(0, 1)$

$f(0) = 1$ is local max

$f(2) = -3$ is local min

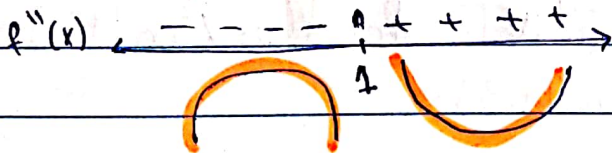


4 inflection point

$f''(x) = 6x - 6$

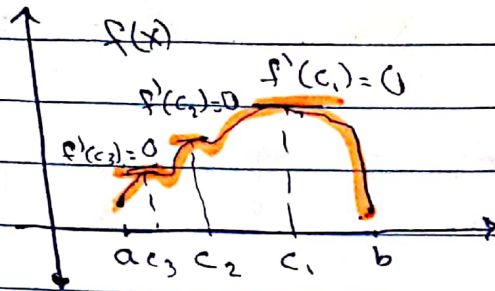
$0 = 6(x-1)$

$x = 1$



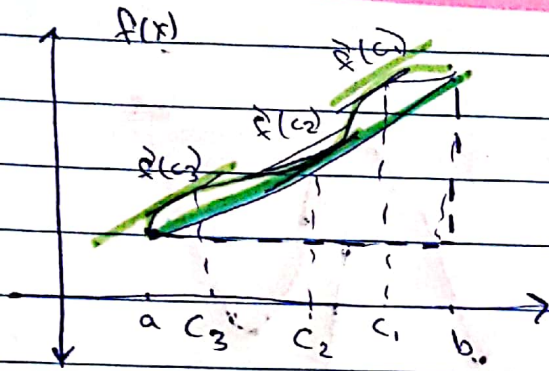
Th (Roll's Th): Assume f is cont on $[a, b]$
and f is diff on (a, b)
with $f(a) = f(b)$

* Then \exists at least one point $c \in (a, b)$ s.t
 $f'(c) = 0$.



Th (Mean Value Th): Assume f is cont. on $[a, b]$
and f is diff on (a, b)

Then, \exists at least one
point $c \in (a, b)$ s.t
 $f'(c) = \frac{f(b) - f(a)}{b - a}$



$f'(c) =$ Slope of secant
 $f'(c) = \frac{f(b) - f(a)}{b - a}$

Ex. Find c that satisfies the MVT for $f(x) = x^2$ on $[1, 3]$

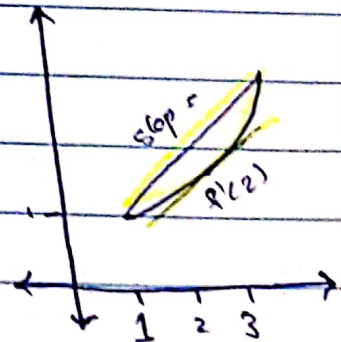
- f cont on $\mathbb{R} \Rightarrow f$ cont on $[1, 3]$
- $f' = 2x$ cont on $\mathbb{R} \Rightarrow f'$ cont on $(1, 3)$

Hence $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$2c = \frac{f(3) - f(1)}{3 - 1}$$

$$2c = \frac{9 - 1}{2}$$

$$c = 2$$



Exp: Find a, b, c so that f satisfies the MVT on $[0, 2]$.

$$f(x) = \begin{cases} 3 & , x=0 \\ -x^2+3x+a & , 0 < x < 1 \\ cx+b & , 1 \leq x \leq 2 \end{cases}$$

f is cont. on $[0, 2]$, f is cont at $x=0$
 $f(0) = \lim_{x \rightarrow 0^+} f(x) \Rightarrow 3 = a$

f is cont at $x=1 \Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$
 $c+b = -1+3+3$
 $c+b = 5$ (1)

f is diff on $(0, 2)$
 $\lim_{x \rightarrow 1^+} f'(x) = \lim_{x \rightarrow 1^-} f'(x)$
 $c = -2(1) + 3 \Rightarrow c = 1$
 $\Rightarrow b = 4$

Exp: Given $y = f(x) = x\sqrt{8-x^2}$
 $y' = \frac{8-2x^2}{\sqrt{8-x^2}}$
 $y'' = \frac{2x^3 - 24x}{\sqrt{(8-x^2)^3}}$

- find (1) Df (2) R(f)
 (3) $\uparrow \downarrow$ (4) critical points
 (5) Extrem Values
 (6) inflection point (7) Concave (up+down)
 (8) graph &

(1) $8-x^2 \geq 0$
 $-2\sqrt{2} \leq x \leq 2\sqrt{2}$
 $D = [-2\sqrt{2}, 2\sqrt{2}]$

(2) wait for (8)

(3) $y' = 0 \Leftrightarrow 8-2x^2 = 0$

(4) $\sqrt{x^2} = \sqrt{4}$

$x = 2$ or $x = -2, \in D(f)$

critical points

$(2, 4), (-2, -4)$

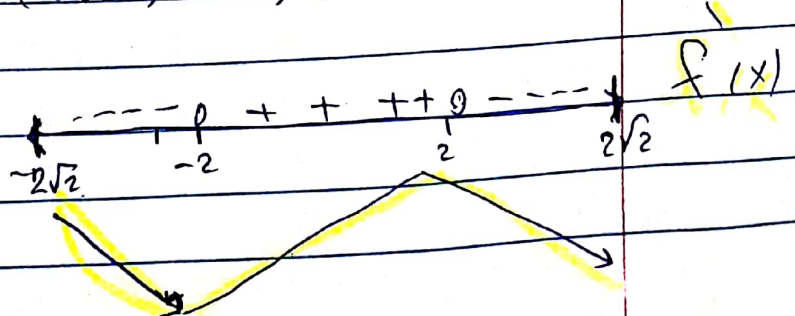
(3) y' is undefined:

$8-x^2 = 0$
 $x^2 = 8$

$x = 2\sqrt{2}, -2\sqrt{2} \in D(f)$

$(2\sqrt{2}, 0), (-2\sqrt{2}, 0)$

f is decreasing on $[-2\sqrt{2}, -2] \cup [2, 2\sqrt{2}]$
 f is decreasing on $[-2, 2]$



5) Extrem Value $\Rightarrow (-2, -4)$ is local min (Abs Min)

$(2\sqrt{2}, 0)$ is local min

$(2, 4)$ is local max (Abs Max)

$(-2\sqrt{2}, 0)$ is local max

6) inflection point $(0, 0)$

7) $y'' = 0 \Rightarrow 2x^3 - 24x = 0$

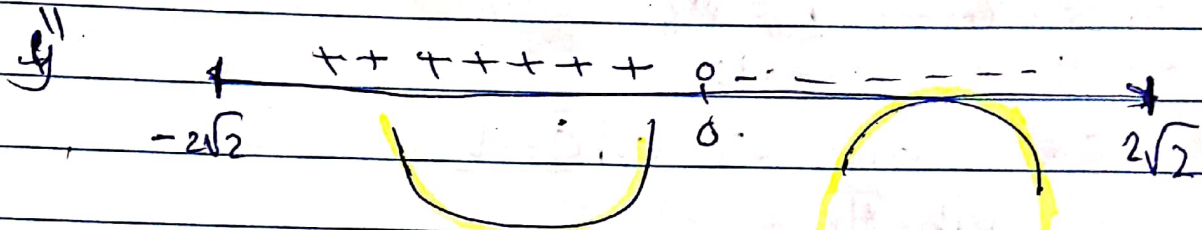
$$2x [x^2 - 12] = 0$$

$$x = 0 \in D(f)$$

or $x^2 - 12 = 0$

$$x^2 = 12$$

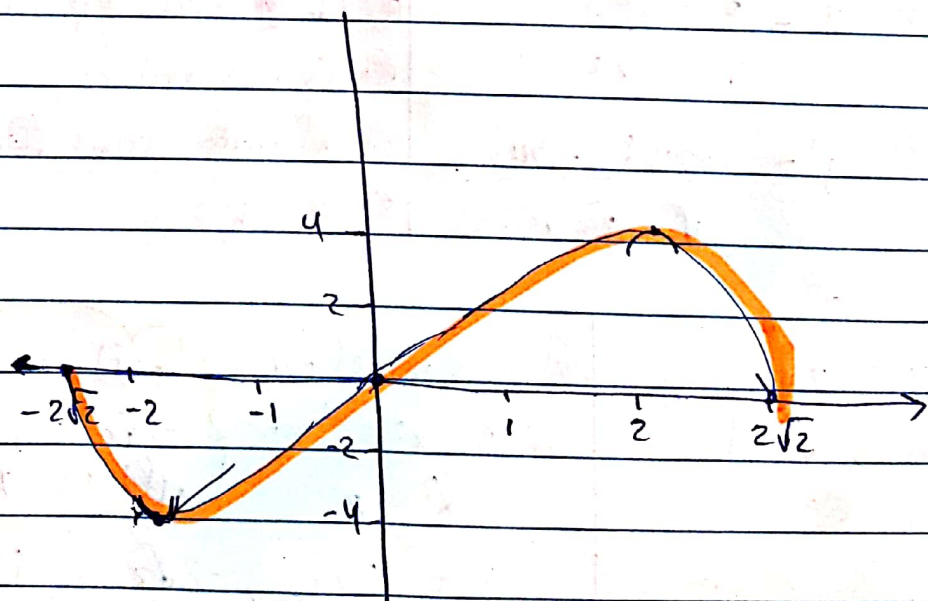
$$|x| = \sqrt{12} = 2\sqrt{3}, -2\sqrt{3} \notin D(f)$$



• f is concave up on $[-2\sqrt{2}, 0]$

• f is concave down on $[0, 2\sqrt{2}]$

8)



Integration

Def: The function $F(x)$ is called anti-derivative for the function $f(x)$ on interval I .

$$i.f \quad F'(x) = f(x) \quad \forall x \in I$$

Exp Give 4 antiderivatives for $f(x) = 2x$,

$$ii \quad f(x) = x^2 + 5 \Rightarrow \textcircled{1} x^2 + 4 \quad \textcircled{3} x^2 + 9$$

$$\textcircled{2} x^2 + \sqrt{3} \quad \textcircled{4} x^2 - 5$$

$$F(x) = x^2 + C$$

Def

The set of all anti-derivatives for $f(x)$ is called indefinite integral

$$\int 2x \, dx = x^2 + C$$



$$\int f(x) \, dx = F(x)$$

$$f(x) = F'(x)$$

$$\textcircled{1} \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\textcircled{2} \int \cos x \, dx = \sin x + C$$

$$\textcircled{3} \int \sin x \, dx = -\cos x + C$$

$$\textcircled{4} \int \sec^2 x \, dx = \tan x + C$$

$$\textcircled{5} \int \csc^2 x \, dx = -\cot x + C$$

$$\textcircled{6} \int \sec x \tan x \, dx = \sec x + C$$

$$\textcircled{7} \int \csc x \cot x \, dx = -\csc x + C$$

Exp: $\int \sin^2 x \, dx$
 $= \int \frac{1 - \cos 2x}{2} \, dx$

$$= \frac{1}{2} x - \frac{1}{2 \times 2} \sin 2x + c = \frac{1}{2} x - \frac{1}{4} \sin 2x + c$$

$$\begin{aligned} \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos^2 x &= \frac{1 + \cos 2x}{2} \end{aligned}$$

Exp: $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx$

$$= \frac{x}{2} + \frac{1}{4} \sin 2x + c$$

Exp: $\int \frac{\csc \theta}{\csc \theta - \sin \theta} \, d\theta = \int \frac{1/\sin \theta}{1/\sin \theta - \sin \theta} \, d\theta$
 $= \int \frac{1/\sin \theta}{\frac{1 - \sin^2 \theta}{\sin \theta}} \, d\theta = \int \frac{1}{\cos^2 \theta} \, d\theta = \int \sec^2 \theta \, d\theta = \tan \theta + c$

Exp: $\int \frac{\sec z \cdot \tan z \, dz}{\sqrt{\sec z}}$

$$\begin{aligned} u &= \sec z \\ du &= \sec z \cdot \tan z \, dz \end{aligned}$$

$$= \int \frac{du}{\sqrt{u}} = \int u^{-1/2} \, du = 2\sqrt{u} + c = 2\sqrt{\sec z \cdot \tan z} + c$$

Exp:

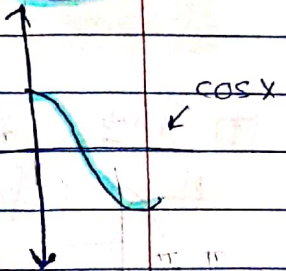
$$\int_a^b (\cos x + |\cos x|) dx$$

↑ absolute value

النسبة
في
الفترة
الموجبة
لـ cos x
القيمة
الموجبة

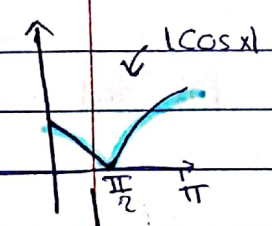
$$\int_a^b \cos x + \int_a^b |\cos x| dx$$

$$0 + \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x dx$$



$$\sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\pi}$$

$$1 - (0 - 1) = 2$$

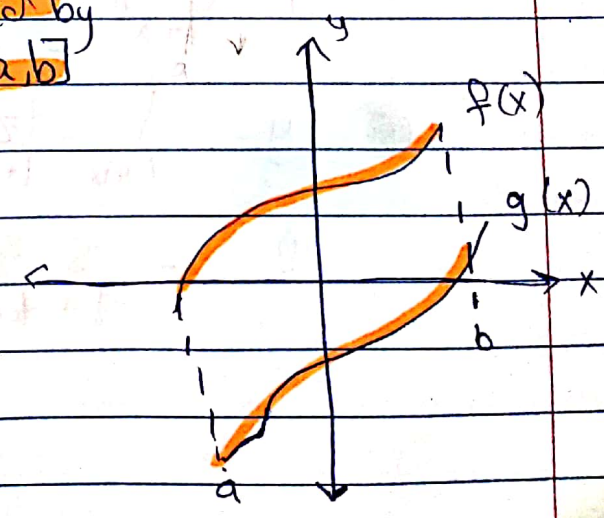


Remark: If $f(x) \geq 0 \forall x \in [a, b]$, then

$\int_a^b f(x) dx$ represents the area bounded between the x axis and the curve of $f(x)$ on $[a, b]$

Remark If $f(x) > g(x) \forall x \in [a, b]$

then: $\int_a^b (f(x) - g(x)) dx$ represents the area bounded by the 2 curve on $[a, b]$



1 h. Fundamental theorem of Calculus:

Assume $f(x)$ is Cont on $[a, b]$ and

1. If $F(x)$ is anti derivative for $f(x)$

then $\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$

∴ definite integral \Rightarrow gives real number.

2. If $F(x) = \int_a^x f(t) dt$, then $F(x)$ is Cont on $[a, b]$ and $F(x)$ is diff on (a, b) with $F'(x) = f(x)$.

Exp: Find y : $f(x) = \int_0^x \cos t dt$

① $y' = \cos x$

② $y = \int_0^x \cos t dt \Rightarrow y = 0$

③ $y = \int_0^x \frac{dt}{1+t^2}$

$y = \frac{2x}{1+x^4} \Rightarrow$

عند تقاطع الاقتران
الكل ، نزل استراتيجيا
وذاً نجد dt مستقلة بالمتغير

④ $y = \int_{-\tan x}^{0} \frac{dz}{1+z^2}$

$y = - \frac{\sec^2 x}{1+\tan^2 x} - \frac{\sec^2 x}{\sec^2 x} = -1$

Find the area between the curve $y = 3x\sqrt{x^2+1}$ and x-axis on $[0, 1]$

$$A = \int_0^1 3x\sqrt{x^2+1} \cdot dx$$

$$u = x^2 + 1 \quad \begin{cases} x=1 \rightarrow \textcircled{1} \\ x=2 \rightarrow \textcircled{2} \end{cases}$$
$$du = 2x$$

$$\frac{3}{2} \int_0^1 \sqrt{u} = \frac{3}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} =$$

$$= \sqrt{u^3} \Big|_1^2 = \sqrt{(x^2+1)^3}$$

Exp: Find area enclosed by $f(x) = 2 - x^2$ and $g(x) = -x$.

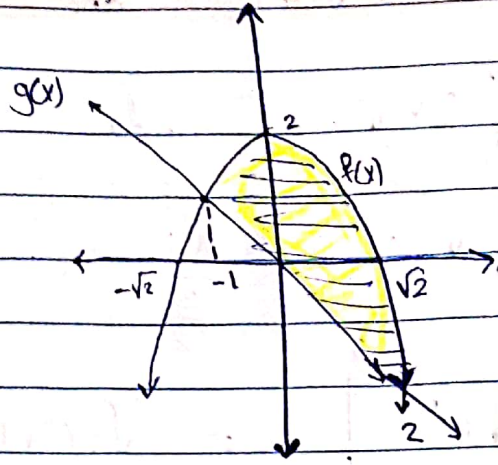
$$f(x) = g(x)$$

$$2 - x^2 = -x$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1, 2$$



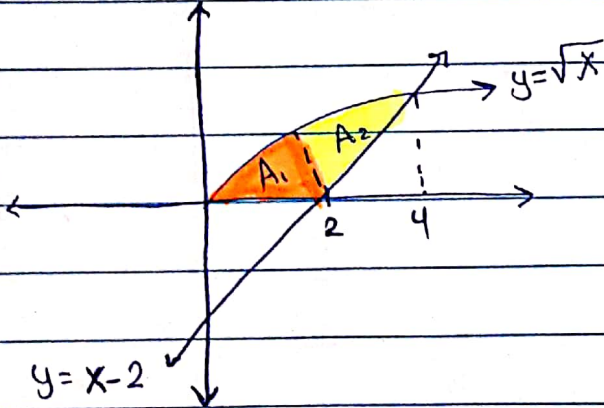
$$\int_{-1}^2 (f(x) - g(x)) \cdot dx$$

$$\int_{-1}^2 (2 - x^2 + x) \cdot dx = 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^2 = \frac{9}{2}$$

Exp: Find the area bounded by x-axis, $y = \sqrt{x}$, $y = x - 2$ using [1] Integral w.r.t x

[2] Integral w.r.t y

w.r.t: with respect to



$$(\sqrt{x} - x - 2)^2$$

$$x - x^2 - 4x + 4$$

$$x^2 - 5x + 4 = 0$$

$$(x-1)(x-4) = 0$$

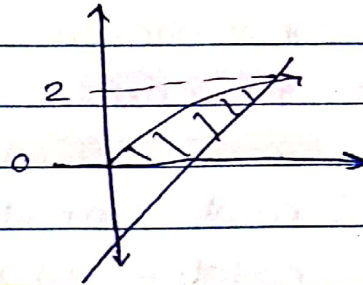
$$x = 4, 1$$

$$\int_0^2 \sqrt{x} \cdot dx + \int_2^4 (\sqrt{x} - (x-2)) \cdot dx = \frac{10}{3}$$

$$\boxed{2} \quad A = \int_0^2 (y+2 - y^2) \cdot dy$$

$$= \frac{10}{3}$$

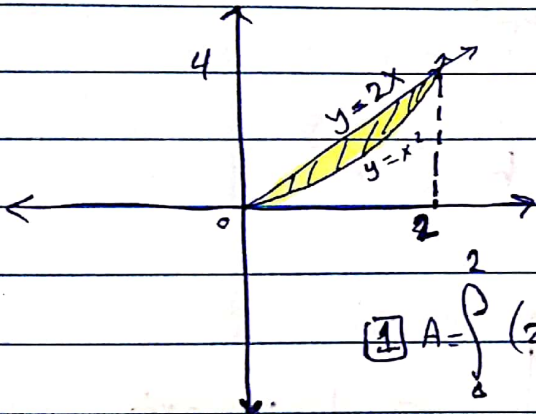
① $y = x - 2 \Rightarrow x = y + 2$
 ② $y = \sqrt{x} \Rightarrow y^2 = x$



Exp: Find the area bounded by $y = x^2$, $y = 2x$

① Integral w.r.t x

② Integral w.r.t y



$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0, 2$$

$$\boxed{1} \quad A = \int_0^2 (2x - x^2) \cdot dx = \frac{4}{3}$$

$$\boxed{2} \quad A = \int_0^4 \left(\sqrt{y} - \frac{y}{2} \right) dy = \frac{4}{3}$$