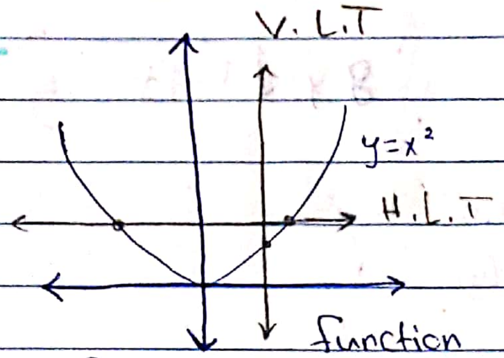
V.L.T ⇒ Vertical Line test
H.L.T ⇒ Horizontal line test

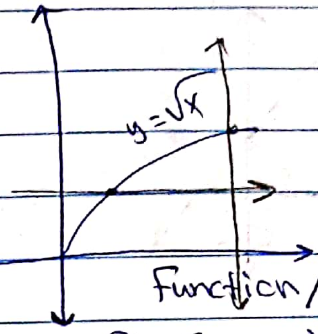
$\boxed{7.1}$ Inverse function:

Exp



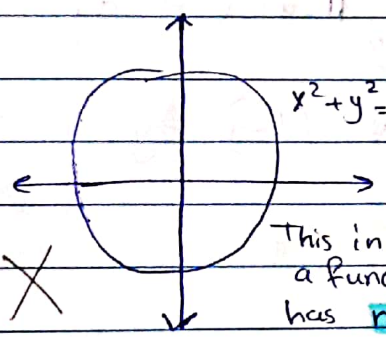
function has no inverse

$D = \mathbb{R}$

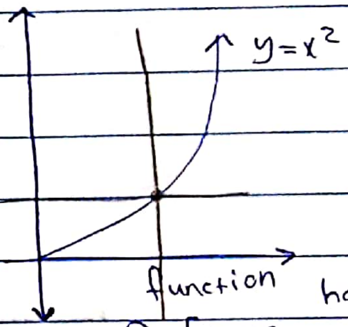


Function / has inverse

$D = [0, \infty)$



This in not a function has no inverse



function has inverse.

$D = [0, \infty$

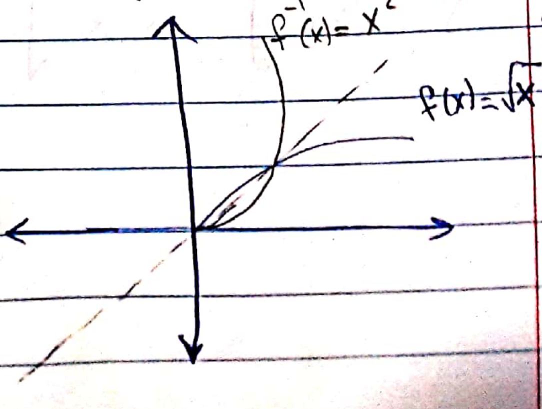
Def: The function $y=f(x)$ is "one to one" on Domain $\underline{\underline{D}}$ if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for all $x_1, x_2 \in D$.

Remark: Only 1-1 function have inverse.

Qustion: How to find $f^{-1}(x)$? • $[f(x)]^{-1} = \dfrac{1}{f(x)}$

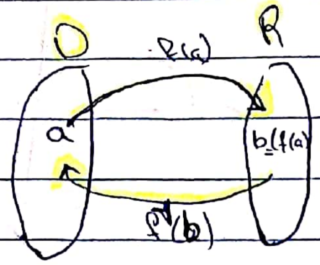
$$\Rightarrow f^{-1}(x) \neq \frac{1}{f(x)}$$

f inverse of x



7.1 → Inverse Functions:

Assume



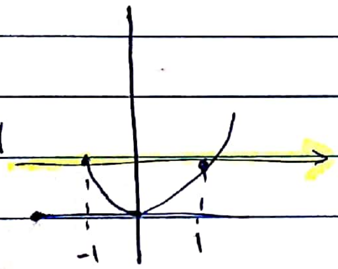
- If f is 1-1 $\Rightarrow$ $f^{-1}(b)$

  $D(f) = R(f^{-1})$

  and $D(f^{-1}) = R(f)$

- f is 1-1 on D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2 \in D$

not 1-1



- other words: f is 1-1 if f cross each HLT at most once.

- Only 1-1 function have inverse.

- the graphs of f and its inverse f^{-1} are symmetries about $x=y$.

Suppose f is $\boxed{1-1}$ How to find it inferse f^{-1} ?

$\boxed{1}$ solve for x

$\boxed{2}$ replace x by y and replace y by x.

$\boxed{3}$ replace y by $f^{-1}(x)$.

Exp: find $f^{-1}(x)$ for ① $f(x) = x^2 \quad x \geq 0$

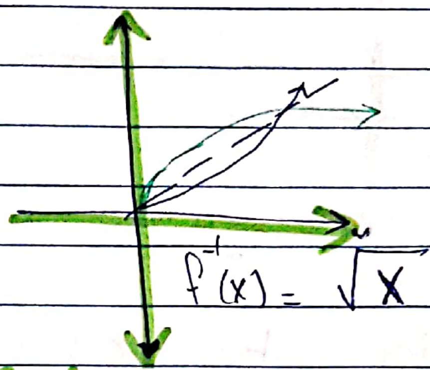
$\boxed{1}$ $y = x^2$

$\sqrt{y} = |x| = x$

$\boxed{2}$ $x = \sqrt{y}$

$y = \sqrt{x}$

$\boxed{3}$ $f^{-1}(x) = \sqrt{x}$



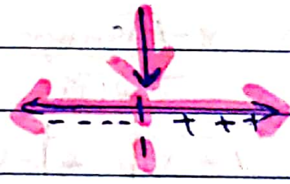
* $(f \circ f^{-1}) = x$
* $(f^{-1} \circ f) = x$

important Note

Exp:

$f(x) = x^2 - 2x \quad x \leq 1$

$y = (x-1)^2 - 1 \rightarrow \sqrt{y+1} = |x-1|$



$\sqrt{y+1} = 1 - x$

$-1 + \sqrt{y+1} = -x$

$\boxed{1}$ $x = 1 - \sqrt{y+1}$

$\boxed{2}$ $y = 1 - \sqrt{x+1}$

$\boxed{3}$ $f^{-1}(x) = 1 - \sqrt{x+1}$

* $D(f) = (-\infty, 1] = R(f^{-1})$
* $R(f) = D(f^{-1}) = [-1, \infty)$
* $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$= f(1 - \sqrt{x+1})$

$= [1 - \sqrt{x+1}]^2 - 2[1 - \sqrt{x+1}]$

$= x$

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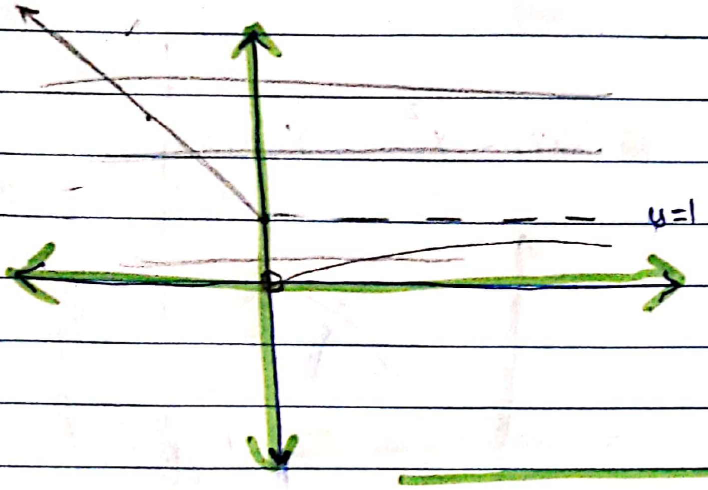
Exp Use $\boxed{\text{HLT}}$ to check if

$$f(x) = \begin{cases} 1 - \frac{x}{2}, & x \le 0 \\ \frac{x}{x+2}, & x > 0 \end{cases}$$

is 1-1 ?!



The HLT cross the function in one point so it is a 1-1 function.

Qustion: How to find derivative of f^{-1}?

$$\frac{df^{-1}}{dx} \text{ or } (f^{-1})'$$

Theorem: Assum $f: D \to R$ is 1-1 and diff and f' never zero. Then f^{-1}.

$f^{-1}: R \longrightarrow D$ is diff.

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a=f^{-1}(b)}}$$

$b = f(a)$

$f^{-1}(b) = f^{-1}(f(a)) = a$

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Esmaeh Jaded

Exp Given $f'(x) = 3x^2$

Find $\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})}$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(\sqrt{2})} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(f(\sqrt{2}))}} = \frac{1}{f'(\sqrt{2})} = \frac{1}{6}$$

Exp: $f(x) = 2x + e^x$

Find $\frac{df^{-1}}{dx}(1)$?

$1 = 2a + e^a$

$\boxed{a=0}$

$$\frac{df^{-1}}{dx}(\underset{b}{1}) = \frac{1}{f'(\underbrace{f^{-1}(1)}_{a})} = \frac{1}{f'(a)} = \frac{1}{2+e^a}$$

$b = f(a) = 1$

$$= \frac{1}{2+1} = \frac{1}{3}$$

$f'(x) = 2 + e^x$

Exp: Find $\left.\dfrac{d f^{-1}}{dx}\right|_{x=0}$ for $f(x) = x^2 - 4x - 5$, $x > 2$

$$\left.\frac{df^{-1}}{dx}\right|_{x=0} = \frac{1}{f'(a)}$$

$$f(a) = 0$$

$$a^2 - 4a - 5 = 0$$

$$(a-5)(a+1) = 0$$

$$a = 5, \underset{\notin\, x>2}{-1}$$

$$a = 5$$

$$f'(x) = 2x - 4 = 2 * 5 - 4 = 6$$

$$\frac{df^{-1}}{dx} = \frac{1}{f'(a)} = \frac{1}{6}$$

Remark: The graph of f and f^{-1} are symmetric about $y = x$

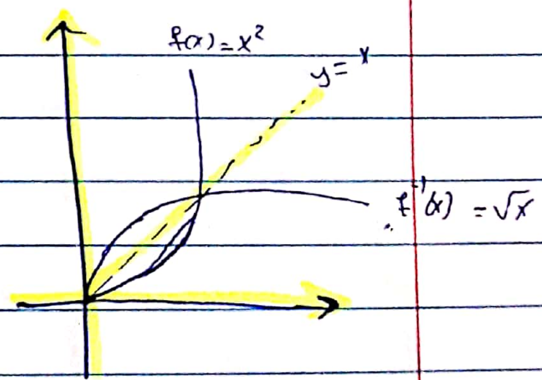
Exp $f(x) = x^2$ $x \geq 0$

1. $\sqrt{y} = \sqrt{x^2}$

$\sqrt{y} = |x|$

$x = \sqrt{y}$

2. $y = \sqrt{x}$
3. $f^{-1}(x) = \sqrt{x}$



$$\left.\frac{df^{-1}}{dx}\right|_{\underset{b}{x=4}} = \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{4}$$

$b = f(a)$

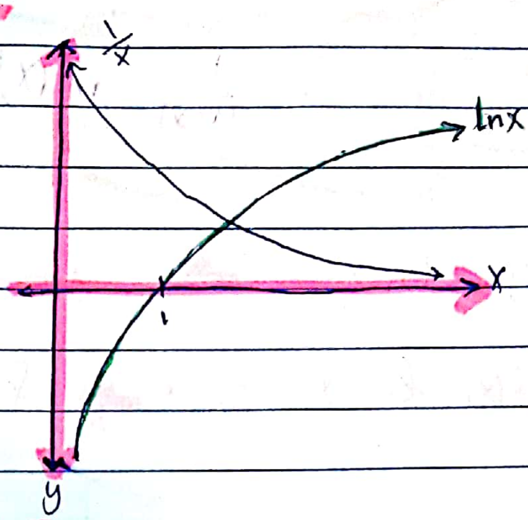
$4 = a^2$

$a = 2$

$$\Rightarrow \quad = \frac{1}{f'(a)} = \frac{1}{f'(2)}$$

$$= \frac{1}{4}$$

7.2 Natural Function:



$y = \ln x$

$D = (0, \infty)$

$R = \mathbb{R}$

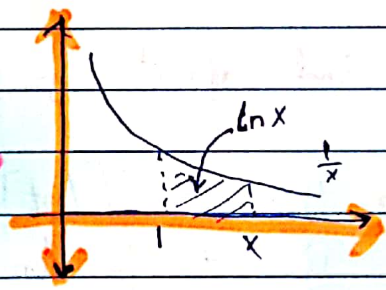
$(\ln x)' = \dfrac{1}{x}$

$\ln x = \int_0^x \frac{1}{t} \, dt$, t

If $x > 1$ then $\ln x > 0$

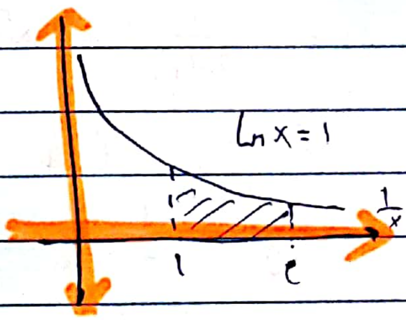
If $0 < x < 1$ then $\ln x < 0$

If $x > 1$ then $\ln x$ is the area below $\frac{1}{t}$ given by $\ln x = \int_1^x \frac{1}{t} \, dt$



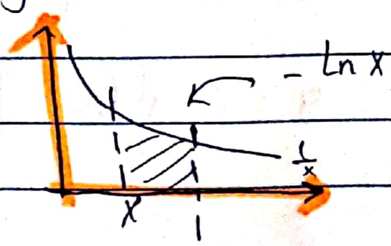
Exp: Take $x = e \approx 2.718$

$\Rightarrow \ln e = 1 = \int_1^e \frac{1}{t} \, dt$



If $x < 1$ then $\ln x$ is negative and given by

$\ln x = \int_x^1 \frac{1}{t} \, dt$



Remark Assume $u(x) > 0$ and diff for $y = \ln(u(x))$

Then $y' = \frac{1}{u(x)} u'(x)$.

Exp ① $y = \ln x \Rightarrow y' = \frac{1}{x}$

② $f(x) \ln x^2 - \sin x \Rightarrow f'(x) = \frac{1}{x^2 - \sin x} * (2x - \cos x)$

③

$\Rightarrow$ Proparties of Natural function:

Assume a and b are Positive constant. Then

① $\ln(ab) = \ln a + \ln b$

② $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

③ $\ln a^b = b \ln a$

④ $\ln \frac{1}{a} = -\ln a$

Exp: Find y' if $y = t\sqrt{\ln t}$

S1 $\Rightarrow$ $y = t\,[\ln t]^{\frac{1}{2}}$

$y' = t\,\frac{1}{2}[\ln t]^{-\frac{1}{2}}\frac{1}{t} + [\ln t]^{\frac{1}{2}}$

$= \frac{1}{2\sqrt{\ln t}} + \sqrt{\ln t}$

S2 $\Rightarrow$ $\ln y = \ln t + \ln(\ln t)^{\frac{1}{2}}$

$\frac{y'}{y} = \frac{1}{t} + \frac{1}{\sqrt{\ln t}}\,\frac{1}{2}(\ln t)^{-\frac{1}{2}}\frac{1}{t}$

$y' = y\left[\frac{1}{t} + \frac{1}{\sqrt{\ln t}}\,\frac{1}{2}(\ln t)^{-\frac{1}{2}}\frac{1}{t}\right]$

$y' = t\sqrt{\ln t}\left[\frac{1}{t} + \frac{1}{2t\ln t}\right]$

② $y = t\,(t+1)(t+2)(t+3)$

$\ln y = \ln t + \ln(t+1) + \ln(t+2) + \ln(t+3)$

$\frac{y'}{y} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}$

$y' = y\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} + \frac{1}{t+3}\right)$

③ $y = \ln\left[\ln(\ln x)\right]$

$= \frac{1}{\ln(\ln x)} * \frac{1}{\ln x} * \frac{1}{x}$

Exp. show that $f(x) = x - \ln x$, $x > 1$ is increasing.

(a) $f'(x) = 1 - \frac{1}{x} \Rightarrow f(x) \; + + + + + + \rightarrow \quad x > 1$

(b) show that $x > \ln x$

$f(1) = 1 - \ln 1$

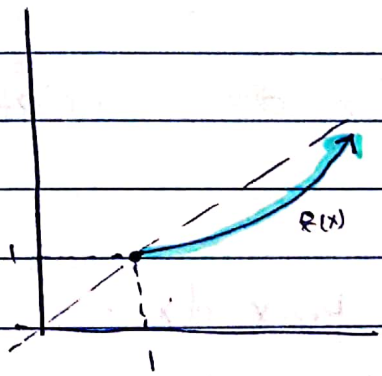
$= 1 - 0 = 1$

and $f \uparrow$

$f(x) > 0$

$x - \ln x > 0$

$x > \ln x$



Exp: Express $\ln\sqrt{13.5}$ interms of $\ln 2$ and $\ln 3$??

$\ln (13.5)^{\frac{1}{2}}$

$= \frac{1}{2} \ln (13.5)$

$= \frac{1}{2} \ln \frac{27}{2}$

$= \frac{1}{2} (\ln 27 - \ln 2)$

$= \frac{1}{2} (3 \ln 3 - \ln 2)$

• Remember :-

① $\int \sec^2 x \, dx = \tan x + c$

② $\int \csc^2 x \, dx = -\cot x + c$

③ $\int \sec x \tan x \, dx = \sec x + c$

④ $\int \csc x \cot x \, dx = -\csc x + c$

⑤ $\int \tan x \, dx = \ln|\sec x| + c$

⑥ $\int \cot x \, dx = + \ln|\sin x| + c$

⑦ $\int \sec x \, dx = + \ln|\sec x + \tan x| + c$

⑧ $\int \csc x \, dx = - \ln|\csc x + \cot x| + c$

Proof

$\boxed{5}$ $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$= - \int \frac{-\sin x}{\cos x} \, dx$

$= - \ln|\cos x| + c$

$= \ln|\sec x| + c$

$\boxed{6}$ نفس الطريقة

$\boxed{7}$ $\int \sec x \, dx = \int \sec x \left[\frac{\sec x + \tan x}{\sec x + \tan x}\right] dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$

$= \ln|\tan x + \sec x| + c$

$\boxed{9}$ نفس طريقة 7

Exp: Find ?

(1) $\int_{-3}^{-2} \frac{dx}{x} = \ln|x| \Big|_{-3}^{-2} = \ln 2 - \ln 3$

$= \ln \frac{2}{3}$

(2) $\int_{0}^{\frac{\pi}{2}} \tan \frac{x}{2} \, dx = 2 \int_{0}^{\frac{\pi}{4}} \tan u \, du$

$u = \frac{x}{2}$

$du = \frac{1}{2} dx$

$x = 0 \rightarrow u = 0$

$x = \frac{\pi}{2} \rightarrow u = \frac{\pi}{4}$

$= 2 \ln|\sec u| \Big|_{0}^{\frac{\pi}{4}}$

$= 2 \ln|\sec \frac{\pi}{4}| - 2 \ln|\sec 0|$

$= 2 \ln \sqrt{2} - 2 \ln 1$

$= \ln 2$

(3) $\int_{2}^{4} \frac{dx}{x \ln x}$

$u = \ln x$

$du = \frac{1}{x} dx$

$x = 2 \Rightarrow u = \ln 2$

$x = 4 \Rightarrow u = \ln 4$

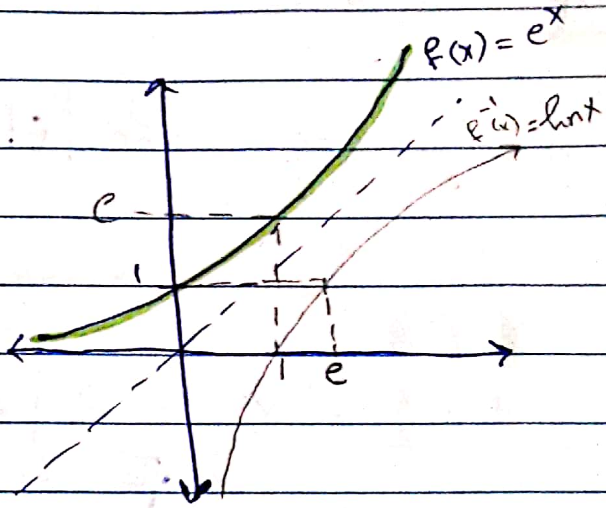
$\int_{\ln 2}^{\ln 4} \frac{du}{u}$

$= \ln u \Big|_{\ln 2}^{\ln 4} = \ln(\ln 4) - \ln(\ln 2)$

$= \ln\left(\frac{\ln 4}{\ln 2}\right) = \ln\left(\frac{2 \cancel{\ln 2}}{\cancel{\ln 2}}\right)$

$= \ln 2$

7.3 Exponensial Function :-



→ $f(x) = e^x$

$D = \mathbb{R}$

$R = (0, \infty)$

→ $y = e^x$

$\ln y = x \ln e$

$\ln y = x$

$x = \ln y$

$y = \ln x$

$f^{-1}(x) = \ln x$

Remarks :

① $e^{\ln x} = x \quad \forall x > 0$

② $\ln e^x = x$

Solve : ① $e^{\ln 2x} = 8 \Rightarrow 2x = 8 \Rightarrow x = 4$

② $e^{(\ln 2)x} = 8$

$\ln e^{(\ln 2)x} = \ln 8$

$(\ln 2)x = \ln 8$

$x = \dfrac{\ln 8}{\ln 2} = \dfrac{\ln 2^3}{\ln 2} = 3$

Remark suppos $a>0$ and $u(x)$ is diff

If $y = a^{u(x)}$ then $y' = a^{u(x)} u'(x) \ln a$

Henec, $\boxed{\int a^{u(x)} u'(x)\, dx = \frac{1}{\ln a} a^{u(x)} + C}$

Exp. Find y' if ① $y = 3^x$

S_1 $\ln y = x \ln 3 \Rightarrow \frac{y'}{y} = \ln 3 \Rightarrow y' = 3^x \ln 3$

S_2 $y' = 3^x (1) \ln 3$

---

② $y = e^{5-7x}$

$y' = e^{5-7x} \cdot (-7)$ ~~$\ln e$~~

$y' = -7 e^{5-7x}$

---

③ $f(x) = \pi^{ex}$

$f'(x) = \pi^{ex} (e) \ln \pi$

$= \pi^{ex} \ln \pi^e$

---

④ $f(x) = x^{\pi + e}$

$f'(x) = (\pi + e) x^{(\pi + e - 1)}$

---

⑤ $y = (\ln 2)\, 2^{\sin 3t}$

$y' = (\ln 2)\, 2^{\sin 3t} \cos 3t\, (3) \ln 2$

$= (\ln 2)^2 (3)\, 2^{\sin 3t} \cos 3t$

Exp. Find ① $\int 5^{\sec\theta} \ln 5 \sec\theta \tan\theta \, d\theta = 5^{\sec\theta} + C$

② $\frac{\ln 5}{\ln 5} \int 5^{\sec\theta + 4} \sec\theta \tan\theta \, d\theta$

$\frac{1}{\ln 5} 5^{\sec\theta + 4} + C$

③ $\int 7^{x} \, dx$

$= \frac{\ln 7}{\ln 7} \int 7^{x} \, dx$

$\frac{1}{\ln 7} 7^{x} + C$

- properties of Expo

$\forall \; x_1, x_2, x_3 \Rightarrow$

$\boxed{1}$ $e^{x_1} e^{x_2} = e^{x_1 + x_2}$

$\boxed{2}$ $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

$\boxed{3}$ $[e^{x_1}]^{x_2} = e^{x_1 x_2}$

$\boxed{4}$ $e^{\ln y} = y \qquad y > 0$

Exp :- $y = 2^x = e^{\ln 2^x}$

$= e^{x \ln 2}$

$y' = e^{x \ln 2} * \ln 2$

Exp :- $\int \frac{dx}{1+e^x} = ?$

$$\int \frac{dx}{1+e^x} * \left(\frac{e^{-x}}{e^{-x}}\right) = \int \frac{e^{-x}}{1+e^{-x}} dx$$

$u = e^{-x} + 1$

$du = -e^{-x} dx$

$$\int \frac{\cancel{e^{-x}} du}{\cancel{(-e^{-x})} u} = -\int \frac{du}{u}$$

$= -\ln u$

$= -\ln(e^{-x} + 1)$

$= \ln \frac{1}{e^{-x}+1} + C$

If $y = \ln u(x) \rightarrow y' = \frac{1}{u(x)} u'(x)$

Exp: Find

① $\int_{\ln 2}^{\ln 3} e^x \, dx = e^x \Big|_{\ln 2}^{\ln 3} = e^{\ln 3} - e^{\ln 2}$

$= 3 - 2 = 1$

② $\int \frac{e^{\sqrt{r}}}{\sqrt{r}} \, dr$ $\quad u = \sqrt{r}$

$du = \frac{1}{2\sqrt{r}} \, dr$

$2 \int \frac{e^u}{\cancel{2\sqrt{r}}} \, du \, \cancel{2\sqrt{r}}$

$= 2e^u + c = 2e^{\sqrt{r}} + c$

③ $\int t e^{-t^2} \, dt$ $\quad u = -t^2$

$du = -2t \, dt$

$-\frac{1}{2} \int e^u \, du$

$= -\frac{1}{2} e^{-t^2} + c$

Exp Find y' if :-

① $\ln xy = e^{x+y}$

$\ln x + \ln y = e^{(x+y)}$

$\frac{1}{x} + \frac{y'}{y} = e^{(x+y)} (1 + y')$

بعد التوزيع =

$\frac{1}{x} + \frac{y'}{y} = e^{(x+y)} + y' e^{(x+y)}$

$y' = \frac{y}{x} \left[\frac{x e^{x+y} - 1}{1 - y e^{x+y}} \right]$

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2) $y = [\ln x]^{\ln x}$

$$\ln y = \ln [\ln x]^{\ln x}$$

$$= \ln x \ \ln(\ln x)$$

$$\frac{y'}{y} = \ln x * \frac{1}{\ln x} * \frac{1}{x} + \ln(\ln x) \frac{1}{x}$$

$$\frac{y'}{y} = \frac{1}{x}\left(1 + \ln(\ln x)\right)$$

$$y' = \frac{(\ln x)^{\ln x}}{x}\left(1 + \ln(\ln x)\right)$$

Exp: Find $\int \frac{x\, 2^{x^2}}{1+2^{x^2}} \, dx$

$u = 1 + 2^{x^2}$

$du = 2^{x^2}(2x)\ln 2 \, dx$

$\frac{du}{2\ln 2} = x\, 2^{x^2} dx$

$$\int \frac{1}{u} \frac{du}{2\ln 2}$$

$$\frac{1}{2\ln 2} \int \frac{1}{u} du = \frac{1}{2\ln 2} \ln|u| + c$$

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Exp Find $f'(e)$

if $f(x) = x^x$

(1) $\ln|f(x)| = x \ln x$

$$\frac{f'(x)}{f(x)} = x\frac{1}{x} + \ln x$$

$$\frac{f'(e)}{e^e} = 1 + 1$$

$$f'(e) = 2e^e$$

(2) $f(x) = e^{\ln x^x}$

$$= e^{x\ln x}$$

$$= e^{x\ln x}\left[x\frac{1}{x} + \ln x\right]$$

$$f'(e) = e^{e\ln e}[1 + \ln e]$$

$$= 2e^e$$

Exp. Find $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = ?$

$$= \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim \frac{\ln(1+x)}{e^x}$$

since exp (e^x) cont $\Rightarrow e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}}$

$$e^{\lim_{x\to 0}\frac{\ln(1+x)}{x}} = e^{\lim_{x\to 0}\frac{\frac{1}{x+1}}{1}} = e^1 = e$$

* $\frac{0}{0} \rightarrow$ lopital

Logarithmic function.

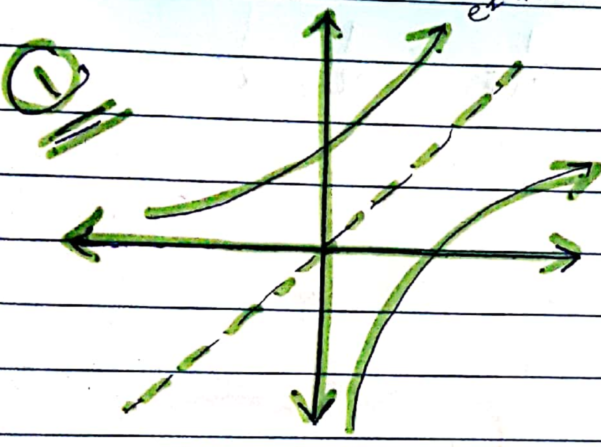
$\log_a u(x) = \dfrac{\ln(u(x))}{\ln a}$ $\quad a>0,\ a\neq 1$, $u(x)>0$

Special case $a=e$

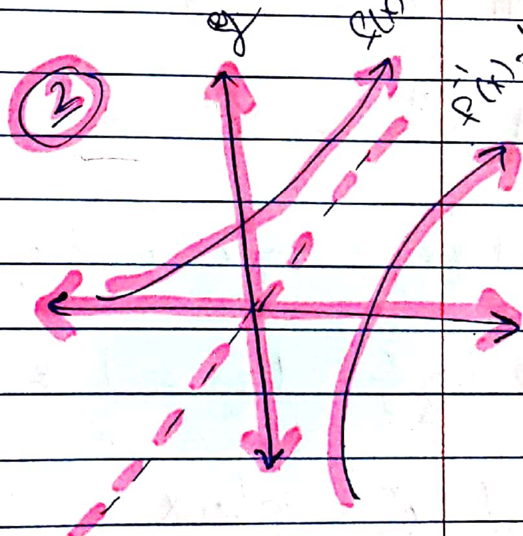
$\rightarrow \log_e u(x) = \dfrac{\ln u(x)}{\ln e} = \ln u(x)$

$* \boxed{\log_e x = \ln x}$

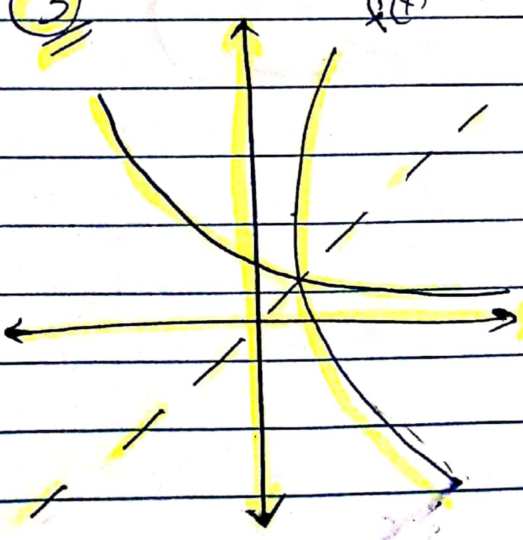
① $e^x = f(x)$ $\quad f^{-1}(x) = \dfrac{\ln x}{\ln e} = \log_e x$



② $f(x) = 2^x$ $\quad f^{-1}(x) = \dfrac{\ln x}{\ln 2} = \log_2 x$



③ $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$



$f^{-1} = -\log_2 x$

$= -\dfrac{\ln x}{\ln 2}$

$= \dfrac{-\ln x}{\ln 2} = \dfrac{\ln x}{\ln \frac{1}{2}} = \log_{\frac{1}{2}} x$

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Re call that if :

$\boxed{1}$ $y = \ln u(x)$ then $y' = \frac{dy}{dx} = \frac{1}{u(x)} u'(x)$

where $u(x) > 0$ and diff

$\boxed{2}$ $y = \log_a u(x) = \frac{\ln u(x)}{\ln a}$ then

$y' = \frac{1}{\ln a} \frac{1}{u(x)} u'(x)$ where $a > 0$ $a \neq 1$

$u(x) > 0$

$u(x)$ is diff

Exp :- $\frac{d}{dx}\left(\log_2 \sqrt{x}\right) = \frac{1}{\ln 2} \frac{1}{\sqrt{x}} \frac{1}{2\sqrt{x}} = \frac{1}{2\ln 2} \frac{1}{x}$

properites :-

$\boxed{1}$ $\log_a a^x = x$ $\forall x$

$\boxed{2}$ $a^{\log_a x} = x$ $\forall x > 0$

$\boxed{3}$ $\log_a xy = \log_a x + \log_a y$

$\boxed{4}$ $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\boxed{5}$ $\log_a x^r = r \log_a x$

$\boxed{6}$ $\log_a \frac{1}{x} = -\log_a x$

Exp: $2^{\log_2 x} = 8$

$x = ?$

$x = 8$

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Exp: $\displaystyle\int \frac{\log_2 x}{x}\,dx = \int \frac{\ln x}{x(\ln 2)}\,dx$

$$= \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$$

$u = \ln x$

$du = \dfrac{dx}{x}$

$$= \frac{1}{\ln 2}\int u\,du$$

$$= \frac{1}{\ln 2}\,\frac{u^2}{2} + c$$

$$= \frac{1}{\ln 4}(\ln x)^2 + c$$

Exp: Find y' for

① $y = \log_2 (8x^{\ln 2})$

$$y' = \frac{1}{\ln 2}\,\frac{1}{8x^{\ln 2}}\left(8\ln 2\; x^{\ln 2 - 1}\right) = \frac{1}{x}$$

② $\displaystyle y = \int_0^{\log_4 x} 2\ln 2\; 4^t\,dt$

$$y' = 2\ln 2\; 4^{\log_4 x}\left(\log_4 x\right)'$$

$$\ln 4\; x \cdot \frac{1}{\ln 4}\,\frac{1}{x}\,(1) = 1$$

Exp: Find y so that

$$\log_2 y = 3$$

$$2^{\log_2 y} = 2^3$$

$$y = 8$$

7.4 Separable Differential Equations:- (DE)

Exp: Solve $\frac{dy}{dx} = \frac{\cos x}{1+3y^2}$

If $(y_{x=0} = 1)$ (initial condition).

DE's: equations with derivatives.

relations — changes (rate)

$$\int (1+3y^2)\,dy = \int (\cos x)\,dx$$

$$y + y^3 = \sin x + c$$

$$1+1 = \sin(0) + c$$

$$c = 2$$

$\boxed{y + y^3 = \sin x + 2} \Rightarrow$ implicit solution.

Remark: (IVP) is DE with (IC)

IVP: inital value problem

IC: intial condition

~~Explicit~~ sol.

Exp: solve the IVP

$$\frac{dy}{dt} = \sqrt{y} \rightarrow y(1) = 2$$

$$\frac{dy}{\sqrt{y}} = dt \Rightarrow \int \frac{dy}{\sqrt{y}} = \int dt$$

$$2y^{1/2} = t + C$$

* to find "c" $y(1) = 2 \Rightarrow 2 * \sqrt{2} = 1 + C$

$$C = 2\sqrt{2} - 1$$

$$\frac{2\sqrt{y} = t + 2\sqrt{2} - 1}{2}$$

$$\left(\sqrt{y} = \frac{t}{2} + \sqrt{2} - \frac{1}{2}\right)^2$$

$$y = \left(\frac{t}{2} + \sqrt{2} - \frac{1}{2}\right)^2 \longrightarrow$$ * Explicit sol^n

Exp $p(t)$: population size at time t

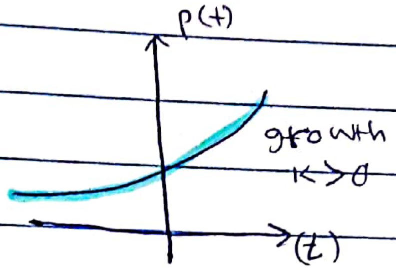
$$\frac{dp}{dt} \propto p$$

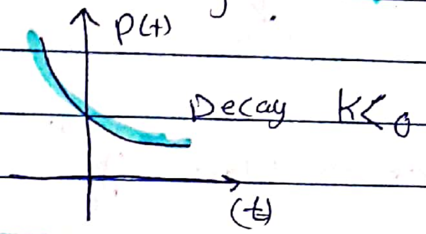
$$\frac{dp}{dt} = kp \quad ① \qquad \text{[where } k \text{ is constant]}$$

$p(0) = P_0$ and called:

① $k > 0$ growth rate.

② $k < 0$ decay rate.





Question: How to solve the (IVP) ①:

$$\frac{dp}{dt} = kp$$

$$\int \frac{dp}{p} = \int k\,dt \Rightarrow \ln|p| = kt + c$$

$$|p| = e^{kt+c}$$

$$|p| = \underbrace{\pm e^{c}}_{D} e^{kt}$$

$$p(t) = D e^{kt}$$

To Find D we use IC: $p(0) = P_0$

$$p(0) = D e^{k(0)}$$

$$P_0 = D$$

Henec the solution of the IVP ① is given by

$$\boxed{p(t) = P_0 e^{kt}} \quad ②$$

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Exp: Solve the IVP.

$$\frac{dy}{dt} = -2y \quad , \quad y(0) = 4$$

solution $y(t) = y_0 e^{kt}$

$y(t) = 4 e^{-2t}$

7.4 complete

IVP: $\boxed{\frac{dy}{dt} = ky}$, $\boxed{y(0) = y_0}$

$\Downarrow$ DE $\qquad \Downarrow$ IC

has solution $y(t) = y_0 e^{kt}$

Exp: (k>0): $y(t)$: populations size at time t

A poupulation increases Exponsially with time such that the population after 3 years is 10,000 and the population after 5 years is 40,000. Find the initial size of population?

$$\frac{dP}{dt} = kP \qquad k > 0$$

$\Downarrow$

$P(t) = P_0 e^{kt}$

$P(3) = P_0 e^{3k}$

$10{,}000 = P_0 e^{3k}$ — ①

$40{,}000 = P_0 e^{5k}$ — ②

$P(3) = 10{,}000$

$P(5) = 40{,}000$

Find P_0?

② ÷ ①

$\hookrightarrow 4 = e^{2k}$

$\ln 4 = \ln e^{2k}$

$\cancel{2} \ln 2 = \cancel{2}k \Rightarrow \boxed{\ln 2 = k}$

k in ① $\rightarrow 10{,}000 = P_0 e^{3\ln 2} = 8P_0$

$P_0 = \frac{10{,}000}{8} = 1250$

Radioactivity :

$Q(t)$: amount of the quantity at time t present

$$\frac{dQ}{dt} = -kQ \quad , \quad Q(0) = Q_0 \qquad \boxed{K<0}$$

$$\boxed{Q(t) = Q_0 e^{-kt}}$$

Half-life time (t^*) is the time that make the initial quantity half.

$$Q(t^*) = \frac{1}{2} Q_0$$

$$\rightarrow Q(t^*) = \frac{1}{2} Q_0 \rightarrow Q_0 e^{-kt^*} = \frac{1}{2} Q_0$$

$$e^{-kt^*} = \frac{1}{2}$$

$$-kt^* = \ln\frac{1}{2} = -\ln 2 \rightarrow t^* = \frac{\ln 2}{k}$$

Exp: The half-life time of Polonuim is 139 days If this sample is not useful after 95% of radioactivity then how many days will it be useful to use ?

$$t^* = 139 = \frac{\ln 2}{k} \Longleftrightarrow k = \frac{\ln 2}{139} \quad (\text{find } \bar{t} \text{ s.t } Q(\bar{t}) = 0.05 Q_0)$$

$$Q_0 e^{-k\bar{t}} = 0.05 Q_0$$

$$-k\bar{t} = \ln 0.05$$

$$\bar{t} = -\frac{\ln 0.05}{k}$$

$$\bar{t} \simeq 600 \text{ days}$$

Exp: A radio active material has half - life time $\ln 8$ years , How many years it will take to decay 80% ??

$$t^* = \ln 8 = \frac{\ln 2}{k} \rightarrow k = \frac{1}{3}$$

$$Q(t) = Q_0 \, e^{-kt}$$

$$\frac{20}{100}\cancel{Q_0} = \cancel{Q_0} \, e^{-kt}$$

$$\ln 0.02 = -kt$$

$$-3 \ln 0.02 = t \rightarrow t = 3 \ln \frac{10}{2} = 3 \ln 5.$$

7.5 Indeterminants' forms :

$\left(\left(\frac{0}{0}\right)\left(\frac{\infty}{\infty}\right)\right)$ → L'Hopital Rule

$\left((\infty \cdot 0)(\infty - \infty)\right)$ → Simple bg «توحيد المقامات»

$\left((0^0)(1^\infty)(\infty^0)\right)$ → we use this idea

$$\lim_{x \to c} f(x) = \lim_{x \to c} e^{\ln f(x)}$$

Th: (L'Hopital Rule):

Assume $g(x)$ and $f(x)$ are diff at c with $g(c) = f(c) = 0$ and $g'(c) \neq 0$

then $\displaystyle\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$ ⇒ then put (c) instead (x)

Find :

(1) $\lim\limits_{x \to -2} \frac{x+2}{x^2-4} \quad \left(\frac{0}{0}\right)$

$\lim\limits_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$

(2) $\lim\limits_{x \to 0} \frac{x - \sin x}{x^3} \quad \left(\frac{0}{0}\right)$

$\lim\limits_{x \to 0} \frac{1-\cos x}{3x^2} \quad \left(\frac{0}{0}\right)$

$\lim\limits_{x \to 0} \frac{\sin x}{6x} \quad \left(\frac{0}{0}\right)$

$\lim\limits_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$

(3) $\lim\limits_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta} \quad \left(\frac{0}{0}\right)$

$\lim\limits_{\theta \to 0} 3^{\sin\theta} \cos\theta \ln 3 = (1)(1) \ln 3$

$= \ln 3$

(4) $\lim\limits_{x \to 0^+} [\ln x - \ln(\sin x)]$

$[-\infty - -\infty]$

$\lim\limits_{x \to 0^+} \ln\left(\frac{x}{\sin x}\right) = \ln \lim\limits_{x \to 0^+} \frac{x}{\sin x}$

$= \ln \lim\limits_{x \to 0^+} \frac{1}{\cos x}$

$= \ln 1$

$= 0$

$\Rightarrow$ we can put the (limit) on the function if it is continuous.

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$\boxed{7.5}$ Indeterminals forms:-

$\left(\frac{0}{0}, \frac{\infty}{\infty}\right) \rightarrow$ Derivative

Revision

$(\infty - \infty)$, $(0 \cdot \infty) \rightarrow$ simplify

0^0 , 1^∞ , $\infty^0 \longrightarrow f(x) = e^{\ln f(x)}$

Exp: Find ① $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x} \right) = \infty - \infty$

$$\lim_{x \to 1^+} \left[\frac{\ln x - (x-1)}{(x-1)\ln x} \right] \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 1^+} \left[\frac{\frac{1}{x} - 1}{\frac{(x-1)}{x} - \ln x} \right] \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 1^+} \frac{\frac{-1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = -\frac{1}{2}$$

Exp: $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}}$ $\qquad 1^\infty$

$$= \lim_{x \to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} e^{\frac{1}{x}\ln(1+x)}$$

$$= e^{\lim_{x \to 0^+} \left(\frac{\ln(1+x)}{x} \right)} = e^{\lim_{x \to 0^+} \frac{\frac{1}{1+x}}{1}} = e$$

Exp: $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$

$$x\ln\left(1+\tfrac{1}{x}\right) \to 0 * \infty$$

$$\lim_{x\to 0^+} e^{x\ln\left(1+\frac{1}{x}\right)} = \lim_{x\to 0^+} e^{\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} \quad \frac{\infty}{\infty}$$

$$e^{\lim_{x\to 0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} = e^{\lim_{x\to 0^+}\frac{\frac{\cancel{-\frac{1}{x^2}}}{1+\frac{1}{x}}}{\cancel{-\frac{1}{x^2}}}}$$

$$e^{\lim_{x\to 0^+}\frac{1}{1+\frac{1}{x}}} = e^0 = 1$$

Exp: $\lim_{x\to 0^+}\sqrt{x}\ \ln x$

① $\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sqrt{x}}} \quad \frac{-\infty}{\infty}$

$$= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}} = -2\lim_{x\to 0^+} x^{-1}x^{\frac{3}{2}}$$

$$= -2\lim_{x\to 0^+}\sqrt{x} = \text{zero}$$

Exp: $\lim_{x\to\infty}\frac{e^x}{x^2}\quad\left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to\infty}\frac{e^x}{2x}\quad\left(\frac{\infty}{\infty}\right)$$

$$\lim_{x\to\infty}\frac{e^x}{2} = \infty$$

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Exp: $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \lim\limits_{x\to\infty} e^{\ln x^{\frac{1}{x}}}$

$= \lim\limits_{x\to\infty} e^{\frac{\ln x}{x}}$

$= e^{\lim\limits_{x\to\infty} \frac{\ln x}{x}} \quad \left(\frac{\infty}{\infty}\right) \quad = e^{\lim\limits_{x\to\infty} \frac{\frac{1}{x}}{1}} = e^{0} = 1$

Exp: $\lim\limits_{x\to-\infty} \dfrac{2^{x} + 4^{x}}{5^{x} - 2^{x}} * \dfrac{2^{x}}{2^{x}}$

* إذا x تقترب من ∞ فنقسم على أكبر عدد
* إذا x تقترب من $-\infty$ فنقسم على أصغر عدد

$\lim\limits_{x\to-\infty} \dfrac{1 + 2^{x}}{\left(\frac{5}{2}\right)^{x} - 1} = \dfrac{1+0}{0-1} = -1$

Exp: $\lim\limits_{x\to\infty} \dfrac{2^{x} + 4^{x}}{5^{x} - 2^{x}} * \dfrac{5^{x}}{5^{x}} = \lim\limits_{x\to\infty} \dfrac{\left(\frac{2}{5}\right)^{x} + \left(\frac{4}{5}\right)^{x}}{1 - \left(\frac{2}{5}\right)^{x}}$

$= \dfrac{0+0}{1-0} = 0$

Exp: Solve $2^{\log_4 5x} = x$

Sol: $2^{\log_2 5x} \;\; = \dfrac{\ln 5x}{\ln 4} = \dfrac{\ln 5x}{2\ln 2} = \dfrac{1}{2}\,\dfrac{\ln 5x}{\ln 2}$

$= \frac{1}{2} \log_2 5x$

$= \log_2 \sqrt{5x}$

Hence $2^{\log_4 5x} = x$

$2^{\log_2 \sqrt{5x}} = x$

$\sqrt{5x} = x$

$5x = x^2 \rightarrow x = 0$ or $x = 5$

S(2) $\ln 2^{\log_4 5x} = \ln x$

$\log_4 5x \ \ln 2 = \ln x$

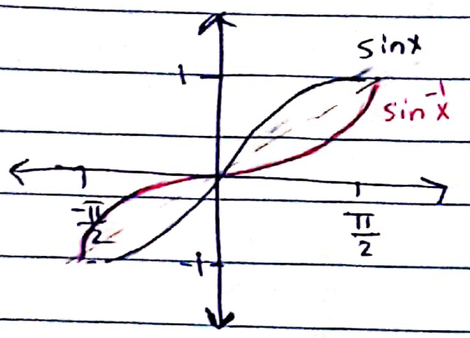
$\frac{\ln 5x}{2\cancel{\ln 2}} \cancel{\ln 2} = \ln x$

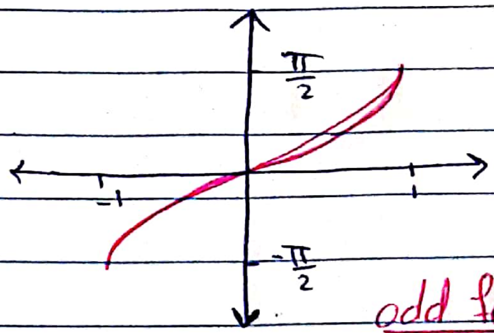
$\ln 5x = \ln x^2$

$5x = x^2 \rightarrow x = \cancel{0}, 5$

7.6 Inverse of Trigorometric Functions:

1 if $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ Then?





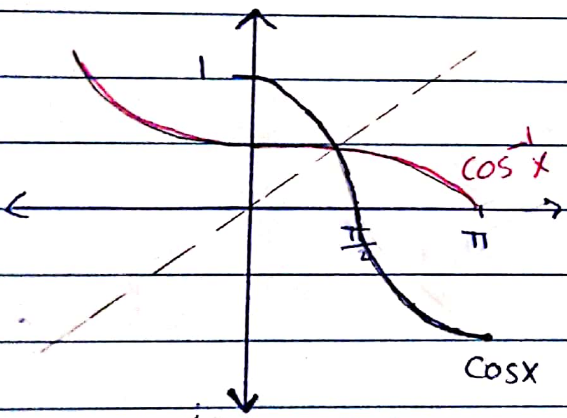
odd function

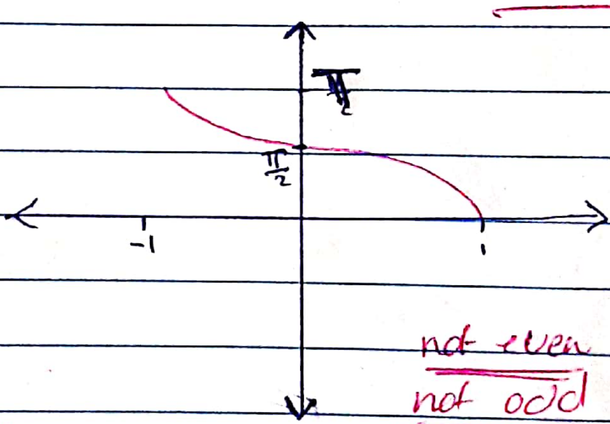
$\Rightarrow f^{-1}(x) = \sin^{-1} x = \text{arc} \sin$ on $[-1, 1]$

Exp: ① $\sin^{-1}(\frac{1}{2}) = \pi/6$ ② $\sin^{-1}(-\frac{1}{2}) = -\pi/6$

③ $\sin^{-1}(\frac{3}{2}) =$ Undefined ($\frac{3}{2}$ not in the domain).

2 if $f(x) = \cos x$ on $[0, \pi]$ Then?





not even
not odd

$\Rightarrow f^{-1}(x) = \cos^{-1} x = \text{arc} \cos x$ on $[-1, 1]$

Exp: ① $\cos^{-1} \frac{1}{2} = \frac{\pi}{3}$ ② $\cos^{-1} -\frac{1}{2} = 2(\pi/3)$

NOTE $\cos^{-1} x + \cos^{-1}(-x) = \pi$

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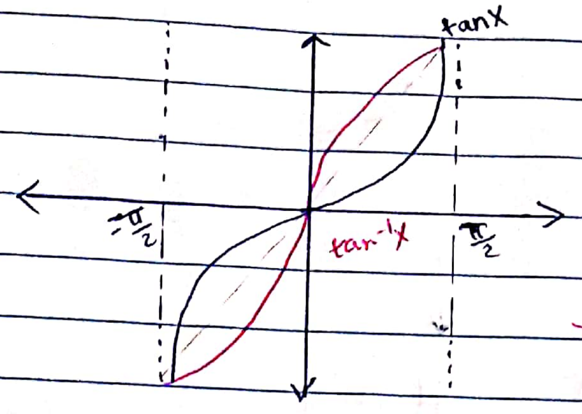
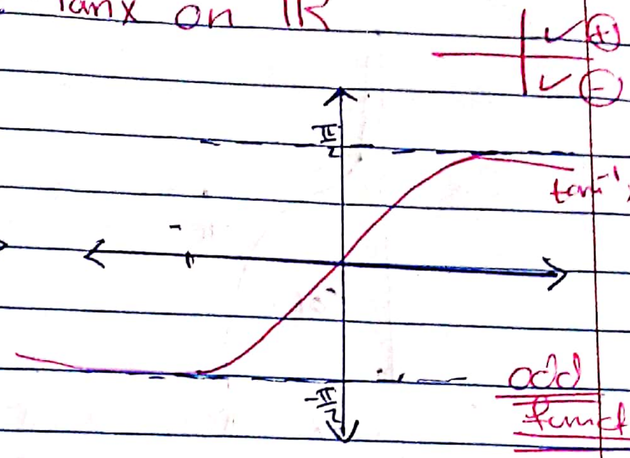
[3] if $f(x) = \tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ Then?

$\Rightarrow f^{-1}(x) = \tan x^{-1} = \text{arc} \tan x$ on $\mathbb{R}$





odd function

Exp: ① $\tan^{-1} 1 = \pi/4$ ② $\tan^{-1} -1 = -\pi/4$ ③ $\tan^{-1}\sqrt{3} = \pi/3$

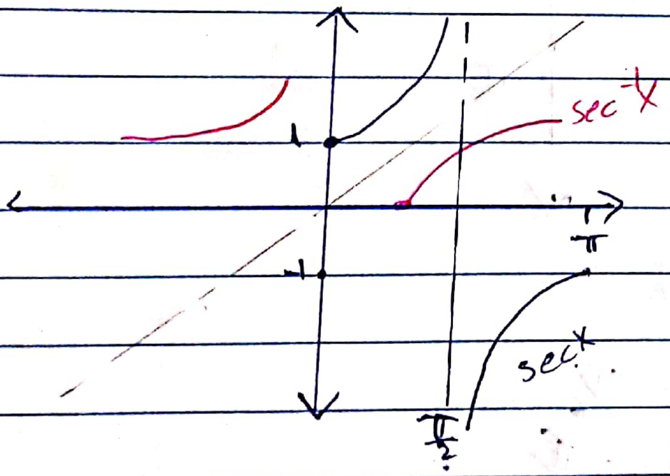
$\rightarrow \lim_{x \to \infty} \tan^{-1}x = \pi/2$

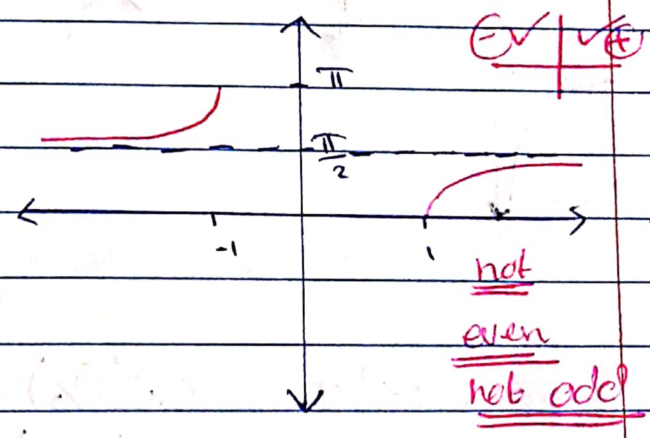
$\rightarrow \lim_{x \to -\infty} \tan^{-1}x = -\pi/2$

H. Asy

[4] if $f(x) = \sec x$ on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ Then.

$\Rightarrow f^{-1}(x) = \sec^{-1}x = \text{arc} \sec$ on $(-\infty, -1] \cup [1, \infty)$





not even
not odd

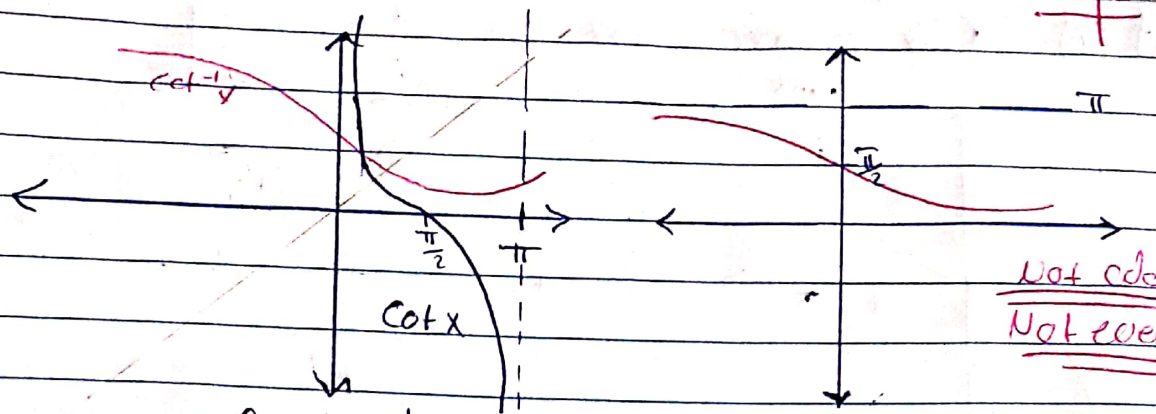
$\rightarrow \lim_{x \to \pm\infty} \sec^{-1}x = \pi/2$ H. Asy

Exp: ① $\sec(1) = \text{zero}$ ② $\sec^{-1}(\frac{1}{2}) = \text{undefined}$

③ $\sec(-1) = \pi$

[5] if $f(x) = \cot x$ on $(0, \pi)$ Then ?

$f^{-1}(x) = \cot x = \text{arc} \cot x$ on $\mathbb{R}$

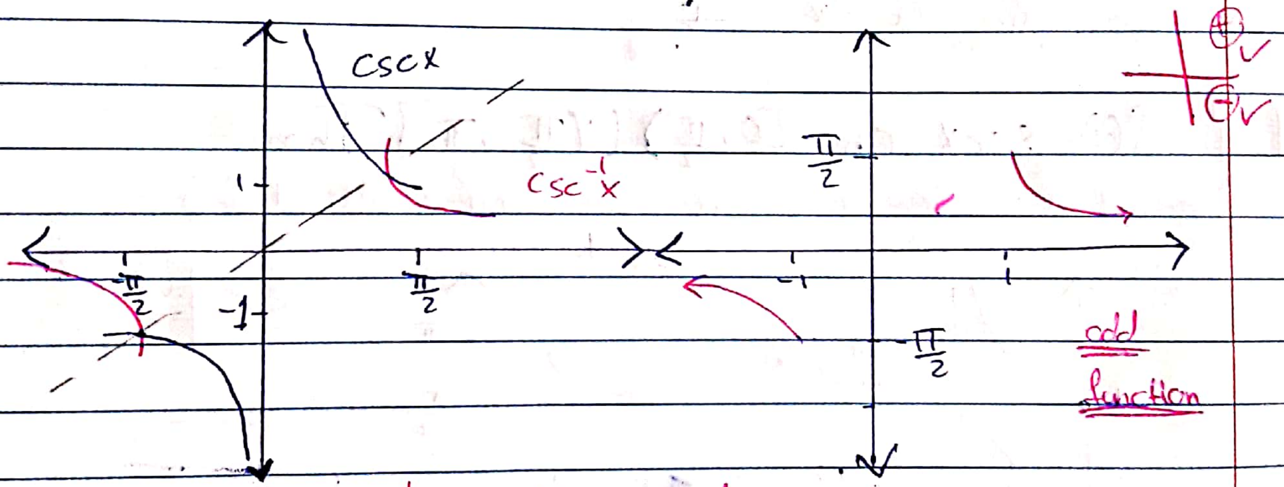


Not odd
Not even

$\lim_{x \to -\infty} \cot^{-1} x = \pi$ } $y = \pi$
$\lim_{x \to \infty} \cot^{-1} x = 0$ } $y = \text{zero}$ H. Asy

[6] if $f(x) = \csc x$ on $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Then? $f^{-1}(x) = \csc^{-1} x = \text{arc} \csc x$ on $(-\infty, -1] \cup [1, \infty)$



odd
function

$\lim_{x \to \pm\infty} \csc^{-1} x = \text{zero}$, H. Asy

Exp: $\csc^{-1} 2 = \sin^{-1}(1/2) = \pi/6$

$\csc^{-1} -2 = \sin^{-1}(-1/2) = -\pi/6$

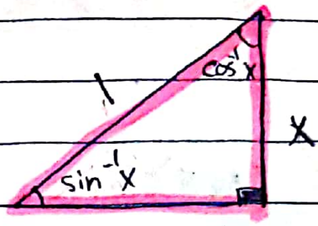
$\csc^{-1}(1) = \pi/2$

$\csc^{-1}(-2/3) =$ Undefind

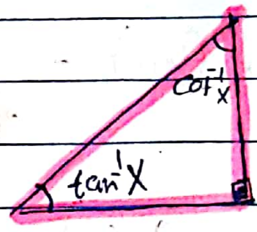
$\csc^{-1}(0) =$ Undefind

NOTE:

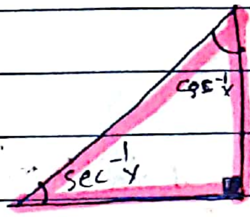
$\boxed{1}$ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$



$\boxed{2}$ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$



$\boxed{3}$ $\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$



Exp: $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = ?$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\cot\frac{-\pi}{3} = \frac{\cos\frac{-\pi}{3}}{\sin\frac{-\pi}{3}} = \frac{-1}{\sqrt{3}}$$

Exp: $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = ?$

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

Exp: Find the domain of $f(x) = \sin^{-1}(\ln x)$

$D = [e^{-1}, e]$

$\ln x = -1$	$\ln x = 1$
e^{-1}	e

Derivatives for the Inverse Of Trigonometric Functions:

let $u(x)$ be a diff function:

1. $\frac{d}{dx}\sin^{-1}(u(x)) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $-1<u<1$

2. $\frac{d}{dx}\cos^{-1}(u(x)) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$

3. $\frac{d}{dx}\tan^{-1}(u(x)) = \frac{1}{1+u^2}\frac{du}{dx}$

4. $\frac{d}{dx}\cot^{-1}(u(x)) = \frac{-1}{1+u^2}\frac{du}{dx}$

5. $\frac{d}{dx}\sec^{-1}(u(x)) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

6. $\frac{d}{dx}\csc^{-1}(u(x)) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

proof ① If $f(x) = \sin x \rightarrow f^{-1}(x) = \sin^{-1} x$

$f'(x) = \cos x$

$$\frac{d f^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)} \sim ①$$

But $\rightarrow \boxed{\sin^2 x + \cos^2 x = 1}$

$\cos x = \sqrt{1 - \sin^2 x} \sim ②$

② in ①

$$\frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}$$

proof ③ If $f(x) = \tan x \longrightarrow f^{-1}(x) = \tan^{-1} x$

$f'(x) = \sec^2 x$

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\tan^{-1} x)} = \frac{1}{\sec^2(\tan^{-1} x)}$$

But $\rightarrow \boxed{1 + \tan^2 x = \sec^2 x}$

$$= \frac{1}{1 + \tan^2(\tan^{-1} x)}$$

$$= \frac{1}{1 + x^2}$$

proof ⑤ $y = \sec^{-1} x \rightarrow \sec y = x$

$\sec y \tan y \; y' = 1$

$\rightarrow y' = \frac{1}{\sec y \tan y}$

$\rightarrow y' = \frac{1}{x \tan y}$

$1 + \tan^2 y = \sec^2 y$

$\tan y = \pm \sqrt{\sec^2 y - 1}$

$= \pm \sqrt{x^2 - 1}$

$$= \frac{1}{\pm x \sqrt{x^2 - 1}}$$

$$= \frac{1}{|u| \sqrt{x^2 - 1}}$$

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Exp: Find $y'\big|_{x=e}$ if $y = \tan^{-1}(\ln x)$

$$y' = \frac{1}{1+(\ln x)^2} * \frac{1}{x}$$

$$y'(e) = \frac{\frac{1}{e}}{1+(\ln e)^2} = \frac{\frac{1}{e}}{2} = \frac{1}{2e} \leftarrow m$$

Find tangent of $y = \tan^{-1}(\ln x)$ at $x = e$

$$y(e) = \tan^{-1}(\ln e) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$y - \frac{\pi}{4} = \frac{1}{2e}(x - e)$$

Inverse Integral:-

$\boxed{1}$ $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + c$$

$\boxed{2}$ $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$\boxed{3}$ $$\int \frac{du}{u^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$$

Exp: Find y' if

① $y = \cos^{-1} \sqrt{2}\, x$

$$y' = \frac{-1}{\sqrt{1-2x^2}} \sqrt{2}$$

Exp:
$$\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$$
$u = 2x-1$
$du = 2dx$
$$\frac{1}{2}\int \frac{du}{u\sqrt{u^2-2^2}} = \frac{1}{2} * \frac{1}{2} \sec^{-1}\left(\frac{u}{a}\right) + C$$
$$= \frac{1}{4}\sec^{-1}\left|\frac{2x-1}{2}\right| + c$$

Exp: $y = \ln(\tan^{-1}\sqrt{x})$ Find $y'(1)$?
$$y' = \frac{1}{\tan^{-1}\sqrt{x}} * \frac{1}{1+x^2} * \frac{1}{2\sqrt{x}}$$
$$y'(1) = \frac{1}{\tan^{-1} 1} * \frac{1}{2} * \frac{1}{2} = \frac{1}{\pi}$$

Exp: $\int \frac{dx}{\sqrt{4x-x^2}}$
$$4x - x^2 = -[x^2 - 4x]$$
$$= -[(x-2)^2 - 4]$$
$$= 4 - (x-2)^2$$
$$= \int \frac{dx}{\sqrt{2^2-(x-2)^2}}$$
$u = x-2$
$du = dx$
$$\int \frac{du}{\sqrt{2-u^2}}$$
$$\sin^{-1}\left(\frac{u}{a}\right) + c = \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

Exp: $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}}$

$3 + 4t - 4t^2 =$
$-[4t^2 - 4t - 3] =$
$-[(2t-1)^2 - 4]$
$[4 - (2t-1)^2]$

$$\int_{1/2}^{1} \frac{6\,dt}{\sqrt{2^2-(2t-1)^2}}$$
$u = 2t-1$
$du = 2dt$

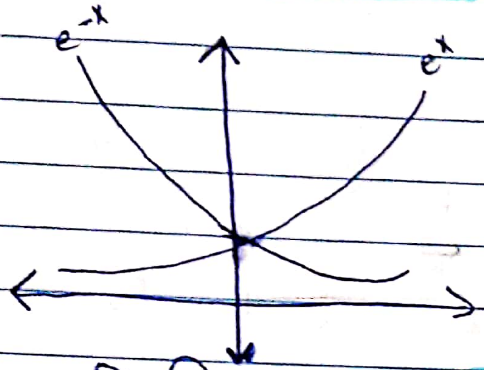
$t = \frac{1}{2} \rightarrow u = 0$
$t = 1 \rightarrow u = 1$
$$\Rightarrow \frac{6}{2}\int_0^1 \frac{du}{\sqrt{2^2-(u)^2}} = 3\sin^{-1}\left(\frac{u}{2}\right)\Big|_0^1$$
$$3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$$

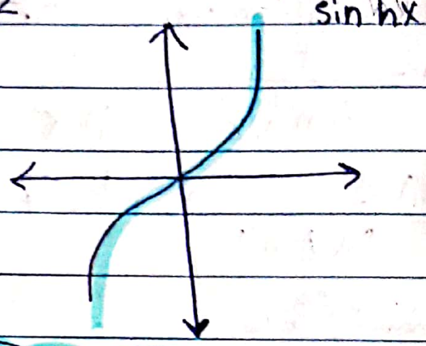
Exp: Find y' if $y = \csc^{-1}(x^2-3x)$

$$y' = \frac{-1}{|x^2-3x|\sqrt{(x^2-3x)^2-1}}(2x-3)$$

7.7 Hyperoplic functions:-

[1] $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$



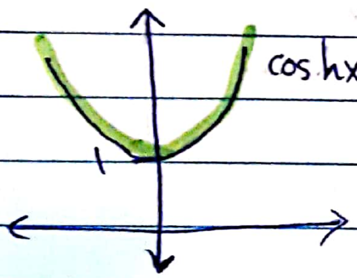


→ $D = \mathbb{R}$

→ $R = \mathbb{R}$

odd:
$\sinh -x = -\sinh x$

[2] $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$



→ $D = \mathbb{R}$

→ $R = [1, \infty)$

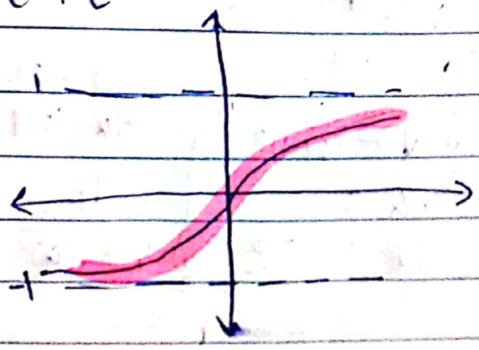
Even
$\cosh -x = \cosh x$

[3] $f(x) = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\lim_{x \to \infty} \tanh x = 1$

$\lim_{x \to -\infty} \tanh x = -1$

$\tanh 0 = \frac{0}{1} = 0$



odd $D = \mathbb{R}$
$R =]-1, 1[$

4. $\operatorname{sech}x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

sechx even

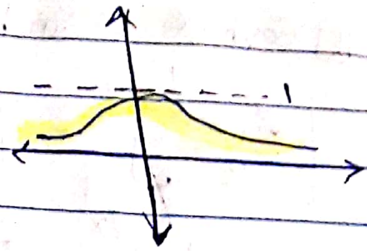
$\operatorname{sech}(0) = 1$

$\operatorname{sech}x > 0$

$y = 0$ is H.Asy

$D = \mathbb{R}$

$R = (0, 1]$

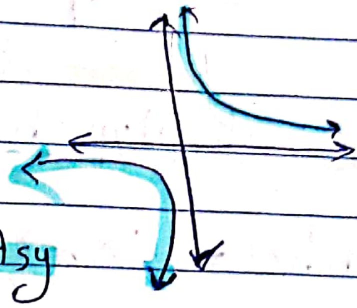


5. $\operatorname{csch}x = \frac{1}{\sinh x}$

odd

$D = \mathbb{R} \setminus \{0\}$

$R = \mathbb{R} \setminus \{0\}$



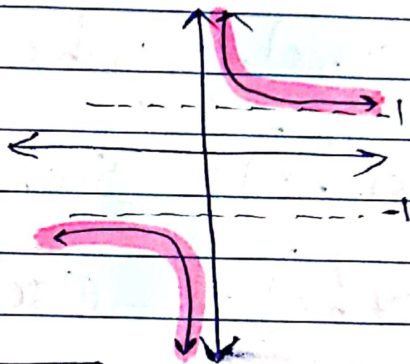
$x = 0$ is a V. Asy

6. $\coth x$

odd

$D = \mathbb{R} \setminus \{0\}$

$R = \mathbb{R} \setminus [-1, 1]$



Properities of hyperbolic functions:

1. $\cosh^2 x - \sinh^2 x = 1$

2. $\sinh 2x = 2 \sinh x \cosh x$

3. $\cosh 2x = \cosh^2 x + \sinh^2 x$
$= 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x$

4. $1 - \tanh^2 x = \operatorname{sech}^2 x$

5. $\coth^2 x - 1 = \operatorname{csch}^2 x$

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proof

$\boxed{1}$ $\cosh^2 x - \sinh^2 x =$

$$\left[\frac{e^x+e^{-x}}{2}\right]^2 - \left[\frac{e^x-e^{-x}}{2}\right]^2$$

$$\frac{\cancel{e^{2x}}+2+\cancel{e^{-2x}}-(\cancel{e^{+2x}}-2+\cancel{e^{-2x}})}{4} = \frac{4}{4} = 1$$

Derivatives of Hyperbolic Functions :.

① $\frac{d}{dx}(\sinh x) = \cosh x$

② $\frac{d}{dx}(\cosh x) = \sinh x$

③ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

④ $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

⑤ $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

⑥ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

proof

$\boxed{2}$ $(\cosh x)'$

$$\frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sinh x$$

Integrals of Hyperbolic functions:

① $\int \cosh x \, dx = \sinh x + c$

② $\int \sinh x \, dx = \cosh x + c$

③ $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

④ $\int \operatorname{csch}^2 x \, dx = -\coth x + c$

⑤ $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c$

⑥ $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + c$

Exp: Find y' if:

① $y = \ln \sinh x \rightarrow y' = \frac{1}{\sinh x} \cosh x = \coth x$

② $y = 4 \cosh \frac{x^3}{2} \rightarrow y' = 4 \sinh \frac{x^3}{2} * \frac{3x^2}{2}$

$= 2 \sinh \frac{x^3}{2} * 3x^2$

③ $y = \ln(\sinh x) - \frac{1}{2} \coth^2 x$

$y' = \coth x - \frac{1}{2}(2) \coth x (-\operatorname{csch}^2 x)$

$= \coth x [1 + \operatorname{csch}^2 x]$

$= \coth^3 x$

Exp: Find (1) $\int_2^4 2\cosh(\ln x)\,dx$

$$= \cancel{2}\int_2^4 \frac{e^{\ln x} + e^{-\ln x}}{\cancel{2}}\,dx$$

$$= \int_2^4 x + \frac{1}{x}\,.\,dx$$

$$= \frac{x^2}{2} + \ln x\Big|_2^4 = 6 + \ln 2$$

(2) $\int_0^{\ln 2} 4e^{-\theta} \sinh\theta \, d\theta$

$$= 4\int_0^{\ln 2} e^{-\theta}\left(\frac{e^{\theta} - e^{-\theta}}{2}\right) d\theta$$

$$= 2\int_0^{\ln 2} \left(1 - e^{-2\theta}\right) d\theta$$

$$= 2\left[\theta + \frac{1}{2}e^{-2\theta}\right]\Big|_0^{\ln 2}$$

$$= 2\ln 2 + e^{-2\ln 2}\left(0 + e^{0}\right)$$

$$= 2\ln 2 + 2^{-2} - 1$$

$$= \ln 4 + \frac{1}{4} - 1$$

$$= \ln 4 - \frac{3}{4}$$

7.8 Relative Rates of Growth:

* Assume (f) and (g) are Positive for large X.

* If $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$

① $\to \infty$, then f(x) grows faster than g as $x \to \infty$

② $\to 0$, then f(x) grows slower than g as $x \to \infty$

③ $\to 0 < L < \infty$, then both (f) and (g) grow in the same rate as $x \to \infty$

Exp: ① 4^x ; e^x

$$\lim_{x\to\infty} \frac{4^x}{e^x} = \lim_{x\to\infty} \left(\frac{4}{e}\right)^x = \infty$$

Hence, 4^x grows faster than e^x as $x \to \infty$

② $\left(\frac{3}{2}\right)^x$, e^x

$$\lim_{x\to\infty} \frac{e^x}{\left(\frac{3}{2}\right)^x} = \lim_{x\to\infty} \left(\frac{2e}{3}\right)^x = \infty$$

Hence, e^x grows faster than $\left(\frac{3}{2}\right)^x$ as $x \to \infty$

③ $\ln x$, 2^x

$$\lim_{x\to\infty} \frac{\ln x}{2^x} \left(\frac{\infty}{\infty}\right) \Rightarrow \lim_{x\to\infty} \frac{\frac{1}{x}}{2^x \ln 2} \Rightarrow \lim_{x\to\infty} \frac{1}{x\, 2^x \ln 2} = \left(\frac{1}{\infty}\right) = \text{zero}$$

Hence, $\ln x$ grows slower than 2^x as $x \to \infty$

4. $\log_3 x$, $\ln x$

$$\lim_{x\to\infty} \frac{\log_3 x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\cancel{\ln x}}{\ln 3}}{\cancel{\ln x}} = \frac{1}{\ln 3}$$

Henec, $\log_3 x$, $\ln x$ grows in the same rate as $x \to \infty$

Exp: Order the following functions from slowest to fastest as $x \to \infty$

* $\lim_{x\to\infty} \frac{e^x}{e^{\frac{x}{2}}} = \lim_{x\to\infty} e^{\frac{x}{2}} = \infty$

$e^x > e^{\frac{x}{2}}$

$(\ln x)^x > x^x$

* $\lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{e}{\ln x}\right)^x \to (0^\infty)$

$$\lim_{x\to\infty} e^{\ln\left(\frac{e}{\ln x}\right)^x} = e^{\lim_{x\to\infty} x \ln\left(\frac{e}{\ln x}\right)} = e^{\overset{\infty}{\lim_{x\to\infty} x}\ \lim_{x\to\infty} \ln\left(\frac{e}{\ln x}\right)}$$

$$e^{\lim_{x\to\infty} x\ \lim_{x\to\infty} [\ln e - \ln(\ln x)]} = e^{(-\infty)\infty} = e^{-\infty} = 0$$

$\ln x > e^x$

$\Rightarrow$ slowet to faster

$e^{\frac{x}{2}}, e^x, (\ln x)^x, x^x$

* $\lim_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{x}{\ln x}\right)^x$

$$= \lim_{x\to\infty} e^{\ln\left[\frac{x}{\ln x}\right]^x}$$

$$= e^{\lim_{x\to\infty} x\ \lim_{x\to\infty} \left[\ln \frac{x}{\ln x}\right]} = e^{\infty \lim_{x\to\infty} \ln \frac{1}{\frac{1}{x}}}$$

$$= e^{\infty(\infty)} = e^{\infty} = \infty$$

Exp : $5, x^3, x, x^5$

frome slowest to faster

$5, x, x^3, x^5$

Note: in polunomin functions the largest degree's function is the faster one