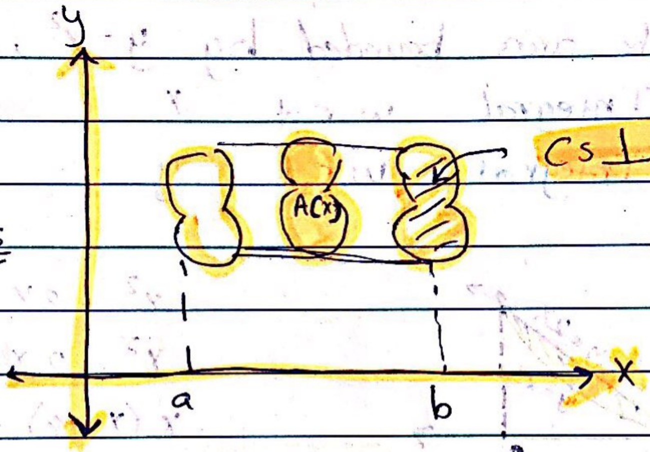
Volumes Using Cross Sections

- $\perp$: perpendicular "normal"
- // : parallel.
- Cs : cross section.
- : rotation about x-axis.
- : rotation about y-axis.

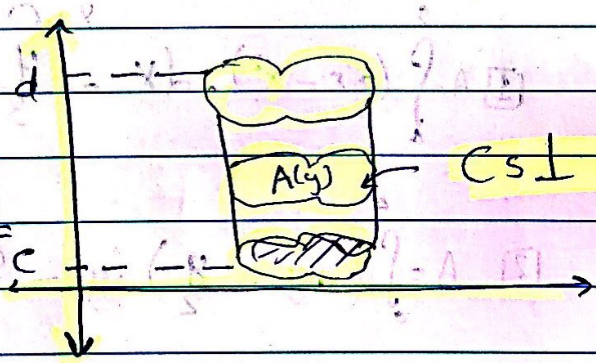
• The volume of the Solid whose cs $\perp$ x-axis is

$$V = \int_a^b A(x)\,dx$$

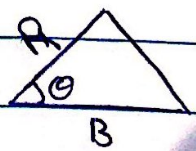


• The volume of the Solid whose cs $\perp$ y-axis is

$$V = \int_c^d A(y)\,dy$$



- ✓ Cs is A disk → $A(r) = \pi r^2$
- ✓ Cs is a squar → $A(r) = r^2$
- ✓ Cs is a triangle → $A = \frac{1}{2} \sin\theta \, R \, B$



$\boxed{6.1}$ Volumes using Cs:

[1] DM [2] WM [3] SM

6.1

[1] Disk method

- In this method the Cs is $\boxed{\text{disk}}$

$\Rightarrow A(x) = \pi R^2(x)$

$\Rightarrow A(y) = \pi R^2(y)$

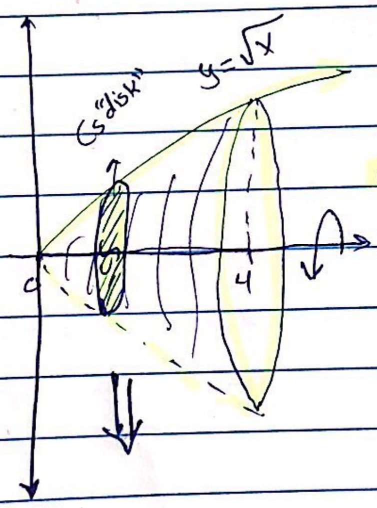
* The volume of the solid generated by rotating the curve about:

[1] x-axis $\Rightarrow$ $V = \int_a^b A(x)\,dx = \int_a^b \pi R^2(x)\,dx$, Cs $\perp$ x-axis

[2] y-axis $\Rightarrow$ $V = \int_c^d A(y)\,dy = \int_c^d \pi R^2(y)\,dy$, Cs $\perp$ y-axis

Exp: Find the volume of the solid generated by revolving the curve:-

[1] $y = \sqrt{x}$, $0 \le x \le 4$, X-axis about x-axis



$V = \int_a^b A(x)\,dx$

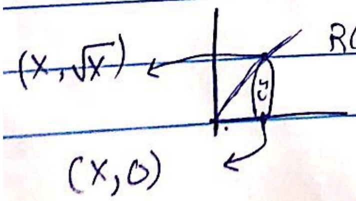
* Cs is disk $\Rightarrow A(x) = \pi R^2(x)$

$V = \int_0^4 \pi R^2(x)\,dx$

$V = \int_0^4 \pi (\sqrt{x})^2\,dx$

$V = \pi \int_0^4 x\,dx$

$= 8\pi$



$R(x) = y_2 - y_1$

$= \sqrt{x} - 0$

$= \sqrt{x}$

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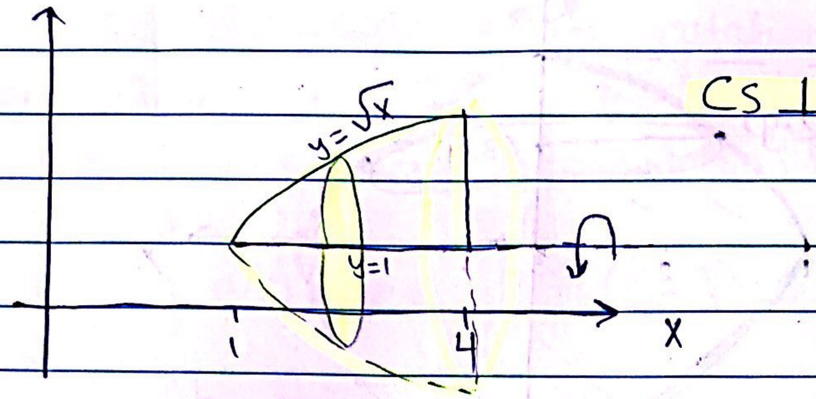
$\boxed{2}$ $y = \sqrt{x}$, Lines $y = 1$, $x = y$ about 1



CS ⊥ x-axis

CS is Disk $\Rightarrow A(x) = \pi R^2(x)$

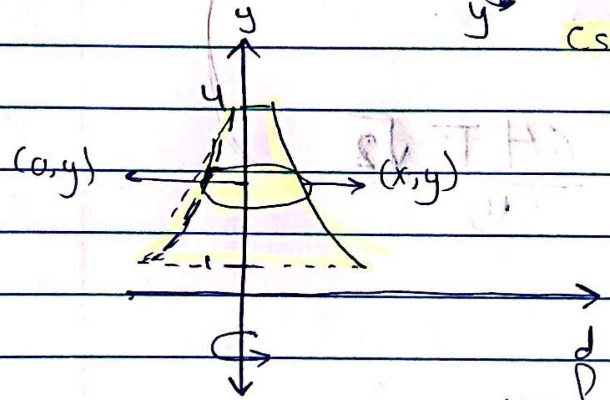
$$V = \int_a^b \pi R^2(x)\, dx$$

$$V = \int_1^4 \pi \left[\sqrt{x} - 1\right]^2 dx$$

$$V = \pi \int_1^4 \left(x - 2\sqrt{x} + 1\right) dx = \frac{7\pi}{6}$$

* $\boxed{3}$ y-axis, $x = \frac{2}{y}$, $1 \leq y \leq 4$ about y-axis

CS ⊥ y-axis, CS is disk.



$R(y) = x_2 - x_1$

$= x - 0$

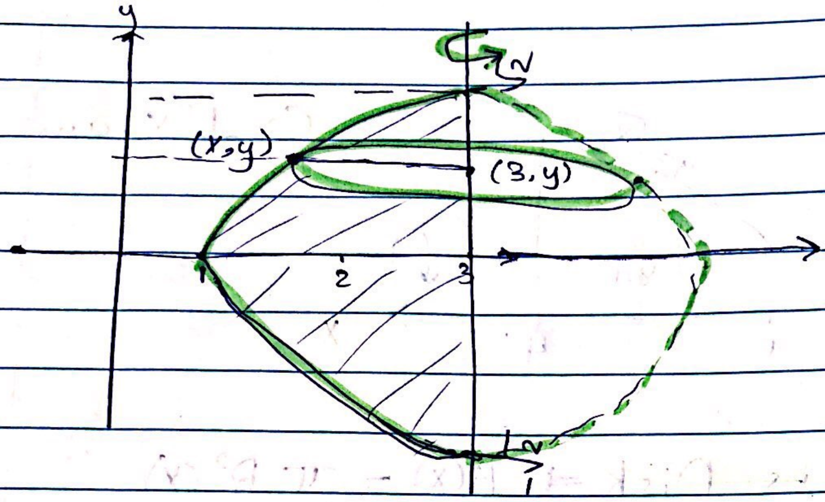
$= x$

$= \frac{2}{y}$

$$v = \int_c^d A(y)\, dy = \int_1^4 \pi R^2(y)\, dy$$

$$= \boxed{3\pi}$$

4. $x = y^2 + 1$, Line $x = 3$ about $x = 3$.



* Cs ⊥ y-axis ⟹ Cs disk ⟹ $A(y) = \pi R^2(y)$

* $y^2 + 1 = 3$
$y^2 = 2$
$|y| = \sqrt{2}$
$y = \pm\sqrt{2}$

* $R(y) = x_2 - x_1$
$= 3 - x$
$= 3 - (y^2 + 1)$
$= 2 - y^2$

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi R^2(y)\,dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi (2 - y^2)^2\,dy = \frac{64\pi\sqrt{2}}{15}$$

2 washer method (WM) :_ Cs is disk

⇒ If CS ⊥ x-axis with rotation about x-axis, Outer raduis $R(x)$, Inner radius $r(x)$ then the volume of the resulted solid

is $V = \int_a^b A(x)\,dx = \int_a^b \left(\pi R^2(x) - \pi r^2(x)\right) dx$

$$= \pi \int_a^b \left(R^2(x) - r^2(x)\right) dx.$$

⇒ If Cs ⊥ y-axis with rotation about y-axis, Outer raduis $R(y)$, Lenner radius $r(y)$ then the Volume of the resulted Solid

is $V = \int_c^d A(y)\,dy = \int_a^b \left(\pi R^2(y) - \pi r^2(y)\right) dy$

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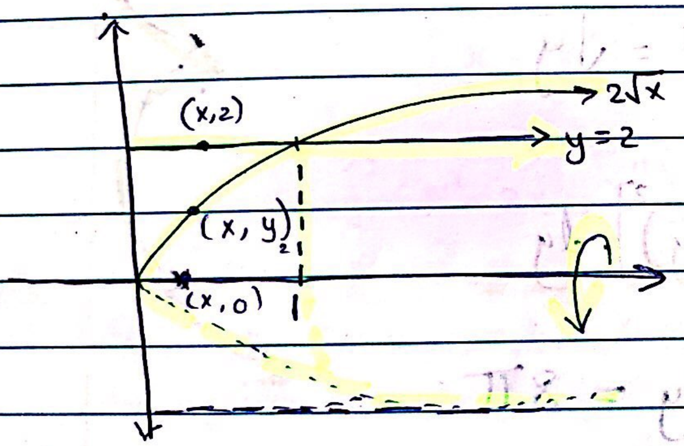
6.1 Washer method

- If the cs $\perp$ x-axis, with outer radius $R(x)$, inner radius $r(x)$, then $V = \int_a^b \pi \left[R^2(x) - r^2(x)\right] dx$

- If the cs $\perp$ y-axis, with outer radius $R(y)$, inner radius $r(y)$, then $V = \int_c^d \pi \left[R^2(y) - r^2(y)\right] dy$

Exp: find the volume of the solid generated by rotating the region bounded by:

① $y = 2\sqrt{x}$, $y = 2$, $x = 0$ about x-axis?



$* \ 2 = 2\sqrt{x}$

$x = 1$

$* \ V = \int_a^b A(x)\, dx$

$* \ R(x) = 2$

$r(x) = y_2 - y_1 = 2\sqrt{x} - 0$

$= 2\sqrt{x}$

$\Rightarrow R(x) = 2$

$\Rightarrow r(x) = 2\sqrt{x}$

$$\pi \int_0^1 \left(R^2(x) - r^2(x)\right) dx$$

$$= \pi \int_0^1 (4 - 4x)\, dx = \pi \left(4x - 2x^2\right)\Big|_0^1$$

$$= 2\pi$$

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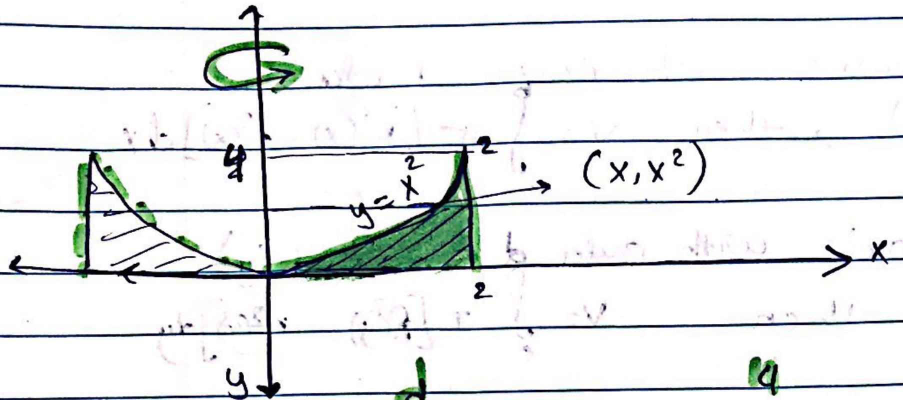
② 1^{st} quadrant , $y=x^2$, x-axis , $x=2$ about y-axis?



$$V=\int_c^d A(y)\,dy = \int_0^4 \pi\,[R^2(y) - r^2(y)]\,dy$$

* $R(y) = 2$
* $r(y) = x_2 - x_1 = x - 0 = x = \sqrt{y}$

$$\int_0^4 \pi\,\left[2^2 - (\sqrt{y})^2\right]\cdot dy$$

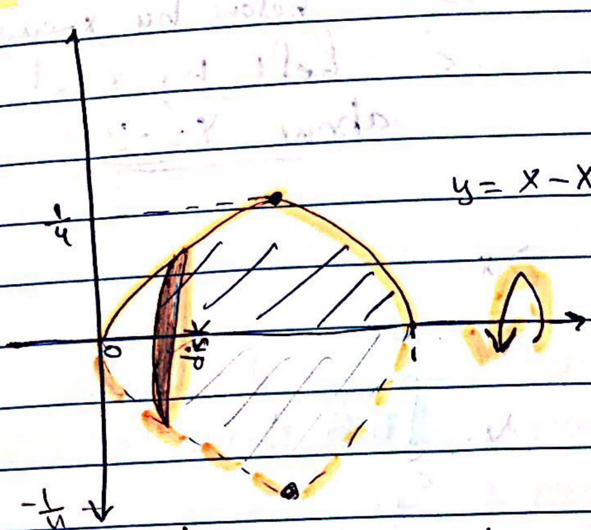
$$= \pi \int_0^4 (4-y)\,dy = 8\pi$$

Q23: $y = x - x^2$, $y=0$ about x-axis.

$y = x - x^2$

$= -[x^2 - x]$

$= -\left[(x-\tfrac{1}{2})^2 - \tfrac{1}{4}\right]$



$$V = \int_a^b A(x)\,dx = \int_0^1 \pi R^2(x)\,dx$$

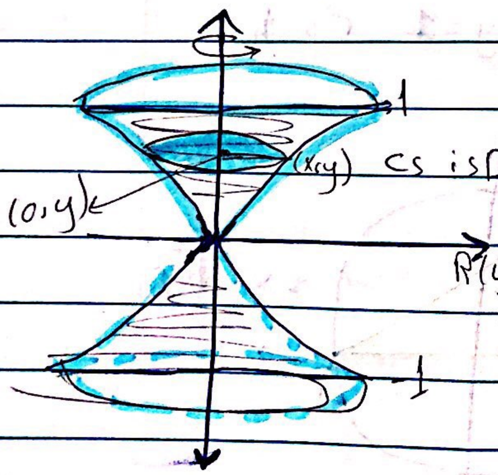
$$= \int_0^1 \pi\,[x - x^2]^2\,dx$$

$$= \pi \int_0^1 (x^2 - 2x^3 + x^4)\,dx = \frac{\pi}{30}$$

Q27 4 $x = \sqrt{5}\;y^2$, $x=0$, $y=-1$, $y=1$

about y-axis.

$$V = \int_c^d A(y)\,dy$$



$R(y) = x_2 - x_1$

$= x - 0$

$= x$

$= \sqrt{5}\,y^2$

$$V = \int_{-1}^{1} \pi R^2(y)\,dy$$

$$= \pi \int_{-1}^{1} (\sqrt{5}\;y^2)^2\,dy$$

$$= \pi \int_{-1}^{1} 5y^4\,dy = 2\pi$$

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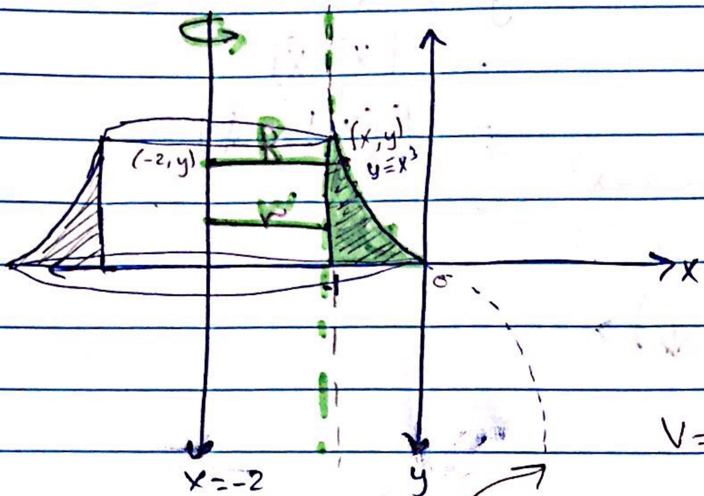
Q_{46} (5) 2^{nd} ~~qsd~~ quadrant, bounded above by $y = -x^3$

" below by x-axis

" left by $x = -1$

about $x = -2$



$$V = \int_c^d \pi A(y)\,dy$$

$$= \pi \int_0^1 [R^2(y) - r^2(y)]\,dy$$

$$= \pi \int_0^1 [((-y)^{1/3} + 2)^2 - 1^2]\,dy$$

$$= \frac{3\pi}{2}$$

$R(y) = x_2 - x_1$

$= x - -2$

$= x + 2$

$= (-y)^{1/3} + 2$

$r(y) = 1$

~~Exp/ Base is isoceles ⊥ y-axis between y = -1 and y = 1 and y = 1 Cs has leg on the disk Cs ⊥~~

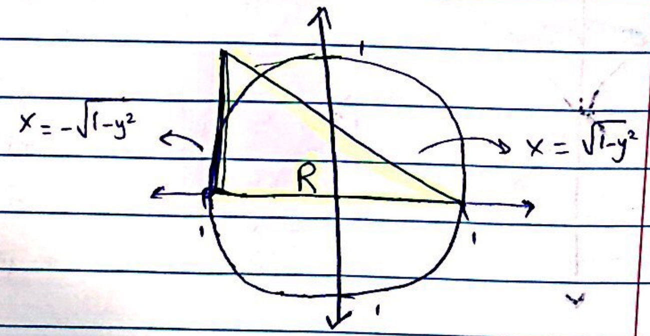
Exp: Base is disk $x^2 + y^2 \leq 1$, Cs is isoceles. ⊥ y-axis, between $y = -1$ and $y = 1$ Cs has leg on disk.

Cs ⊥ y-axis

$$V = \int_c^d A_y \cdot dy$$

$A(y) = \frac{1}{2}$ Bas hieght

$= \frac{1}{2} (\text{leg})^2$



$$\int \frac{1}{2} \left[\sqrt{1-y^2} - -\sqrt{1-y^2}\right]^2$$

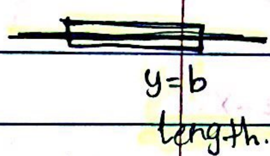
$x^2 + y^2 = 1$

$x = \mp\sqrt{1-y}$

6.2 Shell Method

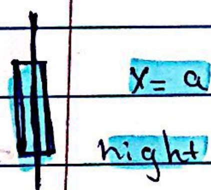
- The volume of the solid generated by revolving the region about x-axis is:-

$$V^* = 2\pi \int_c^d \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{Length}\end{matrix}\right) dy$$



Where → shell radius : is the distance between the shell Length and the axes of revolution,

→ Shell length : is the segment length parallel to the axies of revolution.



- The volume of the solid generated by revolving the region about y-axis is :-

$$V = 2\pi \int_a^b (\text{shell radius}) \left(\begin{matrix}\text{shell}\\ \text{length}\end{matrix}\right) dx$$

6.2 Q9

Exp: Use the shell Method to find the volume of the region bounded by:

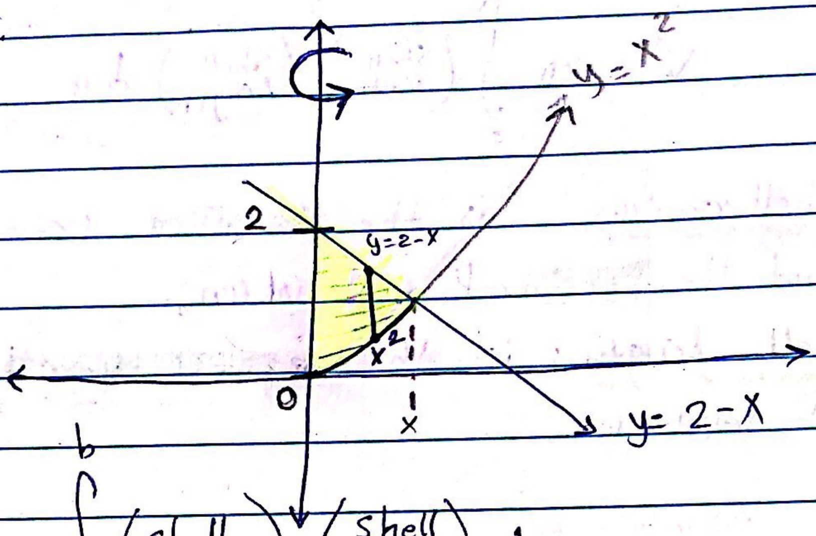
(a) $y = x^2$, $y = 2 - x$, $x = 0$ about y-axis in 1^{st} Quarter.

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = \underset{\alpha}{-2} \,/\, \underset{\checkmark}{1}$



$$y\text{-axis} = 2\pi \int_a^b \left(\begin{matrix}\text{shell}\\ \text{radius}\end{matrix}\right)\left(\begin{matrix}\text{shell}\\ \text{hight}\end{matrix}\right) dx$$

$$= 2\pi \int_0^1 (x)(2 - x - x^2)\, dx$$

$$= 2\pi \int_0^1 (2x - x^2 - x^3)\, dx \quad = \frac{5\pi}{6}$$

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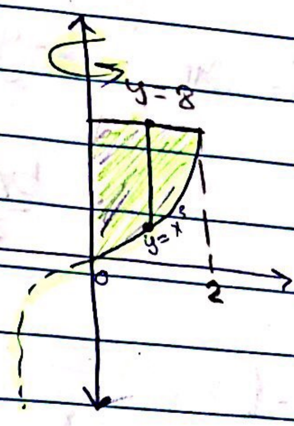
Q24) b) $y=x^3$, $y=8$, $x=0$ about

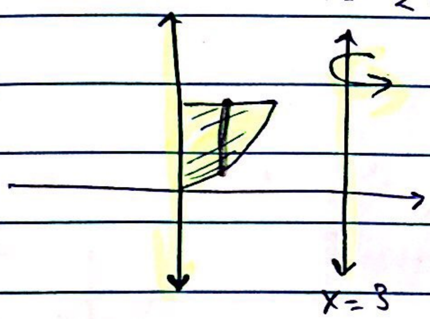
$\boxed{1}$ y-axis :

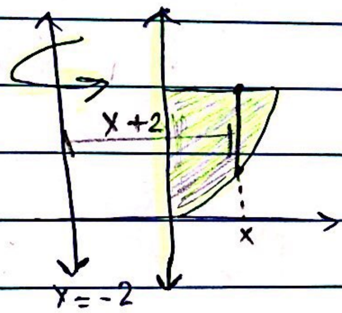
$$\overline{V} = 2\pi \int_0^2 (x)(8-x^3)\,dx = \frac{96\pi}{5}$$



$\boxed{2}$ $x=3$:-

$$\overline{V} = 2\pi \int_0^2 (3-x)(8-x^3)\,dx = \frac{264\pi}{5}$$





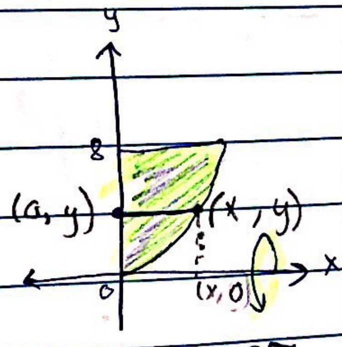
$\boxed{3}$ $x=-2$

$$\overline{V} = 2\pi \int_0^2 (x+2)(8-x^3)\,dx = \frac{336\pi}{5}$$

$\boxed{4}$ x-axis :

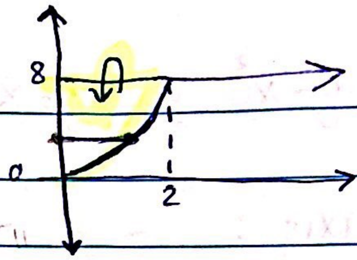
$$V^* = 2\pi \int_0^8 (\ y\ )(\ y^{1/3})\,dy$$

$$= \frac{768\pi}{7}$$



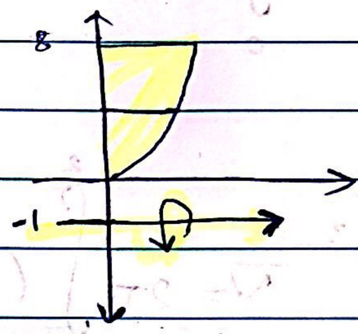
shell $\Delta x = x - 0 = \sqrt[3]{y}$
hight

5 $y = 8$



$$V = 2\pi \int_0^8 (8-y)\left(y^{1/3}\right) dy$$

$$= \frac{576\pi}{7}$$



6 $y = -1$

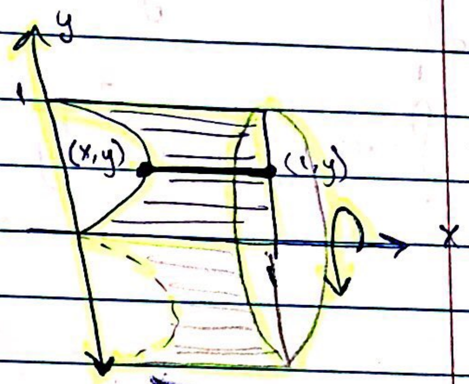
$$V = 2\pi \int_0^8 (1+y)\left(y^{1/3}\right) dy$$

$$= \frac{936\pi}{7}$$

⇒ 6.2

Exp Find the solid generated by rotating the region bounded by 1^{st} quadrant,
$x = y - y^3$, $x = 1$, $y = 1$

(a) about x-axis.



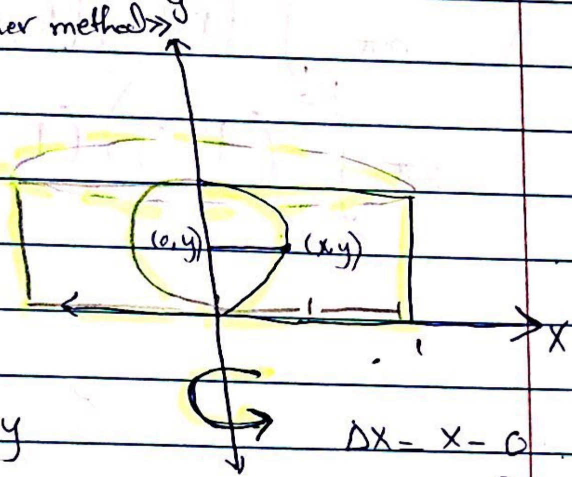
$$V = 2\pi \int_0^1 \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{Length}\end{pmatrix} dy$$

$$= 2\pi \int_0^1 y\left(1 + y + y^3\right) dy$$

$$= \frac{11\pi}{15}$$

$\Delta x = 1 - x$
$= 1 - (y - y^3)$
$= 1 - y + y^3$

(b) about y-axis «washer method»



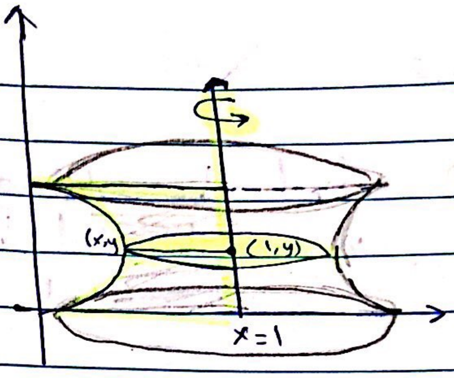
$$V = 2\pi \int_0^1 \left[R^2(y) - r^2(y)\right] dy$$

$$= \pi \int_0^1 \left[1^2 - (y - y^3)^2\right] dy$$

$$= \frac{17\pi}{105}$$

$\Delta x = x - 0$
$= y - y^3$

ⓒ about $x=1$



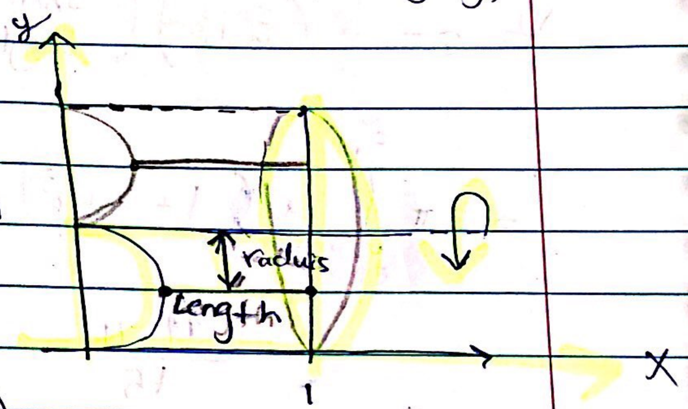
Disk Method

$$V = \int_c^d A(y)\,dy = \int_0^1 \pi R^2(y)\,dy$$

$$= \int_0^1 \pi \left(1-y+y^3\right)^2 dy$$

$\Delta x = 1 - x$
$= 1-(y-y^3)$

ⓓ about $y=1$



Sell Method

$$V = 2\pi \int_c^d \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{length}\end{pmatrix} dy$$

$$2\pi \int_0^1 (1-y)(1-y+y^3)\,dy$$

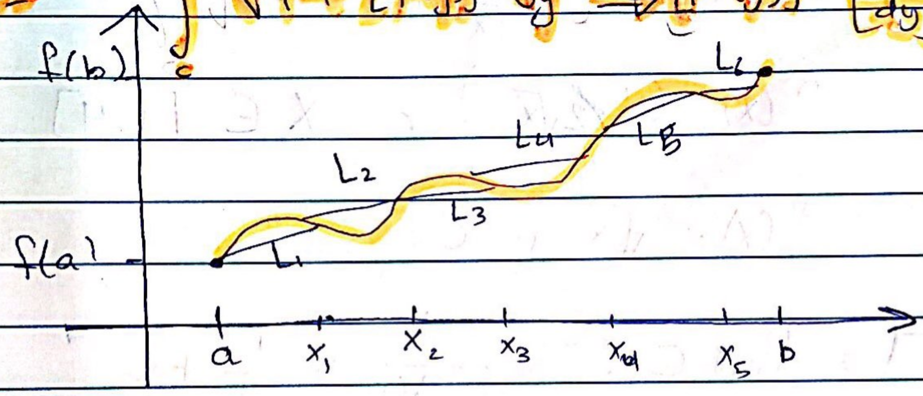
6.3 Arc length

Def: Assume $y = f(x)$ is diff on $[a, b]$

Then, the ~~area~~ length of $f(x)$ from $(a, f(a))$ to $(b, f(b))$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \Rightarrow [f'(x)]^2 = \left[\frac{dx}{dy}\right]^2$$

$$L = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy \Rightarrow [f'(y)]^2 = \left[\frac{dx}{dy}\right]^2$$



Proof:

$\Delta x_k = x(k) - x(k-1)$

$\Delta y_k = y_k - y_{k-1} = f(x_k) - f(x_{k-1})$

Approximated length, $L = \sum_{k=1}^{n} L_k =$

$$= \sum \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$\sqrt{(\Delta x)^2\left(1 + \frac{\Delta y_k^2}{\Delta x_k^2}\right)}$. Since it's diff on $[x_{k-1}, x_k]$, $k = 1, \dots n$

$\rightarrow$ By (MVT), $\exists\, c_k \in (x_{k-1}, x_k)$ s.t

$f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$, $L = \lim_{x \to \infty} \sum_{n=1}^{n} \sqrt{1 + f'(c_k)}\, \Delta x_k$

$$= \int_a^b \sqrt{1 + f(x)^2}\, dx$$

$\Rightarrow$ 6.3 Arc Length

Q19 Exp ① find the curve passes through $(\underset{x}{1},\underset{y}{1})$ and whose length is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}$$

(i) $f'(x)^2 = \frac{1}{4x}$

$f'(x) = \pm \frac{1}{2\sqrt{x}}$

$f'(x) = (1/2\sqrt{x})$, $x \in [1,4]$

$f(x) = \sqrt{x} + c$

To find C ? $\frac{1}{y(1)} = \frac{1}{1} + C$

$1 = 1 + c$

$c = \text{zero}$

② yes, it is unique since it passes (1,1).

Exp: Find the length of the curve:

$$y = \int_0^x \sqrt{\cos 2t}\, dt \qquad x=0 \text{ to } x=\frac{\pi}{4}$$

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx \qquad f'(x) = y' = \sqrt{\cos 2x}$$

$$[f'(x)]^2 = \cos 2x$$

$$= \int_0^{\pi/4} \sqrt{1+\cos 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\cancel{1}+2\cos^2 x - \cancel{1}}\, dx$$

$$L = \int_0^{\pi/4} \sqrt{2\cos^2 x}\, dx = \sqrt{2} \int_0^{\pi/4} |\cos x|\, dx$$

$$= \sqrt{2}\, \sin x \Big|_0^{\pi/4}$$

$$= 1$$

Remark: Note that $y = f(x)$ must be diff on $[a,b]$ otherwise, we try to switch variable $(x = g(y))$

Exp. Find the Length of the curve $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ from $x=0$ to $x=2$

$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$* \frac{dy}{dx} = f'(x) = y'$

$= \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \frac{1}{2}$

$= \frac{2}{3} \times \frac{1}{2} \times \frac{1}{\sqrt[3]{\frac{x}{2}}}$

f' is discont. at $x=0$

$y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$

$y^{\frac{3}{2}} = \frac{x}{2} \Rightarrow 2y^{\frac{3}{2}} = 2\sqrt{y^3}$

$x' = \frac{dx}{dy} = 2\left(\frac{3}{2}\right) y^{1/2} = 3\sqrt{y}$

$(x')^2 = 9y$

$u = 1 + 9y$

$du = 9\, dy$

when $y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=10$

$\Rightarrow \int_0^1 \sqrt{1+9y}\, dy$

$= \frac{1}{9}\int_1^{10} \sqrt{u}\, du = \frac{1}{9} \times \frac{2}{3} u^{\frac{3}{2}} \Big|_1^{10}$

$= \frac{2}{27} \sqrt{u^3} \Big|_1^{10}$

Exp : Find Length of

$$f(x) = \frac{x^3}{12} + \frac{1}{x} \quad \text{on} \quad 1 \le x \le 4 :$$

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

no problem
$0 \notin [1,4]$

$$[f'(x)]^2 = \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}$$

$$1 + [f'(x)]^2 = \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4}$$

$$= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2$$

$$L = \int_1^4 \sqrt{1 + f'^2}\, dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \boxed{6}$$

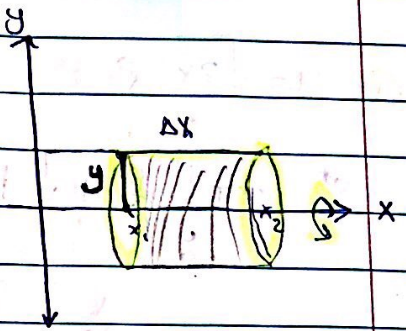
$$L = \frac{x^3}{12} - \frac{1}{x} \Big|_1^4$$

$$= \left(\frac{4(16)}{4(3)} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = 6 \text{ cm}$$

14/11/2018

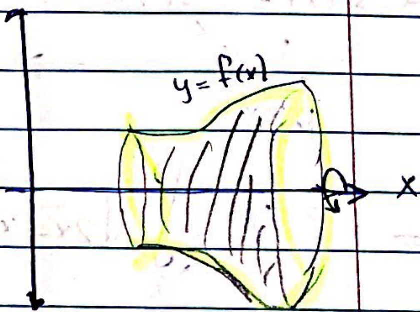
6.4 · Surface Area of revolution.

• The surface area resulted by revolving the segment Line AB with length Δx (about x-axis) is → $S = 2\pi y \, \Delta x$



• The surface area resulted by revolving the segment on [a, b] ~~line~~ $y = f(x)$ about x-axis is

$$S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} \, dx$$



• The surface area resulted by revolving on $[c, d]$ $x = g(y)$ about y-axis is

$$S = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2} \, dy$$

Exp. Find the surface area resulted by revolving the curve

① $y = \sqrt{2x - x^2}$ $\quad \frac{1}{2} \leq x \leq \frac{3}{2}$ about x-axis

$$S^* = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$\Rightarrow f(x) = (2x - x^2)^{\frac{1}{2}}$

$\Rightarrow f'(x) = \frac{1}{2}(2x - x^2)^{-\frac{1}{2}}(2 - 2x)$

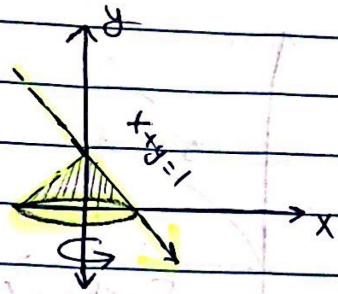
$= \frac{1 - x}{\sqrt{2x - x^2}} \longrightarrow [f'(x)]^2 = \frac{(1 - x)^2}{2x - x^2}$

$$= 2\pi \int_{1/2}^{3/2} \sqrt{2x - x^2} \sqrt{1 + \frac{(1 - x)^2}{2x - x^2}}\, dx$$

$$= 2\pi \int_{1/2}^{3/2} \cancel{\sqrt{2x - x^2}} \sqrt{\frac{\cancel{2x} - \cancel{x^2} + 1 - \cancel{2x} + \cancel{x^2}}{\cancel{2x - x^2}}}\, dx$$

$$= 2\pi \int_{1/2}^{3/2} dx = 2\pi \left(\frac{3}{2} - \frac{1}{2}\right) = 2\pi$$

② $x = 1 - y$ $\quad 0 \leq y \leq 1$ about y-axis



$$\bar{S} = 2\pi \int_c^d g(y) \sqrt{1 + [g'(y)]^2}\, dy$$

$g(y) = 1 - y$

$g'(y) = -1$

$[g'(y)]^2 = 1$

$$\bar{S} = 2\pi \int_0^1 (1 - y) \sqrt{1+1}\, dy$$

$$= 2\pi \int_0^1 \sqrt{2}\,(1-y)\, dy$$

$$= \sqrt{2}\,\pi$$

Exp③ write Integral for the surface area generated by revolving the Curve $y = \cos x$ on $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ about x-axis.

$$S^* = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$f(x) = \cos x$

$f'(x) = -\sin x$

$[f'(x)]^2 = \sin^2 x$

$$S^* = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sqrt{1 + \sin^2 x}\, dx$$