

## 8.1 Integral by parts

$$\int u dv = uv - \int v du$$

$$\begin{array}{c} \left( \frac{d}{dx} \right) u \\ \downarrow \\ du \end{array} \quad \begin{array}{c} \left( \frac{d}{dx} \right) v \\ \downarrow \\ dv \end{array}$$

$du \rightarrow f \rightarrow v$

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

Exp: ①  $\int x \cos x dx$

$$\begin{array}{ll} u = x & dv = \cos x dx \\ du = dx & v = \sin x \end{array}$$

⑤  $\int u dv = uv - \int v du$

$$= x \sin x - \int \sin x dx$$

$$= x \sin x + \cos x + C$$

②  $\int_0^{\pi} x \cos x dx$

$$x \sin x \Big|_0^{\pi} + \cos x \Big|_0^{\pi}$$

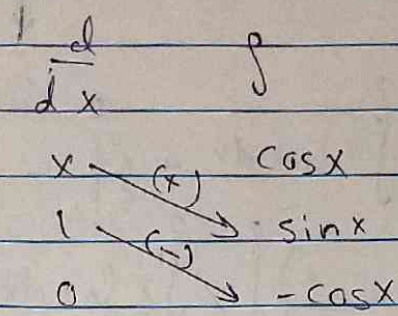
$$\left( (\pi \sin \pi) - (0 \sin 0) \right) + \left( \cos \pi - \cos 0 \right)$$

$$(0 - 0) + (-1 - 1) = -2$$

S2

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

واحد من الطرفين  
الذي يسهل التكامل  
تسمى



$$= x \sin x + \cos x + C$$

Exp:  $\int \ln x \, dx$

$u = \ln x$	$dv = dx$
$du = \frac{1}{x} dx$	$v = x$

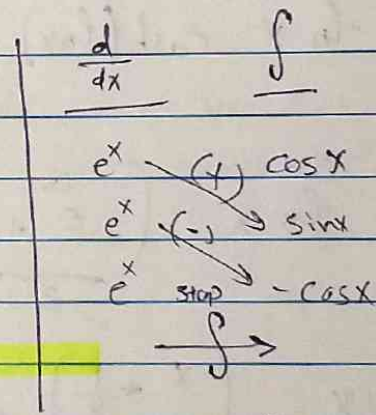
$$= x \ln x - \int \frac{1}{x} x \, dx$$

$$= x \ln x - \int dx = x \ln x - x + C$$

Exp:  $\int e^x \cos x \, dx$

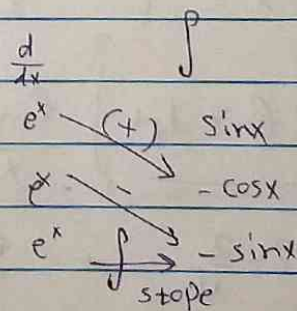
$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + \int -e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = \frac{e^x}{2} [\sin x + \cos x] + C$$



Exp:  $\int e^x \sin x \, dx$

$$= -e^x \cos x + e^x \sin x + \int e^x (-1) \sin x \, dx$$



$$2 \int e^x \sin x \, dx = e^x [\sin x - \cos x]$$

$$\int e^x \sin x \, dx = \frac{e^x}{2} [\sin x - \cos x] + C$$

Exp.: ①  $\int_1^2 x \ln x \, dx$

$x$	$\ln x$
$\downarrow$	$x \ln x - x$
$\cup$	$\int x \ln x \, dx - \frac{x^2}{2}$

$$\int_1^2 x \ln x \, dx = x(x \ln x - x) \Big|_1^2 - \int_1^2 x \ln x \, dx + \frac{x^2}{2} \Big|_1^2$$

$$= (x^2 \ln x - x^2 + \frac{x^2}{2}) \Big|_1^2 - \int_1^2 x \ln x \, dx$$

$$= (x^2 \ln x - x^2 + \frac{x^2}{2}) \Big|_1^2 - \int_1^2 x \ln x \, dx$$

$$= (4 \ln 2 - 2) - (0 - \frac{1}{2}) - \int_1^2 x \ln x \, dx$$

$$2 \int_1^2 x \ln x \, dx = 4 \ln 2 - \frac{3}{2}$$

$$\Rightarrow \int_1^2 x \ln x \, dx = 2 \ln 2 - \frac{3}{4}$$

Exp.  $2 \int_1^2 \ln x \cosh(\ln x) \, dx$

$$2 \int_1^2 \ln x \left( \frac{e^{\ln x} + e^{-\ln x}}{2} \right) dx$$

$$\Rightarrow \int_1^2 \ln x \left( x + \frac{1}{x} \right) dx$$

$$\Rightarrow \int_1^2 x \ln x \, dx + \int_1^2 \frac{\ln x}{x} \, dx$$

$u = \ln x$   
 $du = \frac{1}{x} dx$   
 بالقوة!

$$2 \ln 2 - \frac{3}{4} + \frac{\ln^2 2}{2}$$

Exp:  $\int e^{\sqrt{3x+9}} dx$

$$u = \sqrt{3x+9}$$

$$u^2 = 3x+9$$

$$2u du = 3 dx$$

$$= \frac{2}{3} \int e^u u du$$

$$= \frac{2}{3} [ue^u - e^u] + C$$

$$= \frac{2}{3} e^{\sqrt{3x+9}} [\sqrt{3x+9} - 1] + C$$

$$\begin{array}{l} u \cdot e^u \\ \downarrow \\ 1 \cdot e^u \\ \downarrow \\ 0 \cdot e^u \end{array}$$

Exp:

$$\int_0^{\frac{1}{\sqrt{2}}} 2x \sin^{-1} x^2 dx$$

$$u = \sin^{-1} x^2$$

$$dv = 2x dx$$

$$du = \frac{2x}{\sqrt{1-x^4}} dx$$

$$v = x^2$$

$$\Rightarrow \int_0^{\frac{1}{\sqrt{2}}} u dv = uv \Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} v du$$

$$= x^2 \sin^{-1} x^2 \Big|_0^{\frac{1}{\sqrt{2}}} - \int_0^{\frac{1}{\sqrt{2}}} \frac{2x^3}{\sqrt{1-x^4}} dx$$

$$w = 1 - x^4$$

$$dw = -4x^3 dx$$

$$= \left( \left( \frac{1}{2} \sin^{-1} \frac{1}{2} \right) - (0) \right) + \frac{1}{2} \int_0^{\frac{3}{4}} \frac{-4x^3 dx}{\sqrt{1-x^4}}$$

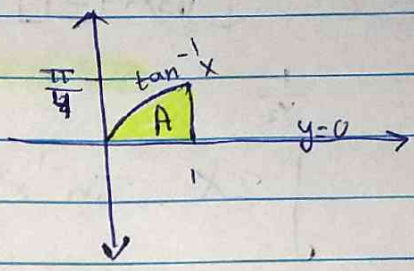
$$= \frac{1}{2} \frac{\pi}{6} + \frac{1}{2} \int_1^{\sqrt{2}} \frac{dw}{\sqrt{w}}$$

$$= \frac{\pi}{12} + \left( \sqrt{w} \Big|_1^{\sqrt{2}} \right) = \frac{\pi}{12} + \frac{\sqrt{2}}{2} - 1$$

Exp: Consider the region bounded by  $y = \tan^{-1} x$ ,  $x=1$ ,  $y=0$ .

- ① find the area of this region.
- ② find the volume of the resulted solid generated by revolving the region about y-axis.

①  $A = \int_0^1 \tan^{-1} x \, dx$



$$u = \tan^{-1} x \quad dv = dx$$

$$du = \frac{1}{1+x^2} \quad v = x$$

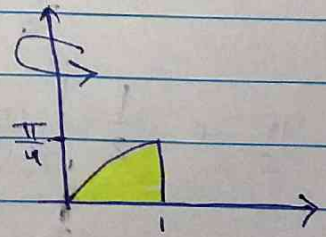
$$A = \int_0^1 u \, dv = UV \Big|_0^1 - \int_0^1 v \, du$$

$$= x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx$$

$$= (\tan^{-1} 1 - 0 \tan^{-1} 0) - \frac{1}{2} \ln |1+x^2| \Big|_0^1$$

$$= \frac{\pi}{4} - \ln \sqrt{2}$$

②  $V \Rightarrow$  (Shell method)



$$V = 2\pi \int_a^b (\text{shell radius}) (\text{shell height}) \, dx$$

$$= 2\pi \int_0^1 x (\tan^{-1} x) \, dx$$

$$u = \tan^{-1} x \quad dv = x \, dx$$

$$du = \frac{1}{1+x^2} \, dx \quad v = \frac{x^2}{2}$$

$$V = 2\pi \left[ \frac{x^2}{2} \tan^{-1} x \Big|_0^1 - 0 - \frac{1}{2} \int_0^1 \frac{x^2+1-1}{1+x^2} \, dx \right]$$

$$= 2\pi \left[ \frac{1}{2} \frac{\pi}{4} - \frac{1}{2} \int \left( 1 - \frac{1}{1+x^2} \right) dx \right] = 2\pi \left[ \frac{\pi}{8} - \frac{1}{2} (1 - \tan^{-1} x) \right]_0^1$$

$$= 2\pi \left[ \frac{\pi}{8} - \frac{1}{2} \left( 1 - \frac{\pi}{4} \right) \right] \neq$$

## 8.2 Trigonometric Integrals:-

$$\int \sin^m x \cos^n x dx$$

Case (I): If  $m$  is odd we use  
 $\Rightarrow 1 - \sin^2 x = \cos^2 x$

Exp:  $I = \int \sin^3 x \cos^2 x dx$

$$= \int \sin^2 x \cos^2 x \sin x dx$$

$$= \int (1 - \cos^2 x) (\cos^2 x) \sin x dx$$

$$= \int (\cos^2 x - \cos^4 x) \sin x dx$$

let  $u = \cos x$   
 $du = -\sin x dx$

$$\therefore = \int (-u^2 + u^4) du = \int u^4 - u^2 du$$

$$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$$

Case (II) : m is even and n is odd.

We use  $1 - \sin^2 x = \cos^2 x$

or  $1 - \cos^2 x = \sin^2 x$

Exp:  $\int \cos^5 x \, dx$

$$= \int \cos^4 x \cdot \cos x \, dx = \int (\cos^2 x)^2 \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx$$

let  $u = \sin x$

$du = \cos x \, dx$

$$= \int (1 - u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2(\sin x)^3}{3} + \frac{(\sin x)^5}{5} + C$$

Exp:  $\int \sin^3 x \cos^9 x \, dx = \int \sin^2 x \cos^9 x \sin x \, dx$

$$= \int (1 - \cos^2 x) \cos^9 x \sin x \, dx$$

$$= \int (\cos^9 x - \cos^{11} x) \sin x \, dx$$

$$\int (-u^9 + u^{11}) \, dx$$

$u = \cos x$   
 $du = -\sin x \, dx$

Case (III) ,  $[m, n]$  are even :-

$$\text{w use } \Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

Exp:  $\int (\sin^4 x) \cos^2 x \, dx$

$$= \int \left( \frac{1 - \cos 2x}{2} \right)^2 \left[ \frac{1 + \cos 2x}{2} \right] dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 - \cos 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 - \cos^2 2x) \, dx$$

$$= \frac{1}{8} \int (1 - \cos 2x)(\sin^2 2x) \, dx$$

$$= -\frac{1}{8} \int \sin^2(2x) \cos 2x \, dx + \frac{1}{8} \int \sin^2(2x) \, dx$$

$$= -\frac{1}{8} \int u^2 \frac{du}{2} + \frac{1}{8} \int \frac{1 - \cos(4x)}{2} \, dx$$

$u = \sin 2x$   
 $du = 2 \cos 2x$

$$= -\frac{1}{16} \frac{u^3}{3} + \frac{1}{16} \left( x - \frac{\sin 4x}{4} \right) + C$$

$$= -\frac{1}{48} (\sin 2x)^3 + \frac{1}{16} x - \frac{\sin 4x}{64} + C$$



$$\int \sec^m x \tan^n x dx$$

We use  $1 + \tan^2 x = \sec^2 x$

or  $\tan^2 x = \sec^2 x - 1$

Exp:  $\int \tan^3 x \sec^3 x dx$

$$= \int \tan^2 x \sec^2 x \sec x \tan x dx$$

$$= \int (\sec^2 x - 1) \sec^2 x \tan x \sec x dx$$

$$= \int (u^4 - u^2) du$$

Let  $u = \sec x$

$$du = \sec x \tan x dx$$

Exp:  $\int \sec^3 x dx = \int \sec^2 x \sec x dx$

$$u = \sec x \quad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \quad \rightarrow \quad v = \tan x$$

$$I = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$I = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$\xrightarrow{I}$

$$2I = \sec x \tan x + \ln |\sec x + \tan x|$$

$$I = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln |\sec x + \tan x| + C$$

## \* Eliminating the square roots

$$\int_0^{\pi} \sqrt{1 + \sin x} \, dx$$

$$\int_0^{\pi} \int_0^{\pi} \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\int_0^{\pi} \sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}$$

$$\int_0^{\pi} \left| \cos \frac{x}{2} + \sin \frac{x}{2} \right|$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos \frac{x}{2} + \sin \frac{x}{2} \, dx$$

$$= \left( 2 \sin \frac{x}{2} - 2 \cos \frac{x}{2} \right) \Big|_0^{\pi}$$

$$= \left( 2(1) - 2(0) \right) - \left( 2(0) - 2(1) \right)$$

$$= 4$$

## 8.2 Trigonometric Integrals

Question: How to find

$$\int \cos^m x \sin^n x dx$$

1 If  $m$  odd  $\Rightarrow m = 2k + 1$

$$\cos^m x = \cos^{2k+1} x = (\cos^2 x)^k \cdot \cos x$$

$$u = \sin x \Rightarrow du = \cos x dx$$

$$= (1 - \sin^2 x)^k \cos x$$

2 If  $n$  odd  $\Rightarrow n = 2k + 1$

$$\sin^n x = \sin^{2k+1} x = [\sin^2 x]^k \sin x$$

$$du = \cos x dx$$

$$du = -\sin x dx \Rightarrow \int (1 - \cos^2 x)^k \sin x dx$$

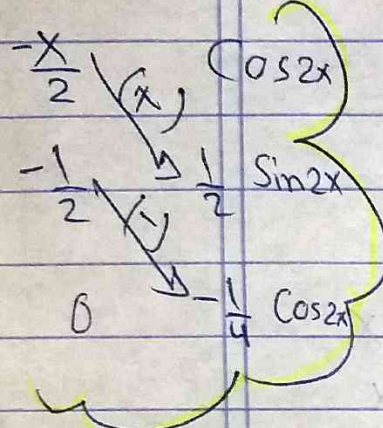
3 If  $n$  and  $m$  are both even then

$$\text{replace } \Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\Rightarrow \cos^2 x = \frac{1 + \cos 2x}{2}$$

exp :  $\int x \sin^2 x \, dx$

$$= \int x \left( \frac{1 - \cos 2x}{2} \right) dx$$

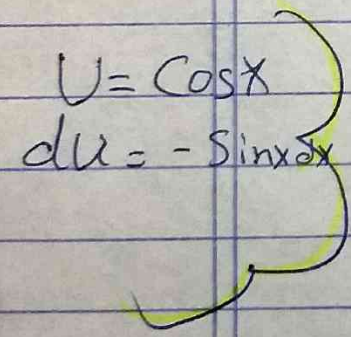


$$= \int \frac{x}{2} dx + \int -\frac{x}{2} \cos 2x \, dx$$

$$\frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C$$

exp :  $\int \sin^3 x \cos^2 x \, dx$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$



$$= \int (1 - \cos^2 x) \cos^2 x \sin^2 x \, dx$$

$$= - \int (1 - u^2) u^2 \, du$$

$$= \int u^4 - u^2 \, du$$

$$\frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

exp  $\int \sin^2 x \cos^2 x dx$

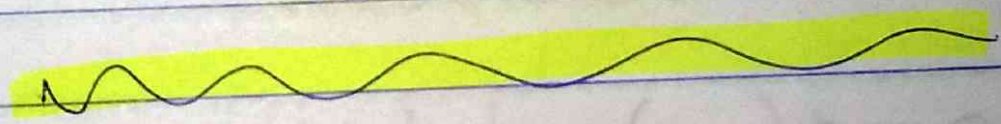
$$\rightarrow \int \left( \frac{1 - \cos 2x}{2} \right) \left( \frac{1 + \cos 2x}{2} \right) dx$$

$$\Rightarrow \frac{1}{4} \int (1 - \cos^2 2x) dx \Rightarrow \frac{1}{4} \int 1 dx - \frac{1}{4} \int \cos^2 2x dx$$

$$\rightarrow \frac{1}{4} \int 1 dx - \frac{1}{4 \times 2} \int (1 + \cos 4x) dx$$

$$\Rightarrow \frac{1}{4} x - \frac{1}{8} x + \frac{1}{4 \times 8} \sin 4x + C$$

$$\Rightarrow \frac{x}{8} - \frac{1}{4} \frac{1}{8} \sin 4x + C$$



Question: ? How to find ?

$$\boxed{1} \int \cos mx \sin nx \, dx$$

$$\boxed{2} \int \cos mx \cos nx \, dx$$

$$\boxed{3} \int \sin mx \sin nx \, dx$$

$$* \cos mx \cos nx = \frac{1}{2} [\cos (m-n)x + \cos (m+n)x]$$

$$* \sin mx \sin nx = \frac{1}{2} [\cos (m-n)x - \cos (m+n)x]$$

$$* \sin mx \cos nx = \frac{1}{2} [\sin (m-n)x + \sin (m+n)x]$$

Exp:  $\int \sin 3x \cos 2x \, dx$

$$= \frac{1}{2} \int (\sin (3-2)x + \sin (3+2)x)$$

$$= \frac{1}{2} \int (\sin x + \sin 5x) \, dx$$

$$= \frac{1}{2} \left( -\cos x - \frac{1}{5} \cos 5x \right) + C$$

Exp:

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{1+\sin x} \cdot dx}{\sqrt{1-\sin x}}$$

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{(1-\sin^2 x)}}{\sqrt{1-\sin x}} dx = \int_0^{\frac{\pi}{6}} \sqrt{\frac{(1-\sin^2 x)}{1-\sin x}}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}} dx = \int_0^{\frac{\pi}{6}} \frac{\cos x}{\sqrt{1-\sin x}} dx$$

$$\int_0^{\frac{1}{2}} \frac{\cos x}{\sqrt{1-u}} \frac{du}{\cos x}$$

$$= \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u}}$$

$$= 2(1-u)^{-\frac{1}{2}} \Big|_0^{\frac{1}{2}} = \left( 2(1-\frac{1}{2})^{-\frac{1}{2}} \right) - \left( 2(1-0)^{-\frac{1}{2}} \right)$$

$$= 2\sqrt{\frac{1}{2}} - 2$$

$$= 2\left(\sqrt{\frac{1}{2}} - 1\right)$$

$u = \sin x$   
 $du = \cos x dx$   
 $\frac{\pi}{6} \rightarrow \frac{1}{2}$   
 $0 \rightarrow 0$

Exp:  $\int 4 \tan^3 x \, dx = 4 \int \tan^2 x \tan x \, dx$

$$= \int 4 \tan x \sec^2 x \, dx - 4 \int \tan x \, dx$$

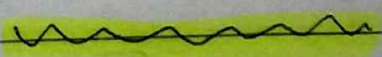
$\tan^2 x = \sec^2 x - 1$

$u = \tan x$   
 $du = \sec^2 x \, dx$

$$= 4 \int u \, du - 4 \ln |\sec x| + C$$

$$= 4 \frac{u^2}{2} - 4 \ln |\sec x| + C$$

$$= 4 \frac{\tan^2 x}{2} - 4 \ln |\sec x| + C$$





Exp.  $\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$

$$= \int \sec^2 x (1 + \tan^2 x) \, dx$$

$$= \int (1 + u^2) \, du$$

$$\begin{cases} u = \tan x \\ du = \sec^2 x \end{cases}$$

$$= u + \frac{u^3}{3} + C$$

$$= \tan x + \frac{1}{3} \tan^3 x + C$$



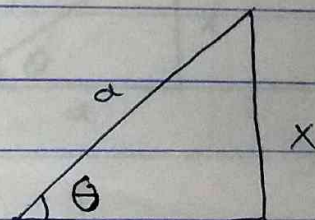
### 8.3 Trigonometric Substitution:

1)  $x = a \sin \theta$

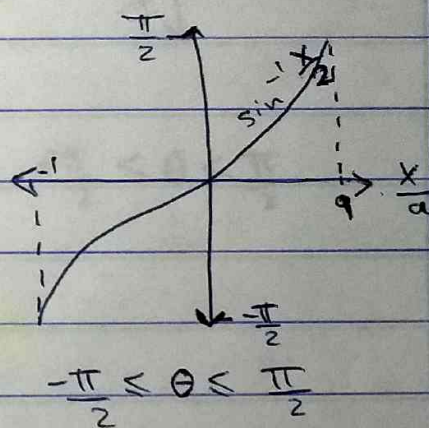
$$dx = a \cos \theta d\theta$$

$$\theta = \sin^{-1} \frac{x}{a}$$

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2 (1 - \sin^2 \theta)} \\ &= a \sqrt{\cos^2 \theta} \\ &= a |\cos \theta| \\ &= a \cos \theta \end{aligned}$$



$$\sqrt{a^2 - x^2}$$



$\int \sqrt{4 - x^2} dx$

$$12 \quad y = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\theta = \tan^{-1} \frac{x}{a}$$

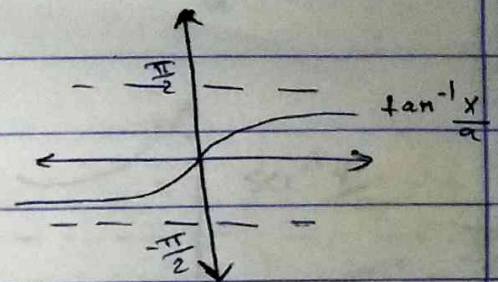
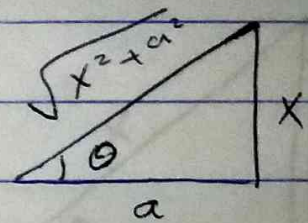
$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta}$$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 (1 + \tan^2 \theta)}$$

$$= a \sqrt{\sec^2 \theta}$$

$$= a |\sec \theta|$$

$$= a \sec \theta$$



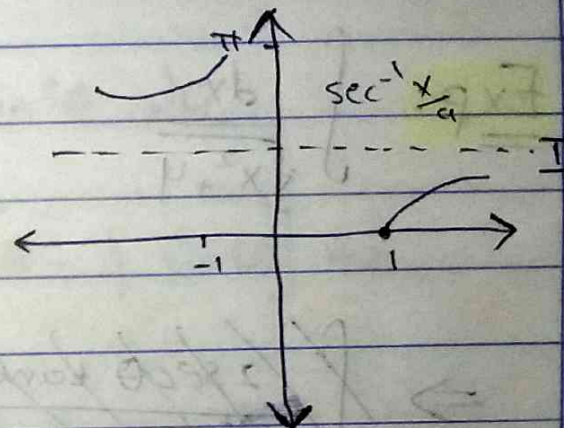
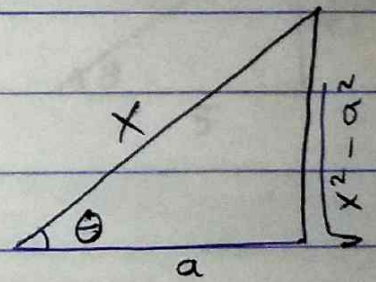
$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\boxed{3} \quad x = a \sec \theta$$

$$dx = a \sec \theta \tan \theta d\theta$$

$$\theta = \sec^{-1} \frac{x}{a}$$

$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= \sqrt{a^2 (\sec^2 \theta - 1)} \\ &= a \sqrt{\tan^2 \theta} \\ &= a |\tan \theta| \\ &= a \tan \theta \end{aligned}$$



$$* \Rightarrow 0 \leq \theta < \frac{\pi}{2} \text{ if } \frac{x}{a} \geq 1$$

$$\Rightarrow \frac{\pi}{2} < \theta \leq \pi \text{ if } \frac{x}{a} \leq -1$$

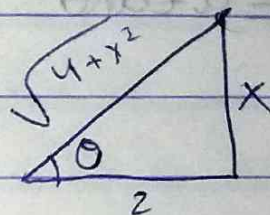


Exp:

$$\int \frac{dx}{\sqrt{x^2+4}}$$

$$a=2$$

$$x = 2 \tan \theta$$

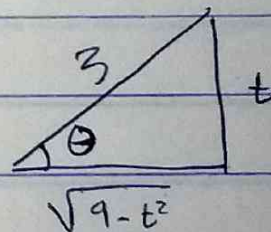


$$\Rightarrow \int \frac{2 \sec \theta \tan \theta d\theta}{\sqrt{4+x^2}}$$

$$\int \frac{2 \sec^2 \theta d\theta}{2 \sec \theta} - \ln |\sec \theta + \tan \theta| + C$$
$$= \ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C$$

Exp:

$$\int \sqrt{9-t^2} dt$$



$$= \int 3 \cos \theta \cdot 3 \cos \theta d\theta$$

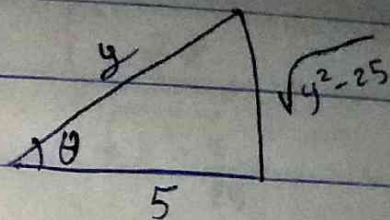
$$= 9 \int \cos^2 \theta d\theta = 9 \int \left( \frac{\cos 2\theta + 1}{2} \right) d\theta$$

$$= \frac{9}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]$$

$$= \frac{9}{2} \left[ \sin^{-1} \frac{t}{3} + \frac{1}{2} \cdot 2 \sin \theta \cos \theta \right] + C$$

$$= \frac{9}{2} \left[ \sin^{-1} \frac{t}{3} + \frac{t}{3} \frac{\sqrt{9-t^2}}{3} \right] + C$$

Exp:  $\int \frac{\sqrt{y^2 - 25}}{y^3} dy$



$$= \int \frac{5 \tan \theta \cdot 5 \sec \theta \cdot \tan \theta d\theta}{(5)^3 \sec^3 \theta}$$

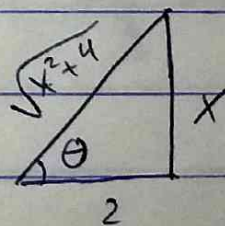
$$\frac{1}{5} \int \frac{\tan^2 \theta}{\sec^2 \theta} \cdot d\theta = \frac{1}{5} \int \sin^2 \theta \cdot d\theta$$

$$= \frac{1}{5} \int \left( \frac{1 - \cos 2\theta}{2} \right) d\theta = \frac{1}{5} \left[ \theta - \frac{1}{2} \sin 2\theta \right] + c$$

$$= \frac{1}{5} (\theta - \sin \theta \cos \theta) + c$$

$$= \frac{1}{5} \left[ \sec^{-1} y - \frac{\sqrt{y^2 - 25}}{y} \cdot \frac{5}{y} \right] + c$$

Exp: (S1)  $\int_{-2}^2 \frac{dx}{x^2 + 4} = \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_{-2}^2 = \frac{\pi}{4}$



(S2)  $\int \frac{2 \sec^2 \theta d\theta}{[2 \sec \theta]^2}$

$$\frac{1}{2} \int d\theta$$

$$\frac{1}{2} \theta = \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_{-2}^2 = \frac{\pi}{4}$$

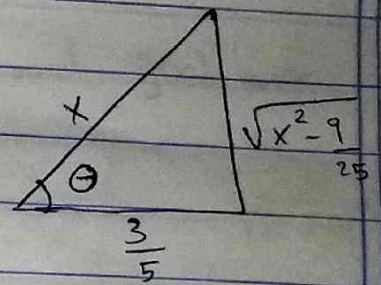
Exp.  $\int \frac{5 dx}{\sqrt{25x^2 - 9}} = \int \frac{5 dx}{5\sqrt{x^2 - \frac{9}{25}}} = \int \frac{dx}{\sqrt{x^2 - \frac{9}{25}}}$

$\int \frac{\frac{3}{5} \sec\theta \tan\theta d\theta}{\frac{3}{5} \tan\theta}$

$\int \sec\theta d\theta$

$= \ln |\sec\theta + \tan\theta| + c$

$= \ln \left| \frac{5x + 5\sqrt{x^2 - \frac{9}{25}}}{3} \right|$



Exp.  $\int \sqrt{\frac{x}{1-x^3}} dx$

$u = x^{3/2}$   
 $du = \frac{3}{2} x^{1/2} dx$

$\frac{2}{3} \int \frac{du}{\sqrt{1-u^2}} = \frac{2}{3} \sin^{-1}\left(\frac{u}{1}\right) + c = \frac{2}{3} \sin^{-1}(x^{3/2}) + c$

## 8.4 Integration of rational functions Using Partial fraction:

- Rational function  $\frac{f(x)}{g(x)}$ ,  $f(x)$  and  $g(x)$  are polys.

The idea of using partial fraction is to simplify the rational function into small parts that are easy to integrate.

- If the degree of  $f <$  degree of  $g$ , and

1  $g$  can be written as linear, distinct factors then we use Cover Method to find the coefficients

2  $g$  can be written as repeated factors then we don't use Cover Method, but we ~~use~~ simplify the factors to find the coefficients.

\* Linear:  $x-1$ ,  $2x+3$ ,  $x$ ,  $1-3x$ , ...

- If the degree of  $f >$  degree of  $g$  then we use Long division.  $\rightarrow$  applied 1 or/and 2



Exp:  $\int \frac{dx}{x^2 - 3x + 2} = \int \frac{1}{(x-2)(x-1)} dx$

Cover Method  
C.M

$$\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$A = \frac{1}{2-1} = 1, \quad B = \frac{1}{1-2} = -1$$

$$\int \frac{dx}{x^2 - 3x + 2} = \int \left( \frac{1}{x-2} + \frac{-1}{x-1} \right) dx$$

$$= \ln|x-2| - \ln|x-1| + C$$

$$= \ln \left| \frac{x-2}{x-1} \right| + C$$



Exp:  $\int \frac{dx}{x^3+x^2-2x} = \int \frac{dx}{x(x^2+x-2)} = \int \frac{dx}{x(x+2)(x-1)}$

$\Rightarrow \frac{1}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$

$A = -\frac{1}{2}, B = \frac{1}{6}, C = \frac{1}{3}$

$\int \frac{dx}{x^3+x^2-2x} = \int \left( \frac{-1/2}{x} + \frac{1/6}{x+2} + \frac{1/3}{x-1} \right) dx$

$= -\frac{1}{2} \ln|x| + \frac{1}{6} \ln|x+2| + \frac{1}{3} \ln|x-1| + c$

Exp:  $\int_{-1}^0 \frac{x^3}{x^2-2x+1} dx$

$= \int_{-1}^0 \left( x+2 + \frac{3x-2}{x^2-2x+1} \right) dx$

$= \frac{x^2+2x}{2} \Big|_{-1}^0 + \int_{-1}^0 \frac{3x-2}{x^2-2x+1} dx$

$= \frac{x^2+2x}{2} \Big|_{-1}^0 + \int_{-1}^0 \left( \frac{3}{x-1} + \frac{1}{(x-1)^2} \right) dx$

$= (0+0) - \left( \frac{1}{2} - 2 \right) + \left[ 3 \ln|x-1| - \frac{1}{x-1} \right]_{-1}^0$

$= 2 - 3 \ln 2$

$\Rightarrow \frac{3x-2}{(x-1)(x-1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$

$\frac{3x-2}{(x-1)^2} = \frac{A(x-1) + B}{(x-1)^2}$

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$A=3 \quad B=1$

Note

Next

• Note: (S1)  $3x - 2 = A(x-1) + B$

$$Ax = 3x \Rightarrow A = 3$$

$$-2 = -A + B$$

$$B = 1$$

معامل  $x$  في الجزيئة الأولى

المطابقت في الجزيئة الثانية

(S2)  $3x - 2 = A(x-1) + B$

$$3x - 2 = A(1-1) + B$$

$x = 1$  تعويض

$$A = 3$$

(S3)  $(3x - 2 = A(x-1) + B)$

$$\boxed{3x = A}$$

$$B = 1$$

by derivative

• Note:  $\rightarrow \frac{1-2x}{(2x-3)^3} = \frac{A}{2x-3} + \frac{B}{(2x-3)^2} + \frac{C}{(2x-3)^3}$

$$\rightarrow \frac{1-2x}{(2x^2-3)^2} = \frac{Ax+B}{2x^2-3} + \frac{Cx+D}{(2x^2-3)^2} + \frac{Ex+F}{(2x^2-3)^3}$$

$$\rightarrow \frac{1-2x}{x(2x^2-3)^3} = \frac{G}{x} + \frac{Ax+B}{2x^2-3} + \frac{Cx+D}{(2x^2-3)^2} + \frac{Ex+F}{(2x^2-3)^3}$$

## 8.4 Integration by partial fraction:

Exp: 
$$\int \frac{x^4}{x^4-1} dx = \int \frac{x^4}{(x^2-1)(x^2+1)} dx = \int \frac{x^4}{(x-1)(x+1)(x^2+1)} dx$$

$$\frac{x^4}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

On cover method  $A = \frac{1}{4}$   
 $B = -\frac{1}{4}$

$$= \frac{\frac{1}{4}}{x-1} - \frac{\frac{1}{4}}{x+1} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^4}{x^4-1} = \frac{1/2}{(x-1)(x+1)} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^4}{x^4-1} = \frac{\frac{1}{2}(x^2+1) + (Cx+D)(x^2-1)}{(x^2-1)(x^2+1)}$$

$$\Rightarrow x^4 = \frac{x^2}{2} + \frac{1}{2} + (C+D)(x^2-1)$$

$$x=0 \Rightarrow 0 = 0 + \frac{1}{2} + D(-1) \Rightarrow D = 1/2$$

$$4x^3 = x + 0 + (Cx+D)(2x) + (x^2-1)c$$

$$x \Rightarrow 0 \Rightarrow 0 = 0 + 0 + 0 - c \Rightarrow c = 0$$

$$\int \frac{x^4}{x^4-1} dx = \int \frac{1/4}{x-1} dx + \int \frac{1/4}{x+1} dx + \int \frac{1/2}{x^2+1} dx$$

$$= \frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| + \frac{1}{2} \tan^{-1} x + c$$

Exp:  $\int \frac{\sqrt{x+1}}{x} dx$

$$y = \sqrt{x+1}$$

$$dy = \frac{1}{2\sqrt{x+1}} dx$$

$$2\sqrt{x+1} dy = dx$$

$$2y dy = dx$$

$$y^2 = x+1$$

$$\int \frac{2y^2}{y^2-1} dy$$

$$\int \frac{2y^2}{y^2-1} dy = \int \left( \frac{1}{y-1} - \frac{1}{y+1} \right) dy = \ln|y-1| - \ln|y+1| + c$$

$$\frac{2y^2}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

$$A = 1, B = -1$$

Exp:  $\int \frac{dx}{x(x^2+1)^2}$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

on cover method A=1

$$= \frac{1}{x} + \frac{cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

$$\frac{1}{x(x^2+1)^2} = \frac{(x^2+1)^2}{x(x^2+1)^2} + \frac{(cx+D)x(x^2+1)}{x(x^2+1)^2} + \frac{x(Ex+F)}{x(x^2+1)^2}$$

$$1 = (x^2+1)^2 + x(cx+D)(x^2+1) + x(Ex+F)$$

$$0 = 2(x^2+1) \cdot 2x + 2x(cx^2+Dx) + (x^2+1)(2cx+D) + 2Ex+F$$

$$x=0 \Rightarrow 0 = E+F$$

$$0 =$$

## 2.7 Improper Integrals:

### Type II

The interval of integrations contain points of discontinuity for  $f(x)$

$$\int_0^2 \frac{dx}{x}$$

### Type I

The upper limit or the lower limit contains  $\infty$  or  $-\infty$

$$\int_1^{\infty} \frac{dx}{x}$$

Q: How to handle Improper Integral of type I?

$f(x)$  is defined on an infinite interval  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) \cdot dx = \lim_{b \rightarrow \infty} \int_a^b f(x) \cdot dx$$

$$\int_{-\infty}^b f(x) \cdot dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) \cdot dx$$

Def (Improper Integral of type (I)) are integrals with infinite limit of integration:

$$\boxed{1} \text{ } f \text{ is cont on } [a, \infty) \Rightarrow \int_a^{\infty} f(x) dx =$$

$$\lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

$$\boxed{2} \text{ } f \text{ is cont on } (-\infty, a] \Rightarrow \int_{-\infty}^a f(x) dx =$$

$$\lim_{b \rightarrow -\infty} \int_b^a f(x) dx.$$

$$\boxed{3} \text{ } f \text{ is cont on } (-\infty, \infty) = \int_{-\infty}^{\infty} f(x) dx$$

$$= \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

$$= \lim_{b \rightarrow -\infty} \int_b^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx$$

• If the limit of the improper integral exists (finite number) then we say that this improper integral converges.

• Other wise we say that the improper integral diverges.

• If  $f(x) \geq 0$   $\forall x$  and the limit exists then the improper integral converges to positive number  $\rightarrow$  Area.

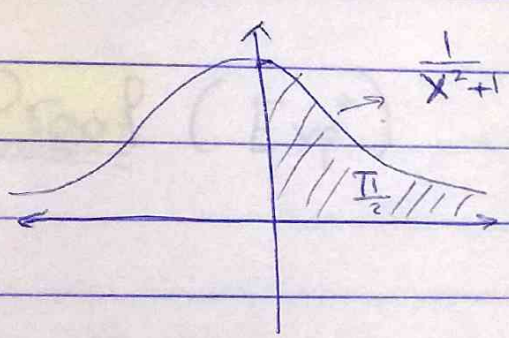
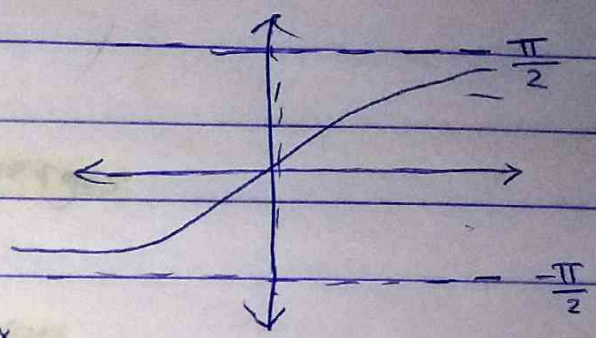
Exp. ①  $\int_0^{\infty} \frac{dx}{x^2+1}$  (I)

$$= \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b \rightarrow \infty} \left. \tan^{-1} x \right|_0^b = \lim_{b \rightarrow \infty} [\tan^{-1} b - \tan^{-1} 0]$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\Rightarrow$  converges to  $\frac{\pi}{2}$

area since  $f(x) \geq 0 \forall x$



②  $\int_{-\infty}^0 \frac{dx}{x^2+1} = \frac{\pi}{2}$

③  $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$



Exp\*

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{1}{x^{p-1}} \quad \begin{array}{l} \text{if } p > 1 \\ \text{if } p \leq 1 \end{array}$$

Exp: ①

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1 \Rightarrow \text{Converges}$$

②

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \infty \Rightarrow \text{diverges}$$

③

$$\int_1^{\infty} \frac{dx}{x} = \infty \Rightarrow \text{diverges}$$

Proof (Exp\*)

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x^p} &= \lim_{c \rightarrow \infty} \int_1^c \frac{dx}{x^p} \\ &= \lim_{c \rightarrow \infty} \int_1^c x^{-p} dx \\ &= \lim_{c \rightarrow \infty} \left. \frac{x^{1-p}}{1-p} \right|_1^c \\ &= \lim_{c \rightarrow \infty} \left[ \frac{c^{1-p}}{1-p} - \frac{1}{1-p} \right] \\ &= \frac{1}{p-1} \left[ 1 - \lim_{c \rightarrow \infty} c^{1-p} \right] \end{aligned}$$

$\Rightarrow$

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

$\neq \leftarrow p \text{ comp}$

if  $p=1 \Rightarrow$

$$\begin{aligned} \int_1^{\infty} \frac{dx}{x} &= \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln|x| \Big|_1^b = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b \rightarrow \infty} \ln b - \ln 1 = \infty - 0 = \infty \end{aligned}$$

Exp:  $\int_{-\infty}^{\infty} \frac{2x dx}{(x^2+1)^2}$

$$= \int_{-\infty}^0 \frac{2x dx}{(x^2+1)^2} + \int_0^{\infty} \frac{2x dx}{(x^2+1)^2}$$

$$= - \int_1^{\infty} \frac{du}{u^2} + \int_{\infty}^1 \frac{du}{u^2}$$

$$= \left( \frac{1}{2-1} \right) + \left( \frac{1}{2-1} \right) = -1 + 1 = 0$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\text{as } x=0 \Rightarrow u=1$$

$$x=-\infty \Rightarrow u=\infty$$

Exp:  $\int_0^{\infty} \frac{16 \tan^{-1} x}{1+x^2} dx$

$$= \int_0^{\frac{\pi}{2}} 16 u du$$

$$= 8 u^2 \Big|_0^{\frac{\pi}{2}}$$

$$8 \left[ \left( \frac{\pi}{2} \right)^2 - 0 \right] = 2\pi^2 \text{ converges}$$

$$u = \tan^{-1} x$$

$$du = \frac{1}{1+x^2} dx$$

$$\text{When } x=0 \Rightarrow u=0$$

$$x=\infty \Rightarrow u=\frac{\pi}{2}$$

## 8.7 Improper Integral of type II:

Def. Improper Integrals of type II are integrals that become infinite at some point within the limit of Integration (V.Asy)

• If  $f$  is discont. at  $a$  then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

• If  $f$  is discont at  $b$  then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

• If  $f$  is discont at  $c \in [a, b]$  then

$$\int_a^b f(x) dx = \lim_{d \rightarrow c^-} \int_a^d f(x) dx + \lim_{e \rightarrow c^+} \int_e^b f(x) dx$$

Exp.  $\int_0^1 \frac{dx}{\sqrt{x}}$

$$= \lim_{c \rightarrow 0^+} 2 \int_c^1 \frac{dx}{2\sqrt{x}} = \lim_{c \rightarrow 0^+} 2\sqrt{x} \Big|_c^1$$

$$2 \lim_{c \rightarrow 0^+} (\sqrt{1} - \sqrt{c})$$

$$2[1 - \sqrt{0}] = 2$$

Exp<sup>+</sup>:  $\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases} \Rightarrow \text{Type I}$

Exp<sup>\*\*</sup>:  $\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases} \Rightarrow \text{Type II}$

Exp<sup>'</sup>:  $\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{x^{1/2}} = \frac{1}{1-1/2} = 2$

Exp:  $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \int_c^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

$$= -2 \lim_{c \rightarrow 0^+} \int_c^1 \frac{e^{-\sqrt{x}}}{-2\sqrt{x}} dx$$

$$\begin{aligned} u &= -\sqrt{x} \\ du &= -\frac{dx}{2\sqrt{x}} \end{aligned}$$

$$= -2 \lim_{c \rightarrow 0^+} \int_{-\sqrt{c}}^{-1} e^u du$$

$$= -2 \lim_{c \rightarrow 0^+} e^u \Big|_{-\sqrt{c}}^{-1} = -2 \lim_{c \rightarrow 0^+} (e^{-1} - e^{-\sqrt{c}})$$

$$= -2 \left[ \frac{1}{e} - e^{-\sqrt{c}} \right]$$

$$= -2 \left[ \frac{1}{e} - 1 \right] = 2 - \frac{2}{e}$$

Exp:  $\int_0^1 x \ln x dx$

Type II

$$\int x \ln x dx = x^2 \ln x - \frac{x^2}{2} \int x \ln x dx + \frac{x^2}{2}$$

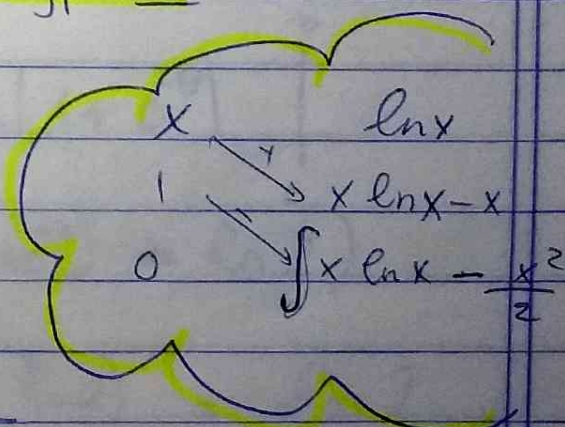
$$2 \int x \ln x dx = x^2 \ln x - \frac{x^2}{2}$$

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$\lim_{c \rightarrow 0^+} \left[ \frac{x^2}{2} \ln x - \frac{x^2}{4} \right] \Big|_0^1 = \lim_{c \rightarrow 0^+} \left[ \left(0 - \frac{1}{4}\right) - \left(\frac{c^2}{2} \ln c - \frac{c^2}{4}\right) \right]$$

$$\lim_{c \rightarrow 0^+} = -\frac{1}{4} + \lim_{c \rightarrow 0^+} \left( \frac{c^2}{4} - \frac{1}{2} c^2 \ln c \right)$$

$$-\frac{1}{4} - \frac{1}{2} \lim_{c \rightarrow 0^+} \frac{\ln c}{\frac{1}{c^2}} = -\frac{1}{4} - \frac{1}{2} \lim_{c \rightarrow 0^+} \frac{\frac{1}{c}}{-2 \frac{1}{c^3}} = -\frac{1}{4}$$



Exp

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}}$$

II

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}}$$

$$= -2 \lim_{c \rightarrow 1^-} \int_0^c \frac{dx}{2\sqrt{1-x}} + 2 \lim_{c \rightarrow 1^+} \int_c^2 \frac{dx}{2\sqrt{x-1}}$$

$$= -2 \lim_{c \rightarrow 1^-} \sqrt{1-x} \Big|_0^c + 2 \lim_{c \rightarrow 1^+} \sqrt{x-1} \Big|_c^2$$

$$= -2 \lim_{c \rightarrow 1^-} [\sqrt{1-c} - 1] + 2 \lim_{c \rightarrow 1^+} [1 - \sqrt{c-1}]$$

$$= 2 + 2 = 4$$

Exp

$$\int_0^{\infty} \frac{d\theta}{1+e^{\theta}} = \lim_{b \rightarrow \infty} \int_0^b \frac{d\theta}{1+e^{\theta}} * \left( \frac{e^{-\theta}}{e^{-\theta}} \right)$$

$$\lim_{b \rightarrow \infty} \int_0^b \frac{db e^{-\theta}}{1+e^{\theta}} = \lim_{b \rightarrow \infty} -\ln|1+e^{-\theta}| \Big|_0^b$$

$$= \lim_{b \rightarrow \infty} (-\ln(1+e^{-b}) + \ln 2)$$

$$= \lim_{b \rightarrow \infty} (-0 + \ln 2) = \underline{\underline{\ln 2}}$$

converge