Birzeit University

Mathematics Department

First Semester 2021/2022

Instructor: Dr. Ala Talahmeh

Course Code: Math 1431

Title: Calculus for Health Sciences

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1.1, 1.2, 1.3

Chapter 1 Preview and Review

1.1 Preliminaries (أساسيات)

1.1.1 The Real Numbers (الأعداد الحقيقية) $\mathbb{R}$

Real number line

فترة مفتوحة

Open interval $= (a,b) = \{x : a < x < b\}$ bounded

closed interval $= [a,b] = \{x : a \le x \le b\}$ bounded

half-open

$[a,b) = \{x : a \le x < b\}$ bounded interval

$(a,b] = \{x : a < x \le b\}$ bounded interval

unbounded intervals

$[a,\infty) = \{x : x \geq a\}$

$(-\infty,a] = \{x : x \le a\}$

$(a,\infty) = \{x : x > a\}$

$(-\infty,a) = \{x : x < a\}$

$\mathbb{R} = \{x : -\infty < x < \infty\}$.

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The Absolute Value

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

$|7| = 7$, $|-7| = -(-7) = 7$.

distance between x_1 and x_2 is $|x_1 - x_2|$

ex. distance between -2 and 4 is

$|-2-4| = |-6| = 6$ or $|4--2| = 6$

Remark. Let $b \geq 0$. Then

① $a \geq 0$, $|a| = b \iff a = b$

② $a < 0$, $|a| = b \iff -a = b$.

ex. Solve $|x-4| = 2$

Sol. $x - 4 = 2$ or $x - 4 = -2$

$\boxed{x = 6}$ or $\boxed{x = 2}$

Solution set $= \{2, 6\}$.

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ex. Solve $\left|\frac{3}{2}x-1\right| = \left|\frac{1}{2}x+1\right|$.

Sol. $\frac{3}{2}x-1 = \frac{1}{2}x+1$ or $\frac{3}{2}x-1 = -\frac{1}{2}x-1$

$\frac{3}{2}x-\frac{1}{2}x = 1+1$ $\qquad$ $\frac{3}{2}x+\frac{1}{2}x = 1-1$

$\boxed{x=2}$ $\qquad$ $2x = 0$

$\boxed{x=0}$

The solution is $\{0,2\}$

Rmk. Let $b>0$. then

(1) $\boxed{|a|<b} \iff \boxed{-b<a<b}$

(2) $|a|>b \iff \boxed{a>b}$ or $\boxed{a<-b}$

ex. Solve $|2x-5|<3$.

Sol. $-3<2x-5<3$
$\;\;+5 \qquad +5 \quad +5$

$\frac{2}{2} < \frac{2x}{2} < \frac{8}{2} \Rightarrow 1<x<4$.

The solution is $(1,4)$.

ex. Solve $|4-3x| \geq 2$

$4-3x \geq 2$ or $4-3x \leq -2$
$-4 \qquad\quad -4 \qquad -4 \qquad\quad -4$

$-3x \geq -2$ $\qquad$ $-3x \leq -6$

$x \leq \frac{2}{3}$ $\qquad$ $x \geq 2$

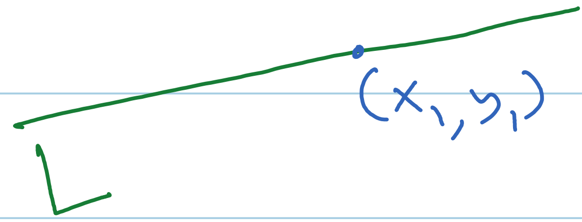
The solution is $\left(-\infty, \frac{2}{3}\right] \cup [2,\infty)$.

Lines in the plane
خط

Linear equation معادلة خطية

point-slope form

ميل slope $= m$



$$\boxed{y - y_1 = m(x - x_1)}$$

$$\boxed{y = mx + b}$$ slope-intercept form مقطع

(x_1, y_1), (x_2, y_2)

$$m = \text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

ميل

ex: Find the equation of the line
يمر passing through $(\underbrace{-2}_{x_1}, \underbrace{1}_{y_1})$ and $(\underset{x_2}{3}, \underset{y_2}{-\frac{1}{2}})$.

Sol.

$$m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-\frac{1}{2} - 1}{3 - -2}$$

$$= \frac{-\frac{3}{2}}{5} = \frac{-3}{10}$$

the eq is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{-3}{10}(x - -2)$$

$$y-1=-\frac{3}{10}(x+2)$$ (point-slope form)

$$y-1=-\frac{3}{10}x-\frac{3}{5}$$
$+1 \qquad +1$

$$y=-\frac{3}{10}x-\frac{3}{5}+1$$

$\to y=mx+b$

$$\boxed{y=-\frac{3}{10}x+\frac{2}{5}}$$ (slope-Intercept form)

slope

y-intercept
المقطع الصادي

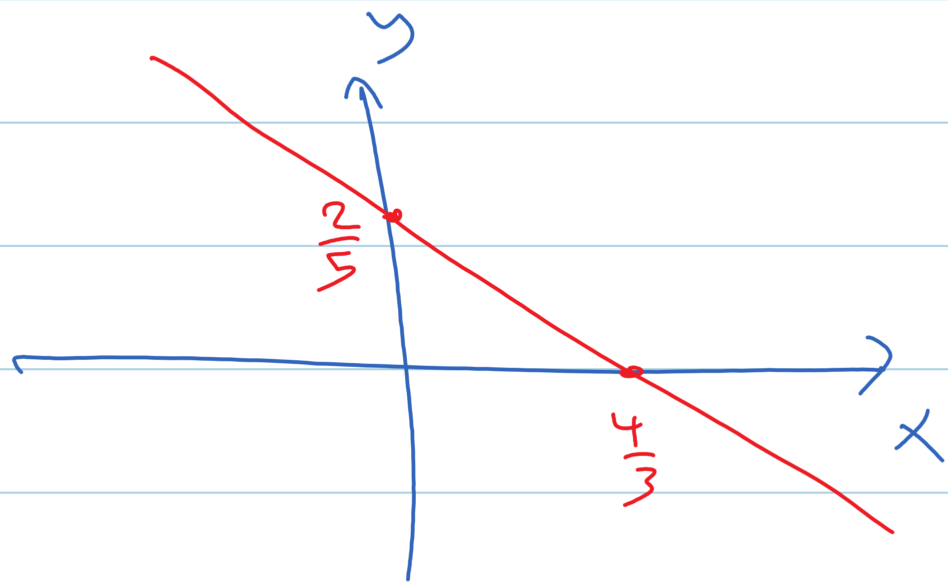
Graph $\boxed{x=0} \Longrightarrow \boxed{y=\frac{2}{5}} \to$ y-intercept
المقطع الصادي

$\boxed{y=0} \Longrightarrow -\frac{3}{10}x+\frac{2}{5}=0$

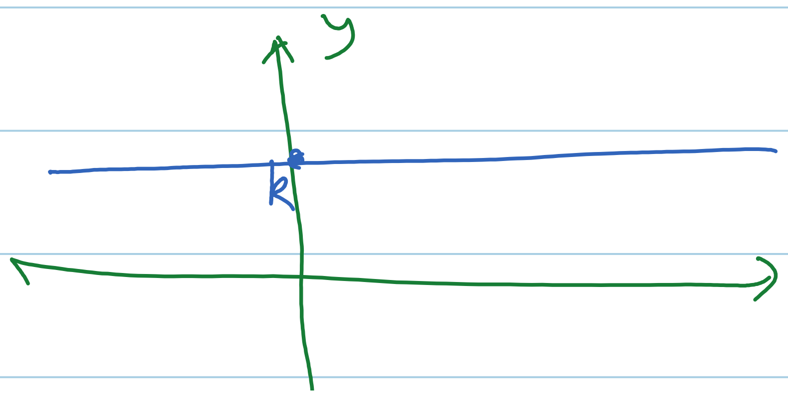
$$\frac{3}{10}x=\frac{2}{5}$$

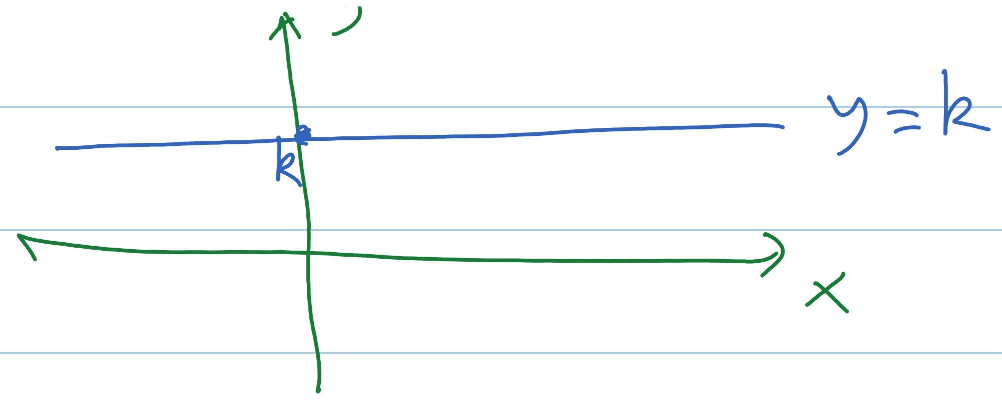
$$x=\frac{2}{5}\left(\frac{10}{3}\right)=\frac{4}{3}$$

x-intercept
المقطع السيني



Remark. $y=k$ (slope 0) horizontal line
خط أفقي

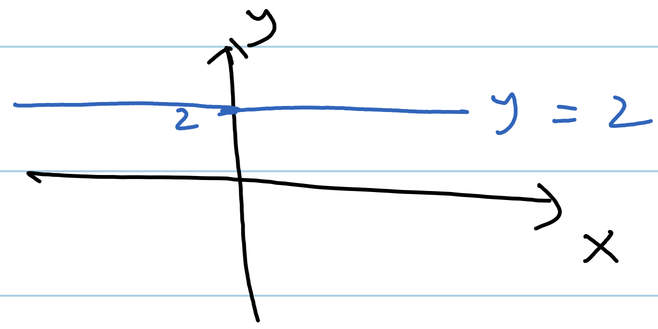




y
k
$y = k$
x

ex. $y = 2$

slope $= 0$



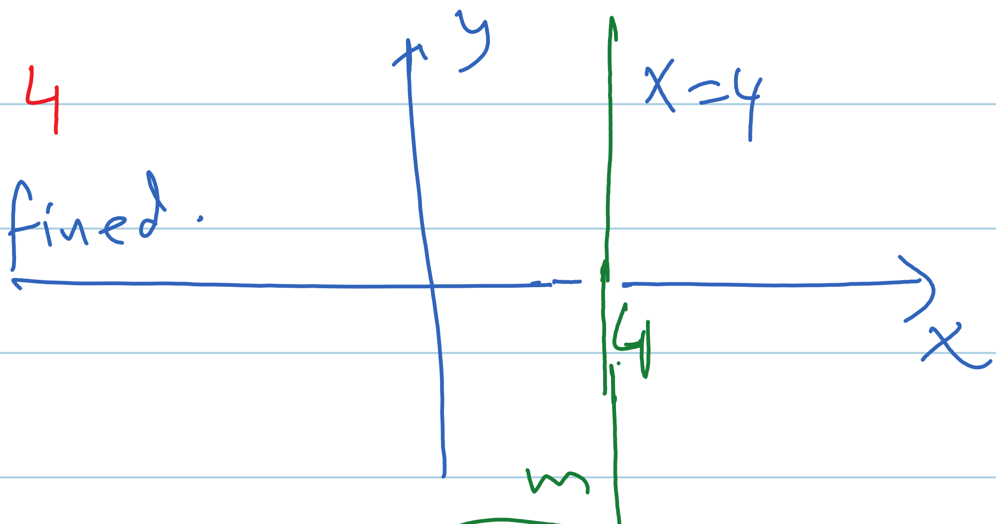
ثابت خط عمودي

$x = h$ " constant " Vertical line

slope undefined
غير معرف

ex. $x = 4$

slope undefined.



ex. Determine the slope and the y-intercept of the line $3y - 2x + 9 = 0$

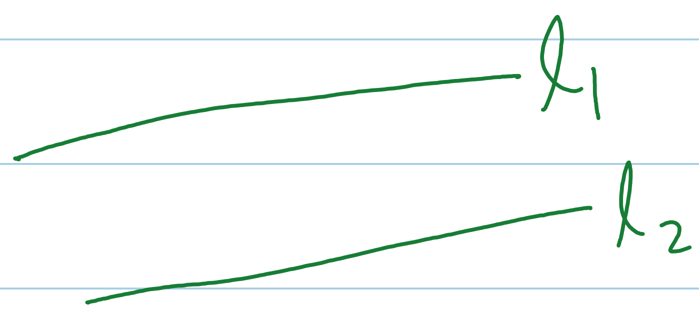
m b

Sol. $3y \underbrace{- 2x + 9} = 0 \longrightarrow \boxed{y = mx + b}$

$$\frac{3y}{3} = \frac{2x}{3} - \frac{9}{3} \Rightarrow \boxed{y = \underset{m}{\frac{2}{3}}x \underset{b}{-3}}$$

slope $= m = \frac{2}{3}$

y-intercept $= b = -3$.



متوازيان

l_1 parallel l_2

$l_1 \parallel l_2 \Leftrightarrow m_1 = m_2$

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متعامدان

l_1 and l_2 are perpendicular

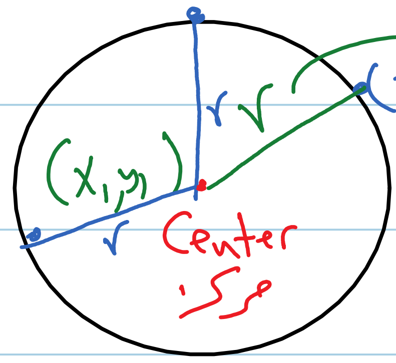
$$l_1 \perp l_2 \iff m_1 m_2 = -1$$

1.1.3 Equation of the circle

CH Section Subsection

معادلة الدائرة

$$r = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

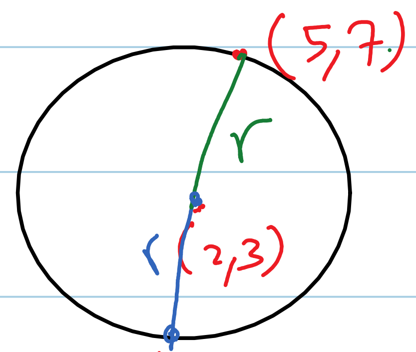


the eq. is $\boxed{(x-x_1)^2 + (y-y_1)^2 = r^2}$

ex. Find the equation of the circle with Center $(\overset{x_1}{2}, \overset{y_1}{3})$ and passing through $(5,7)$

Sol. $r = \sqrt{(5-2)^2 + (7-3)^2}$

$= \sqrt{9+16} = 5$



رسم توضيحي

the eq. is

$$(x-2)^2 + (y-3)^2 = (5)^2$$

$$(x-2)^2 + (y-3)^2 = 25.$$

1.1.4 Trigonometry (المثلثات)

radian measure — التقدير الدائري

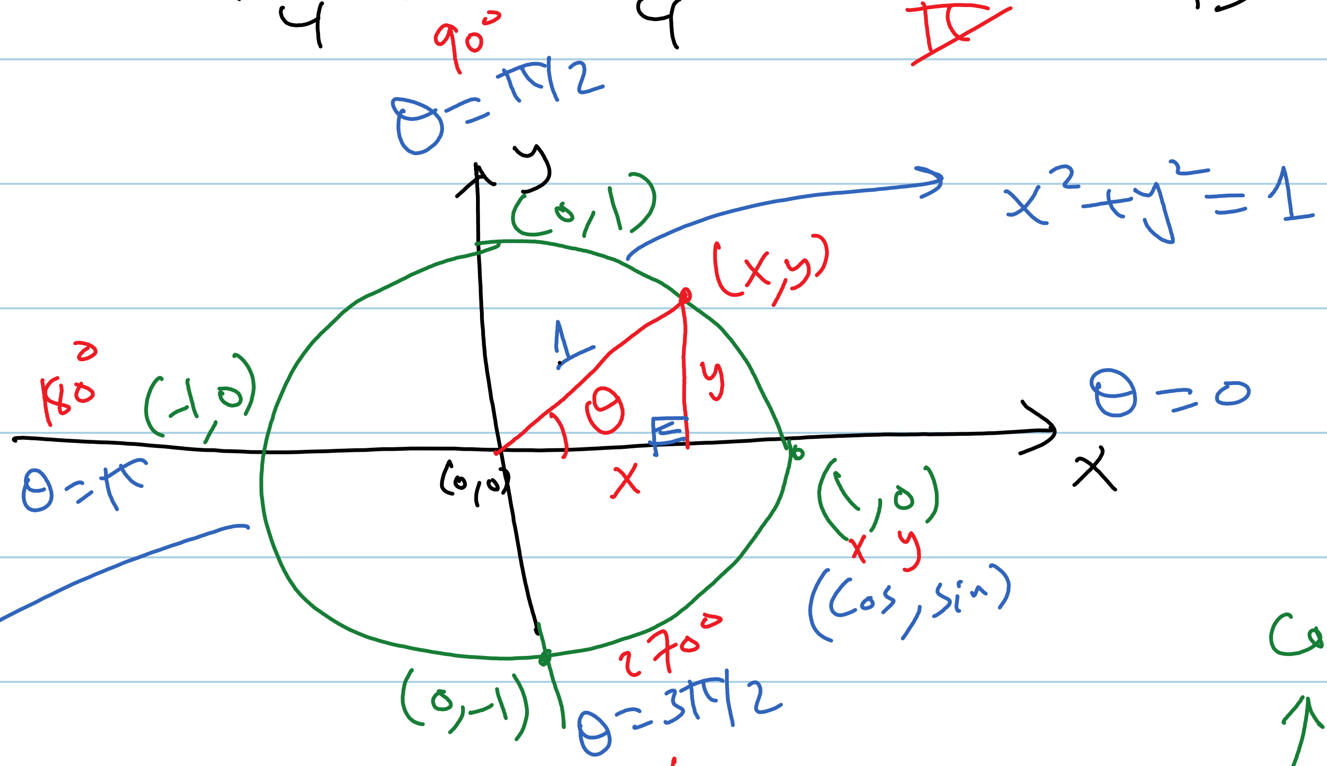
degree measure — التقدير الستيني

ex. convert 60° to radian. deg. $\xrightarrow{\times \frac{\pi}{180}}$ rad

Sol. $60^\circ = 60^\circ \times \frac{\pi}{180^\circ} = \frac{\pi}{3}$

ex. convert $\frac{3\pi}{4}$ to degree rad $\xrightarrow{\times \frac{180}{\pi}}$ deg.

Sol. $\frac{3\pi}{4} = \frac{3\cancel{\pi}}{4} \cdot \times \frac{180}{\cancel{\pi}} = 135^\circ$.



$\sin\theta$ جا → Sine , $\cos\theta$ جتا → Cosine , $\tan\theta$ ظا → tangent , $\sec\theta$ قا → Secant , $\csc\theta$ قتا → Cosecant , $\cot\theta$ ظتا → Cotangent

$\sin\theta = \frac{y}{1} = y$, $\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{y}{x}$, $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$,

Identities متطابقات

1. $\sin^2\theta + \cos^2\theta = 1$.

2. $\sin(2\theta) = 2\sin\theta\cos\theta$

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

3. $\cos(2\theta) = \begin{cases} \cos^2\theta - \sin^2\theta \\ 1 - 2\sin^2\theta \\ 2\cos^2\theta - 1 \end{cases}$

4. $1 + \tan^2\theta = \sec^2\theta$

5. $1 + \cot^2\theta = \csc^2\theta$

6. $\sin(A \mp B) = \sin A\cos B \mp \cos A\sin B$

7. $\cos(A \mp B) = \cos A\cos B \pm \sin A\sin B$

8. $\tan(A \mp B) = \dfrac{\tan A \mp \tan B}{1 \pm \tan A\tan B}$

ملاحظة

$$\sin^2\theta = (\sin\theta)^2$$

$$\sin\theta^2 = \sin(\theta^2)$$

معادلة مثلثية
trigonometric equation

Ex. Solve $2\sin\theta\cos\theta = \cos\theta$, on $[0, 2\pi)$

$[0, 360)$

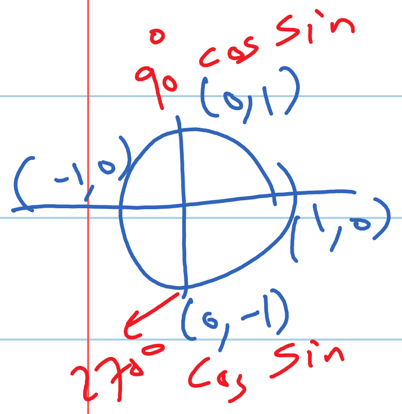
Sol. $2\sin\theta\cos\theta - \cos\theta = 0$

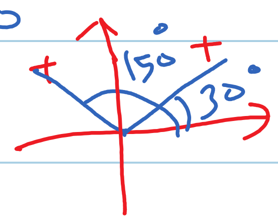
$\cos\theta(2\sin\theta - 1) = 0$

All Students take Calculus
sin tan cosine

$\boxed{\cos\theta = 0}$ or $2\sin\theta - 1 = 0$

$\boxed{\sin\theta = \frac{1}{2}}$





$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

The solution is $\left\{ \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2} \right\}$

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odd Functions

$\sin(-\theta) = -\sin\theta$

$\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$

$\cot(-\theta) = -\cot\theta$

even functions

$\cos(-\theta) = \cos\theta$

$\sec(-\theta) = \sec\theta$

	$0°$	$30°$	$45°$	$60°$	$90°$
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined غير معرف

$\tan\left(-\frac{\pi}{4}\right)$

$= -\tan\frac{\pi}{4} = -1$

$\cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

1.1.5 Exponentials and Logarithms

الأُسّ

a^r: $r \to$ exponent أس ; $a \to$ base أساس

$$a^r a^s = a^{r+s} \qquad (ab)^r = a^r b^r$$

$$\frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$$

$$a^{-r} = \frac{1}{a^r} \qquad (a^r)^s = a^{rs}$$

ex. (a) $3^2 \cdot 3^{\frac{5}{2}} = 3^{2+\frac{5}{2}} = 3^{\frac{9}{2}}$

(b) $\dfrac{2^{-4}\, 2^{3}}{2^2} = \dfrac{2^{-4+3}}{2^2} = \dfrac{2^{-1}}{2^2} = 2^{-1-2} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

(c) $\dfrac{a^k\, a^{3k}}{a^{5k}} = \dfrac{a^{k+3k}}{a^{5k}}$

$= \dfrac{a^{4k}}{a^{5k}} = a^{4k-5k} = a^{-k} = \dfrac{1}{a^k}$

Logarithm

$$\boxed{x = \log_a y \iff y = a^x.}$$

ex. Find $\log_2 8$

Sol. $\log_2 8 = b \Rightarrow \boxed{2}^{b} = 8 = \boxed{2}^{3} \Rightarrow b = 3$

$\therefore \log_2 8 = 3.$

ex. Solve $\log_3 x = -2 \Rightarrow x = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}.$

ex Solve $\log_{\frac{1}{2}} 8 = x$

Sol. $8 = \left(\frac{1}{2}\right)^x \Rightarrow 2^3 = 2^{-x}$

$\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x}$

$\Rightarrow 3 = -x$

$\therefore \boxed{x = -3}$

Properties

1. $\log_a (xy) = \log_a x + \log_a y$

2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.

3. $\log_a x^r = r \log_a x$

Natural Logarithm $\log_e x = \ln x$

where $e \approx 2.7182818\ldots$

4. $\log_a (a^x) = x$ and $a^{\log_a x} = x$

5. $\boxed{\log_a a = 1}$

6. $\log_a 1 = 0$

ex. $\log_3 (9 \cdot x^2) = \log_3 9 + \log_3 x^2$

$= \log_3 3^2 + 2 \log_3 x$

$= 2 + 2 \log_3 x$

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ex. $\log_5\left(\frac{x^2+3}{5x}\right) = \log_5(x^2+3) - \log_5(5x)$

$= \log_5(x^2+3) - [\log_5 5 + \log_5 x]$ 1

$= \log_5(x^2+3) - 1 - \log_5 x$.

ex. $-\ln\frac{1}{2} = \ln\left(\frac{1}{2}\right)^{-1} = \ln(2^{-1})^{-1} = \ln 2$

$(2^{-1})^{-1}$

ex. Solve $e^{2x} = 3$

Sol. $\ln e^{2x} = \ln 3$

$2x \ln e = \ln 3$, $\quad \ln e = \log_e e = 1$.

$2x = \ln 3 \Rightarrow \boxed{x = \frac{\ln 3}{2}}$

ex. Solve $5^{2x-1} = 2^x$

Sol. $\ln(5^{2x-1}) = \ln(2^x)$

$(2x-1)\ln 5 = x\ln 2$

$\underbrace{(2\ln 5)x - \ln 5}_{} = \underbrace{x\ln 2}_{}$

$(2\ln 5)x - x\ln 2 = \ln 5$

$x(2\ln 5 - \ln 2) = \ln 5$

$x = \frac{\ln 5}{2\ln 5 - \ln 2}$.

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$\log_y x = b \Leftrightarrow y^b = x$

ex. Solve $\ln(x+1) = 5$

Sol. $\log_e(x+1) = 5$

$x+1 = e^5 \Rightarrow \boxed{x = e^5 - 1}$

$-1 \quad -1$

1.1.6 Complex Numbers and Quadratic Equations

الاعداد المركبة المعادلات التربيعية

$\sqrt{-1} = i \Rightarrow \boxed{i^2 = -1}$

i : imaginary unit الوحدة التخيلية

ex. $\sqrt{-16} = \sqrt{16}\, i = 4i$

$\sqrt{-5} = \sqrt{5}\, i$

Complex number $z = a + bi$

a : real part, b : imaginary part

$Re(z) = a \quad , \quad Im(z) = b$

ex. $z = -3 + 7i$

$Re(z) = -3 \quad , \quad Im(z) = 7$

ex. $z = 6i \quad , \quad Re(z) = 0 \quad , \quad Im(z) = 6$

Rmk.

$z = bi$ Purely imaginary numbers.

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$a+bi = c+di \Leftrightarrow a=c$ and $b=d$

$(a+bi)(c+di) = ac + adi + (bc)i + bd\,i^2$ -1

$= ac - bd + (ad+bc)i$

ex. $i^3 = i^2 \cdot i = (-1)i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

ex: Find (a) $(2+3i) - (5-6i)$

$= 2 + 3i - 5 + 6i$

$= (2-5) + 3i + 6i$

$= -3 + 9i$

(b) $(2+3i) - (5-6i)$

$= 2 + \underbrace{3i} - 5 + \underbrace{6i}$

$= -3 + 9i$ (نفس a)

(c) $(5-3i)(1+2i)$

$= (5)(1) + 5(2i) - 3i(1) - (3i)(2i)$

$= 5 + 10i - 3i - 6i^2$ -1

$= 5 + 7i + 6 = 11 + 7i$

Conjugate مرافق

$z = a+bi \Rightarrow \bar{z} = a-bi$

ex. $z = 7-3i \Rightarrow \bar{z} = 7+3i$

خصائص properties

$\bar{\bar{z}} = z \quad , \quad \overline{z+w} = \bar{z}+\bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$z\bar{z} = (a+bi)(a-bi) = a^2-(bi)^2$

$= a^2 - b^2 i^2 \quad -1$

$= a^2+b^2$

ex. $z = 2-3i \Rightarrow z\bar{z} = (2)^2+(-3)^2 = 4+9$

$= 13$

Quadratic eq. $ax^2+bx+c=0 \quad , a \neq 0$

(quadratic formula) $x_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

ex. Solve $x^2+4x+5=0$

Sol. $\boxed{a=1}$, $\boxed{b=4}$ $\boxed{c=5}$

$x_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-4 \pm \sqrt{4^2-4(1)(5)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{16-20}}{2}$

$= \dfrac{-4 \pm 2i}{2} = -2 \pm i$

الاقترانات

1.2 Elementary Functions

1.2.1 What Is a Function?

Definition A **function** f is a rule that assigns each element x in the set A exactly one element y in the set B. The element y is called the **image** (or **value**) of x under f and is denoted by $f(x)$ (read "f of x"). The set A is called the **domain** of f, the set B is called the **codomain** of f, and the set $f(A) = \{y : y = f(x) \text{ for some } x \in A\}$ is called the **range** of f.

المجال

المجال المقابل

المدى

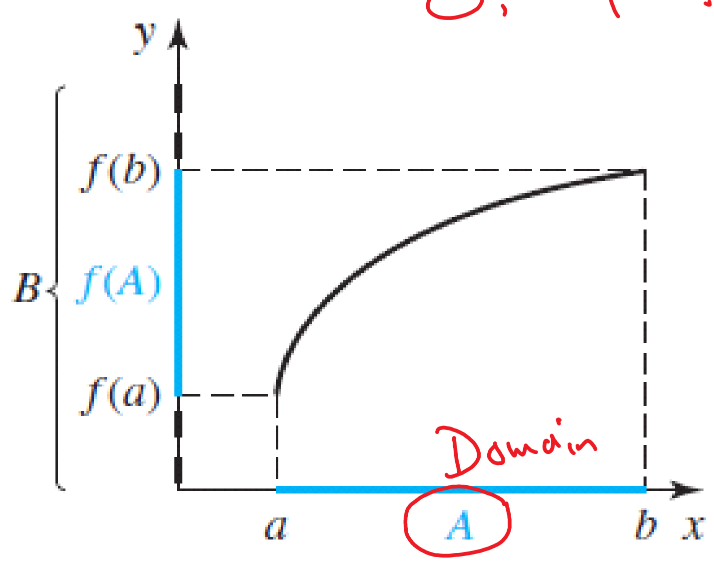
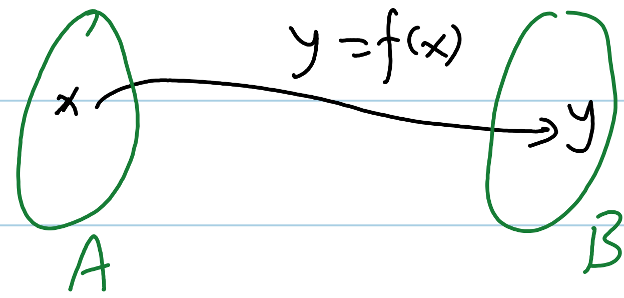
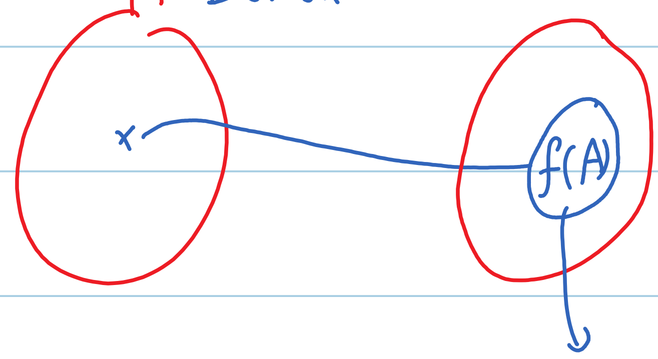


Figure 1.9 A function $f(x)$ with domain A, codomain B, and range $f(A)$.



$$f : A \to B$$
$$x \to f(x)$$



ex. $y = f(x) = x^2 \geq 0$ $\quad f : \mathbb{R} \longrightarrow \mathbb{R}$

$$x \longrightarrow x^2$$

Domain $= (-\infty, \infty)$, Codomain $= (-\infty, \infty)$

Range $= [0, \infty)$

$y = f(x)$ $\quad x$: independent variable متغير مستقل

y : dependent variable متغير تابع

f and g are equal if and only if

1) $f(x) = g(x)$ for all $x \in$ Domain

2) f and g are defined on the same domain.

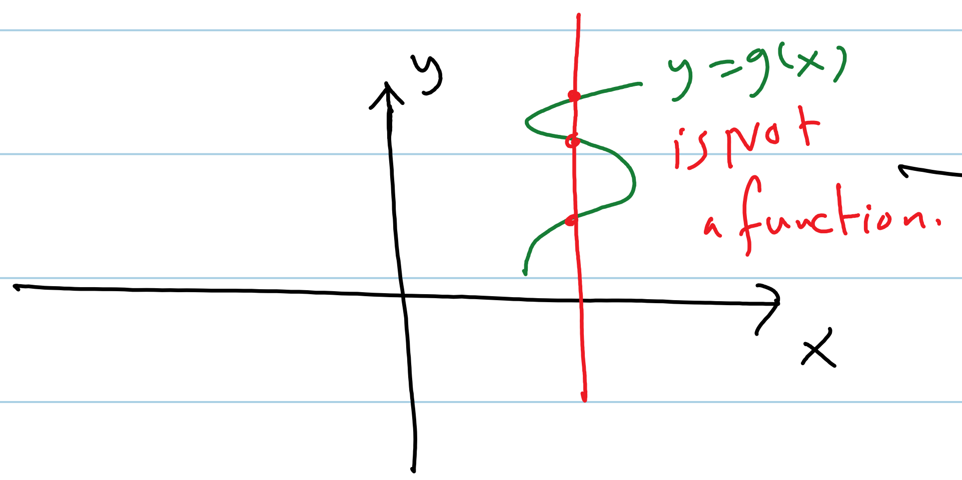
ex. $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = x^2$

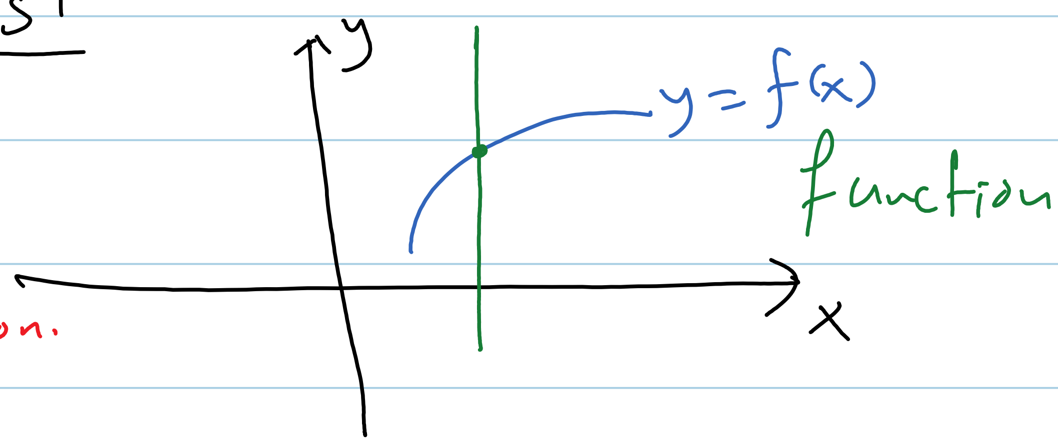
$g : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad g(x) = x^2$

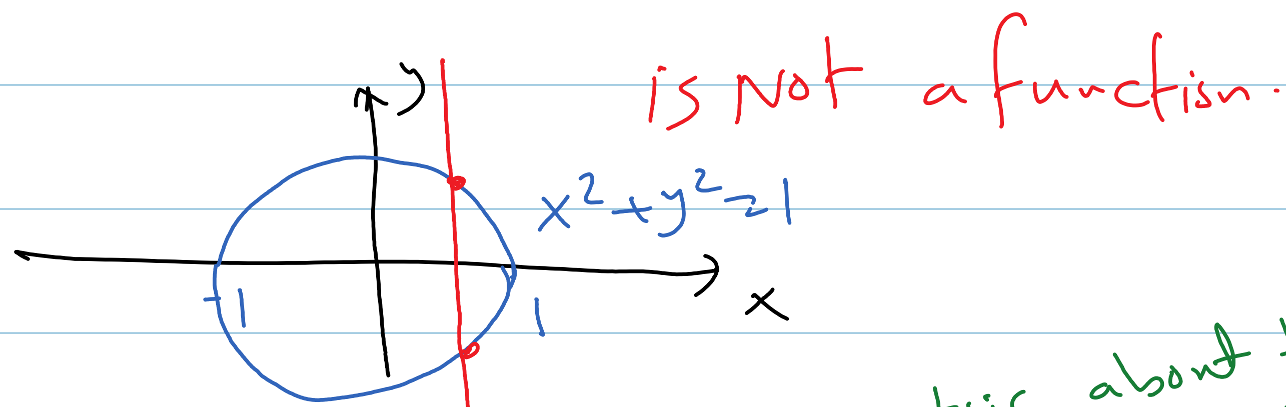
Is $f = g$??

Sol. $f \neq g$ since $\text{domain}(f) = [0,1]$ $\neq$ $\text{domain}(g) = \mathbb{R}$

Vertical line test







$f : A \longrightarrow B$ $\quad$ f is even if $f(-x) = f(x)$, $x \in A$ $\quad$ Symmetric about the y-axis

Sym. about the origin. $\quad$ odd if $f(-x) = -f(x)$, $x \in A$.

ex. $f(x) = x^2$ is even since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$g(x) = x^3$ is odd since $g(-x) = (-x)^3 = -x^3 = -g(x)$

ex. $h(x) = x^3 + x^2$

$h(-x) = (-x)^3 + (-x)^2$

$= -x^3 + x^2 \neq h(x)$

$\neq -h(x)$

$\Rightarrow$ h is neither even nor odd

Df. Composite function تركيب اقترانات

$(f \circ g)(x) = f(g(x))$, $x \in \text{Domain}(g)$

$g(x) \in \text{Domain}(f)$

ex. $\boxed{f(x) = \sqrt{x}}$ and $g(x) = x^2 + 1$, $x \in \mathbb{R}$. Find

(a) $(f \circ g)(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$

(b) Domain $(f \circ g)$

$x \in D_g = (-\infty, \infty)$

$\overset{x^2+1}{g(x)} \in D_f = [0, \infty)$

$D_f = [0, \infty)$, $D_g = (-\infty, \infty)$

$\Rightarrow$ $D_{f \circ g}$ is $(-\infty, \infty)$.

ex. $\boxed{f(x) = 2x^2}$, $\boxed{x \geq 2}$ and $\boxed{g(x) = \sqrt{x}}$, $x \geq 0$

find $(f \circ g)(x)$ and its domain.

Sol. $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = 2(\sqrt{x})^2 = 2x$

$x \in D_g = [0, \infty)$

$\sqrt{x} = g(x) \in D_f = [2, \infty)$

$\Rightarrow$ $x \in [0, \infty)$ and

$\sqrt{x} \in [2, \infty) \Rightarrow x \in [4, \infty)$

$\Rightarrow D_{f \circ g} = [4, \infty)$

كثيرات الحدود

1.2.2 Polynomial functions.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \quad , a_n \neq 0$$

leading coefficient

$n = 0, 1, 2, 3, \ldots$ معاملات

$a_0, a_1, a_2, \ldots, a_n$ Coefficients $\in \mathbb{R}$

$n = \text{degree}(f)$, Domain $= (-\infty, \infty)$

درجة

ثابت

$f(x) = c$ constant function $\Rightarrow \deg(f) = 0$

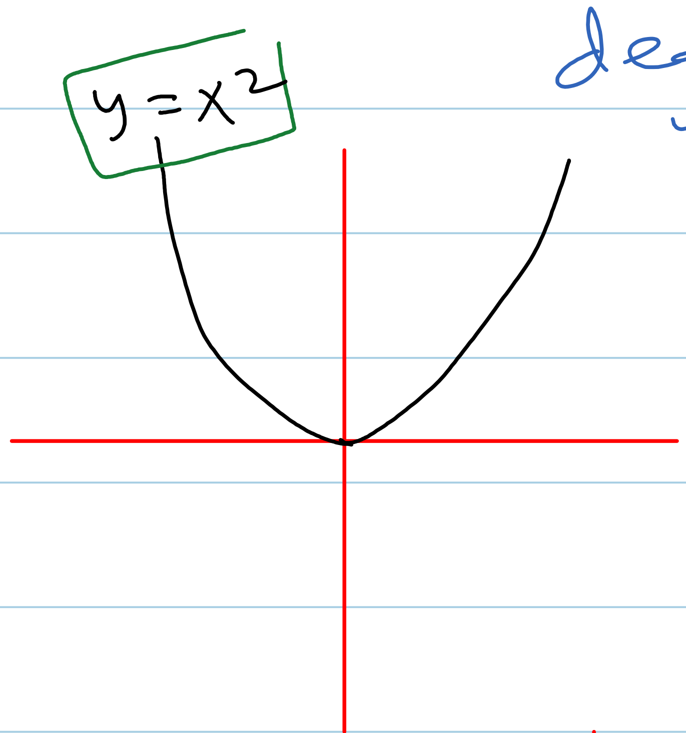
$f(x) = mx + b$ linear function $\deg(f) = 1$

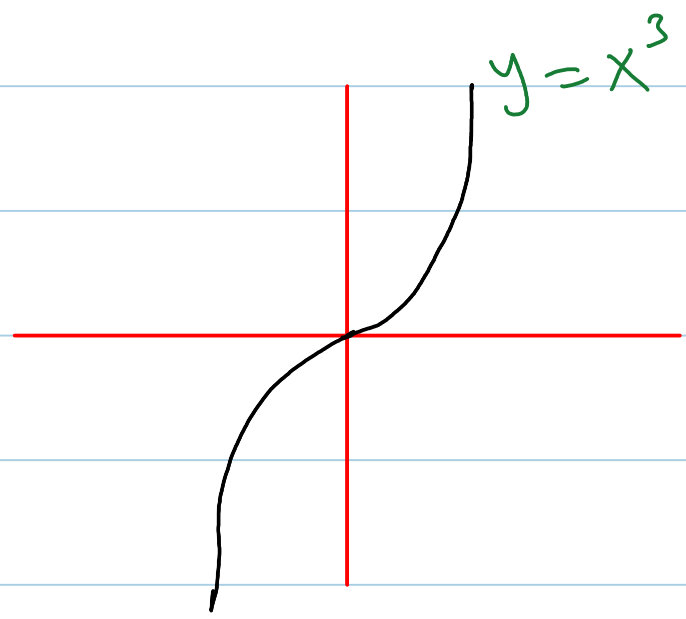
خطي

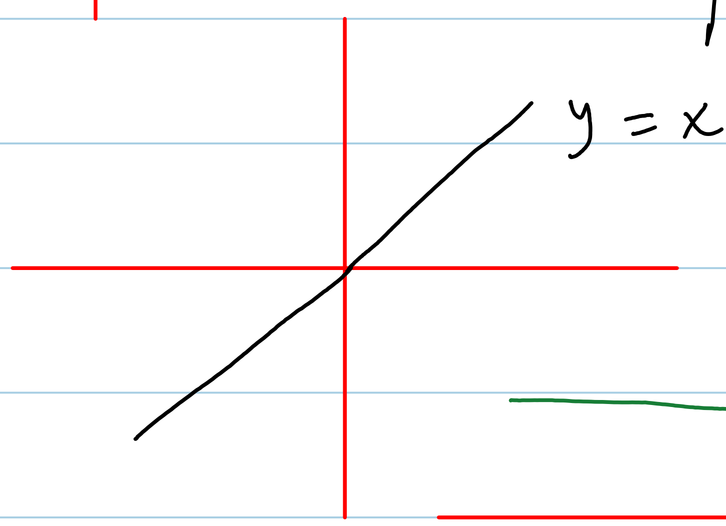
تربيعي

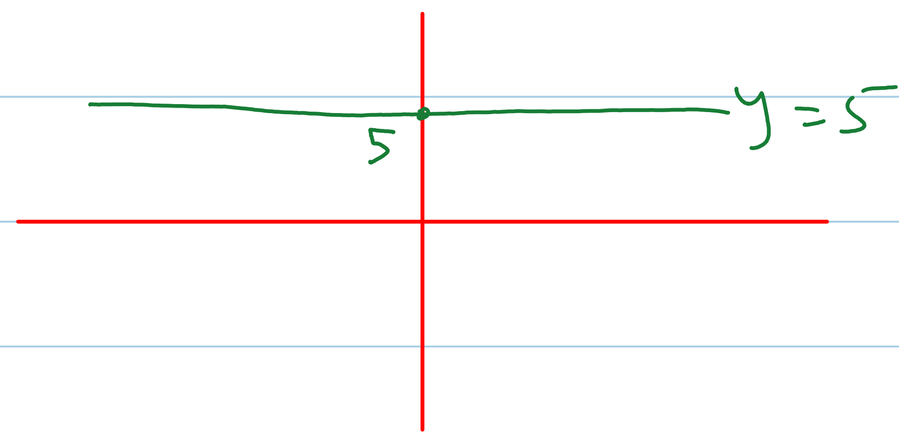
$f(x) = ax^2 + bx + c$ Quadratic function

$\deg(f) = 2.$









ex. $y = 2 - x^7$ is a poly of degree 7, leading coef $= -1$.

ex. $f(x) = 4x^3 - 3x + 1 \Rightarrow \deg(f) = 3$, leading $= 4$.

ex. $y = x^2 + x^{-1} + 4$ is not a poly.

ex. $y = x^{\frac{1}{2}} + x^4$ is not a poly.

$y = x^n$ is even function if n is even.
is odd function if n is odd

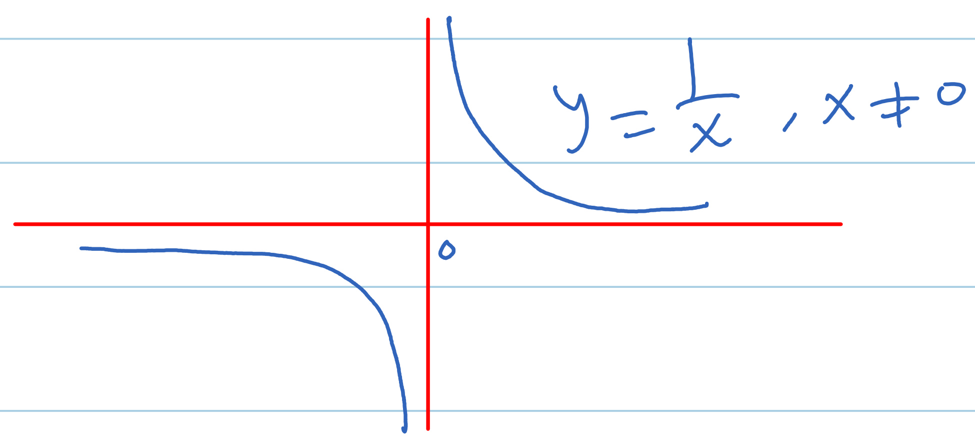
ex. $y = x^{2021}$ is odd, $y = x^{2020}$ is even.

1.2.3 Rational Functions الاقترانات النسبية

$$\boxed{f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0, \quad p, q \text{ polynomials}}$$

ex. $f(x) = \frac{5}{x}, x \neq 0$ is rational function

$y = \frac{x^2 + 2x - 1}{x - 3}, x \neq 3$ is rational function.



$y = \frac{\sqrt{x} + 1}{x - 1}, x \neq 1$ $x^{\frac{1}{2}}$
is not rational function.

1.2.4 Power Functions

power function $f(x) = x^r$, r is any real number

ex. $y = x^{\frac{1}{3}}$, $x \in \mathbb{R}$ is power function (not poly).

$y = x^{-\frac{5}{2}}$, $x \in \mathbb{R}$ is power function. (not poly.)

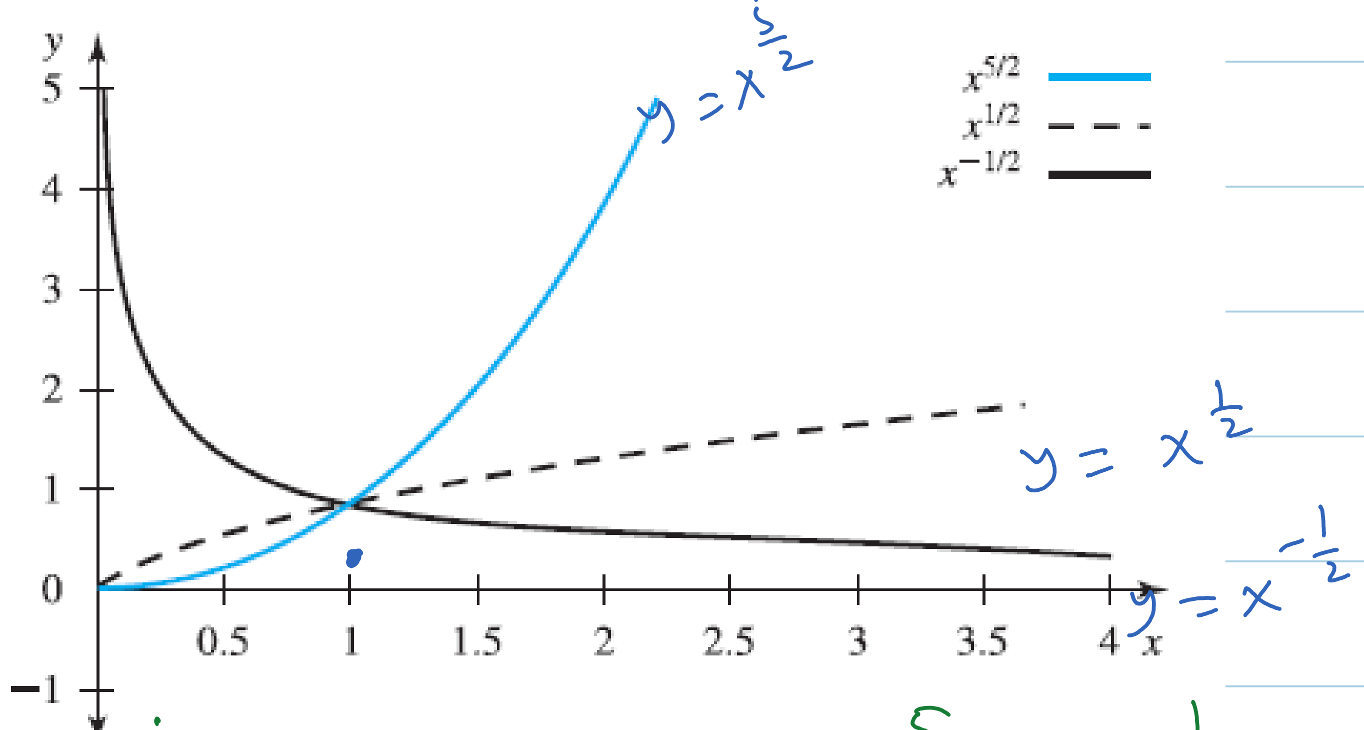
$y = x^3$, $x \in \mathbb{R}$ is power function (Poly.)

$y = \dfrac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$ is power function.

ex. Is $x^{\frac{5}{2}} > x^{\frac{1}{2}}$?? $x \in \mathbb{R}^+$

Sol. if $x > 1 \Rightarrow x^{\frac{5}{2}} > x^{\frac{1}{2}}$

$0 < x < 1 \Rightarrow x^{\frac{5}{2}} < x^{\frac{1}{2}}$.



if $0 < x < 1 \Rightarrow x^{\frac{5}{2}} < x^{\frac{1}{2}} < x^{-\frac{1}{2}}$

$x > 1 \Rightarrow x^{5/2} > x^{\frac{1}{2}} > x^{-\frac{1}{2}}$

1.2 Continue الأسي الاقتران

1.2.5 Exponential Functions

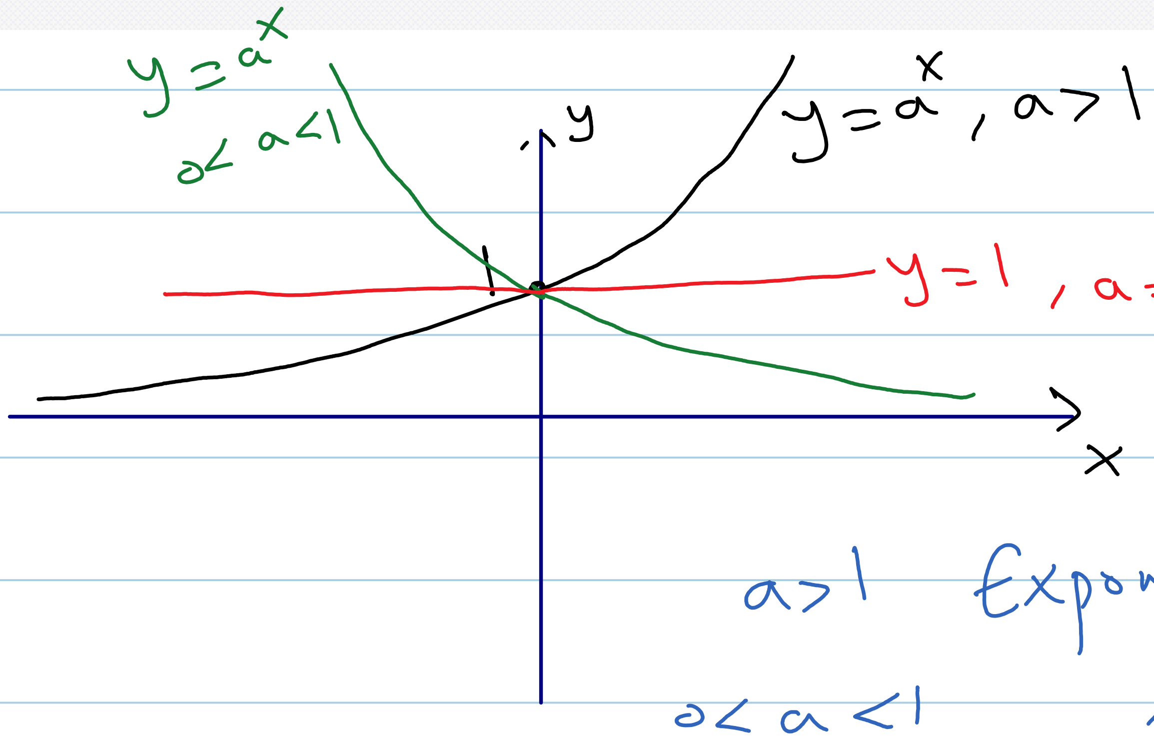
Definition The function f is an **exponential** function with base a if

$$\boxed{f(x) = a^x}$$

أساس

where a is a positive constant other than 1. The largest possible domain of f is $\mathbf{R}$.

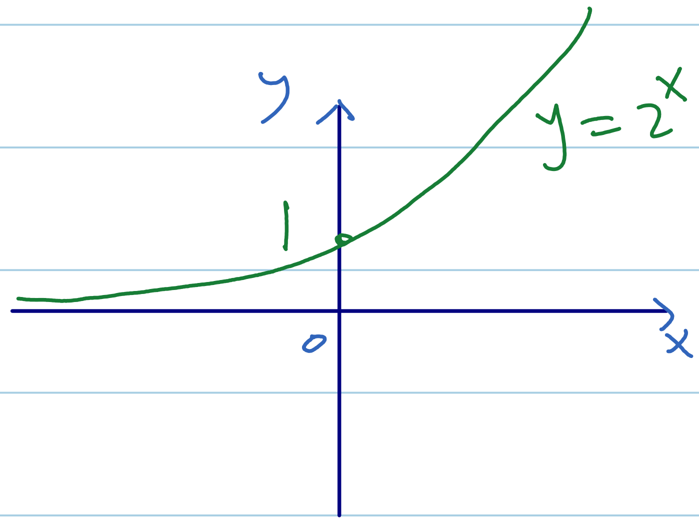
$D = (-\infty, \infty)$
$R = (0, \infty)$



$\boxed{x = 0} \Rightarrow y = a^0 = 1$

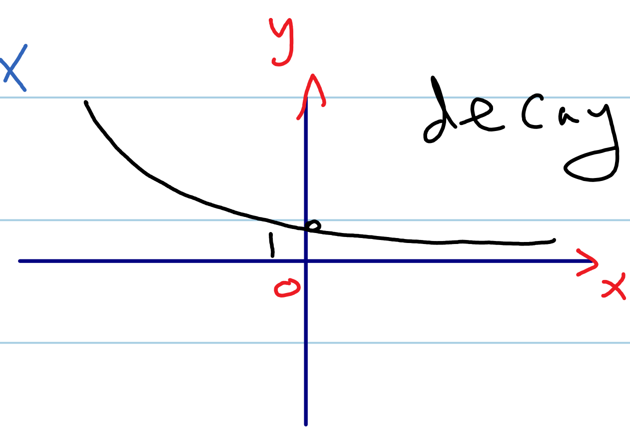
$a > 1$ Exponential growth
$0 < a < 1$ // decay

ex. $y = 2^x$
$a = 2 > 1$



growth
Domain $= (-\infty, \infty)$
Range $= (0, \infty)$.

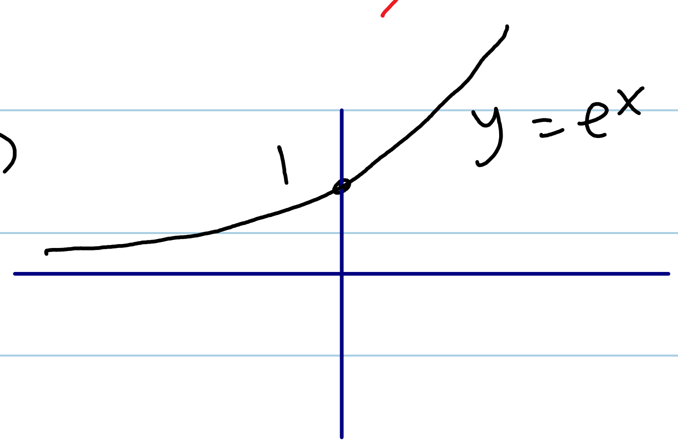
ex. $y = \left(\frac{1}{3}\right)^x$



decay

ex. $y = e^x$, $e = 2.718\ldots\ldots > 1$

or $y = \exp(x)$



exponential growth.

تذكرة

Rules $a^r a^s = a^{r+s}$

$$\frac{a^r}{a^s} = a^{r-s}$$

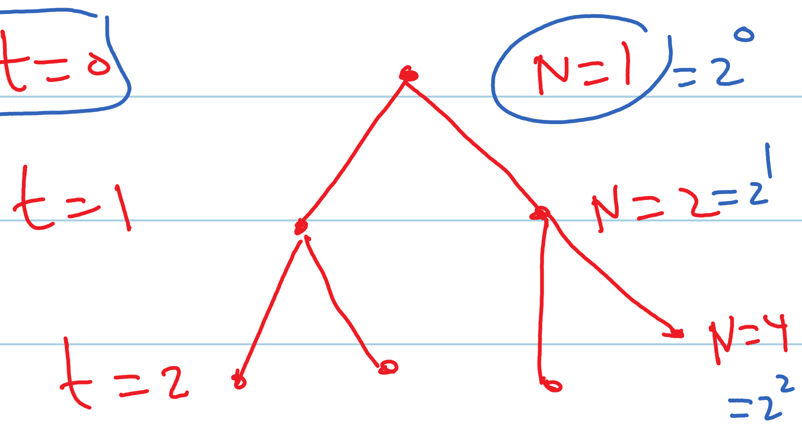
$$a^{-r} = \frac{1}{a^r}$$

$$(a^r)^s = a^{rs}$$

$$a^{\frac{k}{n}} = \sqrt[n]{a^k}$$

Ex. **Exponential Growth** Bacteria reproduce asexually by cellular fission, in which the parent cell splits into two daughter cells after duplication of the genetic material. This division may happen as often as every 20 minutes; under ideal conditions, a bacterial colony can double in size in that time.

$$\boxed{N(t) = 2^t, \quad t = 0, 1, 2, \ldots}$$

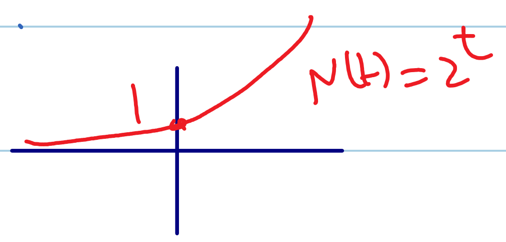


$N(t)$ = the population at time t.

$N(0) = 2^0 = 1 = N_0$

$N(1) = 2^1 = 2$

$N(2) = 2^2 = 4.$



exponential growth

If $\boxed{N(0) = N_0} \Rightarrow \boxed{N(t) = N_0 \cdot 2^t, \; t = 0, 1, 2, \ldots}$

ex. $N(0) = 40$

$t = 0 \longrightarrow N = 40$

$t = 1 \longrightarrow N(1) = 2^1 \cdot 40 = 80$

$t = 2 \longrightarrow N(2) = 2^2 \cdot 40$
$= 160.$

$$N(t+1) = N_0 \cdot 2^{t+1}$$
$$= \boxed{N_0 \cdot 2^t} \cdot 2^1$$

$N(t+1) = 2N(t)$

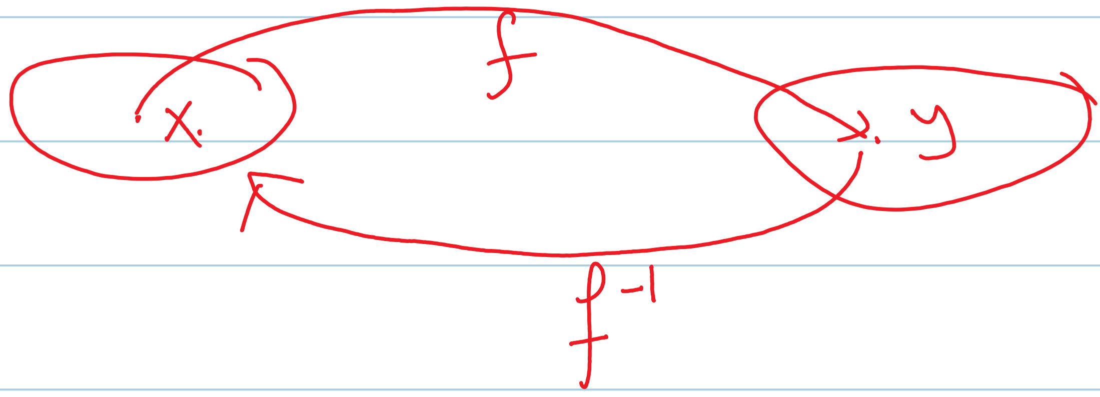
الاقترانات العكسية

1.2.6 Inverse Functions

f is $\underline{1-1}$ (one to one) function

if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $\boxed{f(x_1) = f(x_2) \longrightarrow x_1 = x_2}$

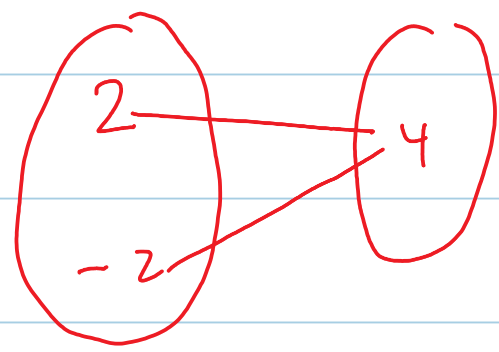


ex. $y = x^3$ is $1-1$ since

Sol. let $\boxed{f(x_1) = f(x_2)} \Rightarrow x_1^3 = x_2^3 \Rightarrow \boxed{x_1 = x_2}$

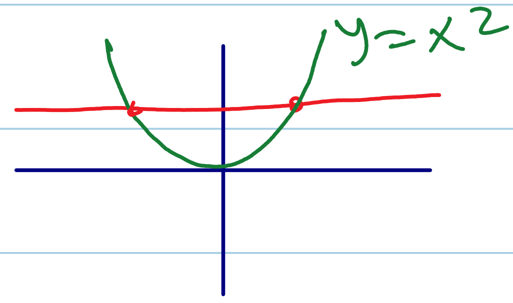
ex. $y = x^2$ is Not $1-1$ since

$(2)^2 = (-2)^2$ but $2 \neq -2$

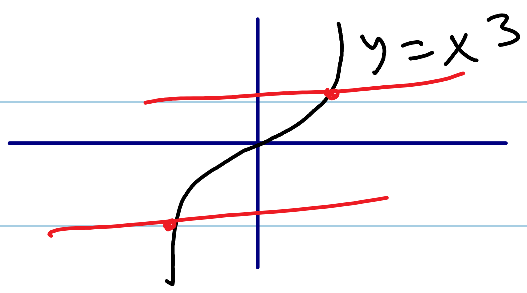


$2 \neq -2$ but $f(2) = f(-2)$

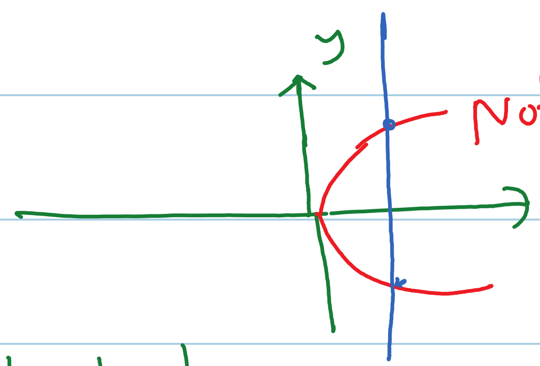
Horizontal line test



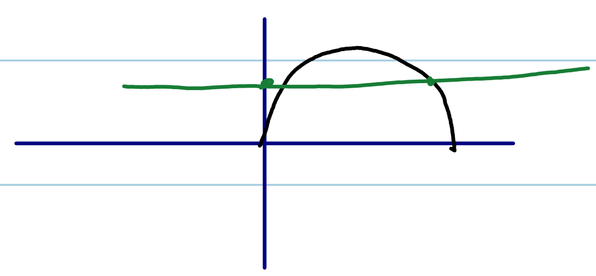
is not $1-1$



is $1-1$



Not function.



is Not $1-1$.

$D = A$, Range $= f(A)$

$f : A \longrightarrow B$ is 1-1

$f^{-1} : \underbrace{f(A)}_{D} \longrightarrow \underbrace{A}_{\text{Range}}$

$x \xrightarrow{f} y$, $y \xrightarrow{f^{-1}} x$

ex. Find the inverse of

$$f(x) = x^3 + 1, \quad x \geq 0.$$

$f(x) = y \Leftrightarrow x = f^{-1}(y)$,

$y \in f(A)$.

Sol. ① 1-1 $\boxed{f(x_1) = f(x_2)} \Rightarrow x_1^3 + \cancel{1} = x_2^3 + \cancel{1}$
$\quad \cancel{-1} \qquad \cancel{-1}$

$\Rightarrow x_1^3 = x_2^3 \Rightarrow \boxed{x_1 = x_2}$

$\therefore f$ is 1-1

② $\boxed{y = x^3 + 1 \,, \; x \geq 0}$

$x^3 \geq 0 \Rightarrow \underbrace{1 + x^3}_{y} \geq 1$

$x \geq 0 \Rightarrow y \geq 1$

Domain $(f) = [0, \infty)$

Range $(f) = [1, \infty)$

Solve for x : $x^3 = y - 1$

$(x^3)^{\frac{1}{3}} = (y-1)^{\frac{1}{3}}$

$\boxed{x = \sqrt[3]{y-1}}$, $y \geq 1$

$x \leftrightarrow y$: $y = \sqrt[3]{x-1}$, $x \geq 1$

$f^{-1}(x) = \sqrt[3]{x-1}$, $x \geq 1$

Dom $(f^{-1}) = [1, \infty) =$ Range (f).

$\Rightarrow f(x) = x^3 + 1$, $f^{-1}(x) = \sqrt[3]{x-1}$

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = (f^{-1}(x))^3 + 1 = (\sqrt[3]{x-1})^3 + 1$ ✓
$= x - 1 + 1 = x$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = \sqrt[3]{f(x) - 1} = \sqrt[3]{x^3 + \cancel{1} - \cancel{1}} = \sqrt[3]{x^3} = x.$

In general, $f: A \to B$ has inverse,

$$(f^{-1} \circ f)(x) = x \;, \quad \forall x \in A$$

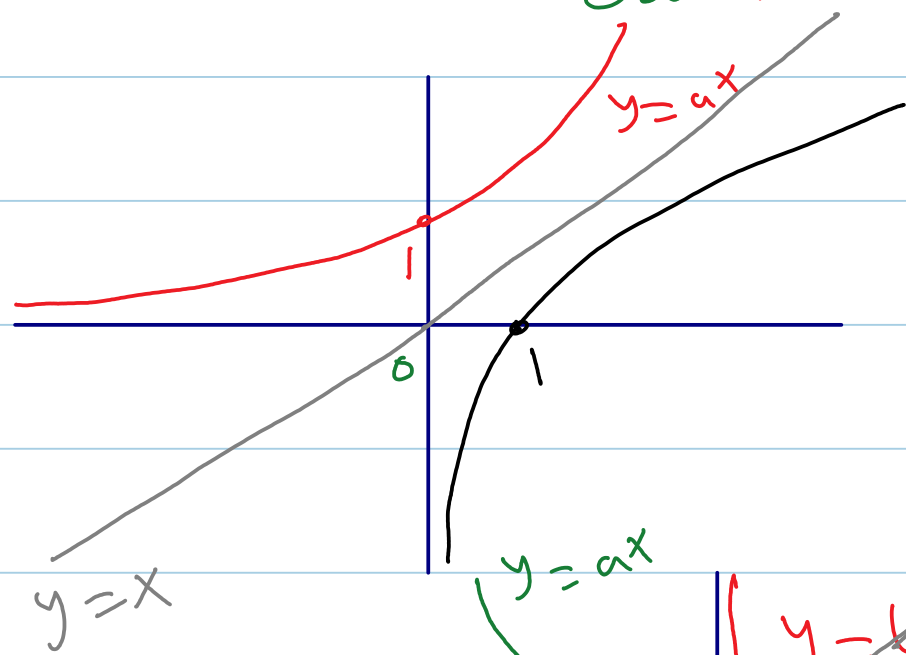
$$(f \circ f^{-1})(x) = x \;, \quad \forall x \in f(A)$$

Warning: $\underbrace{f^{-1}}_{\text{the inverse of } f} \neq \dfrac{1}{f}$

1.2.7 Logarithmic Functions

Domain $= (0, \infty)$, Range $= (-\infty, \infty)$

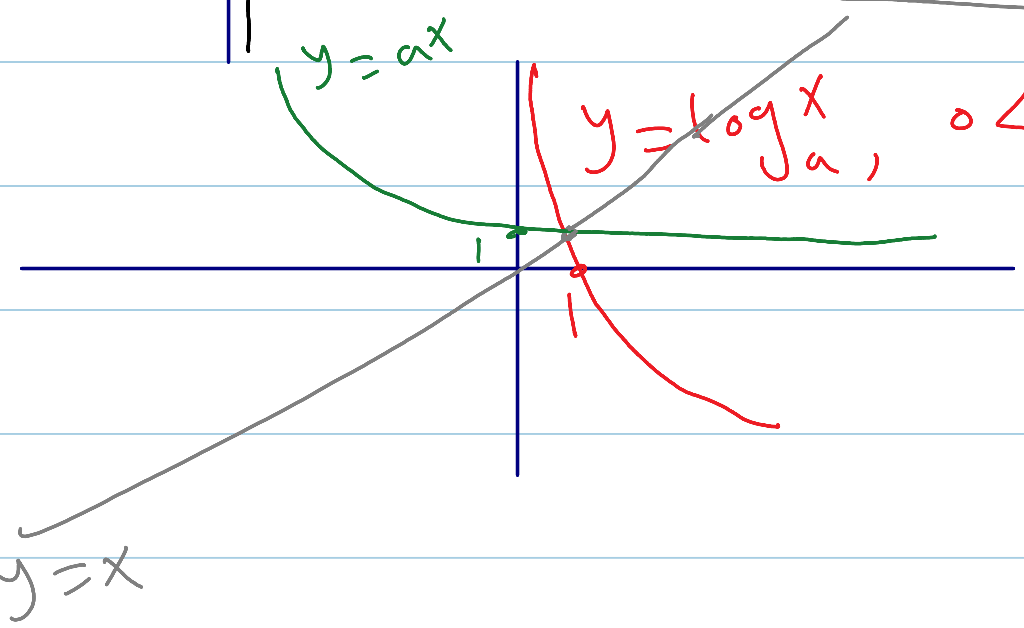
$f(x) = \log_a x \;, \; a \neq 1$



$y = \log_a x \;, \; a > 1$

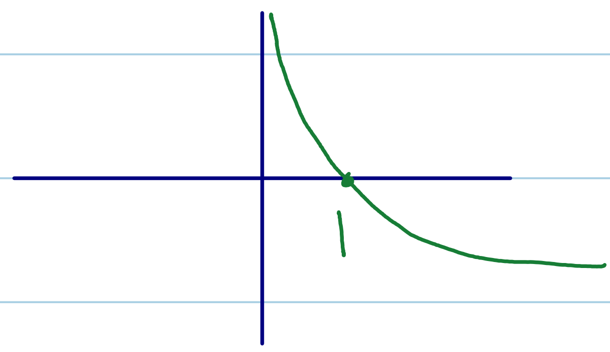
$\boxed{f(x) = a^x}$

$\boxed{f^{-1}(x) = \log_a x}$



$y = \log_a x \;, \quad 0 < a < 1$

$\underline{\underline{\text{ex.}}}$ $y = \log_{\frac{1}{2}} x$



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$f(x) = e^x \Rightarrow f^{-1}(x) = \log_e x = \ln x$

Rules.

$$\boxed{a^{\log_a x} = x, \quad x > 0}$$ $e^{\ln x} = x, x > 0$

$$\boxed{\log_a(a^x) = x, \quad x \in \mathbb{R}}$$ $\ln e^x = x, x \in \mathbb{R}$

$$\boxed{\log_a(st) = \log_a s + \log_a t}$$

$$\boxed{\log_a\left(\frac{s}{t}\right) = \log_a s - \log_a t}$$

$$\boxed{\log_a(s^r) = r \log_a s}$$

$$\boxed{a^x = e^{\ln(a^x)} = e^{x \ln a}}$$

ex. $2^x = e^{x \ln 2}$, $4^x = e^{x \ln 4}$

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

ex. $\log_5 7 = \frac{\ln 7}{\ln 5}$

$\boxed{\log} \to \log_{10}$

$$\boxed{y = \log_a x \Longleftrightarrow a^y = x}$$

$\boxed{\ln} \to \log_e$

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ex. Simplify

(a) $\log_2 8(x-2) = \log_2 8 + \log_2 (x-2)$

$= \frac{\ln 8}{\ln 2} + \log_2 (x-2)$

$= \frac{\ln 2^3}{\ln 2} + \log_2 (x-2)$

$= \frac{3\cancel{\ln 2}}{\cancel{\ln 2}} + \log_2 (x-2)$

$= 3 + \log_2 (x-2)$

(b) $\log_3 9^x = x\log_3 9 = x\log_3 3^2 = x \cdot 2 = 2x.$

(c) $\ln\left(e^{3x^2+1}\right) = 3x^2+1$ since $\boxed{\ln e^A = A}$

ex. write in terms of base e $2.7....$

(a) $2^x = e^{x\ln 2}$

(b) $10^{x^2+1} = e^{(x^2+1)\ln 10}$

(c) $\log_3 x = \frac{\ln x}{\ln 3}$

(d) $\log_2 (3x-1) = \frac{\ln(3x-1)}{\ln 2}$

الاقترانات المثلثية

1.2.8 Trigonometric Functions

دوري

Definition A function $f(x)$ is **periodic** if there is a positive constant a such that
$$f(x+a) = f(x)$$
for all x in the domain of f. If a is the smallest number with this property, we call it the **period** of $f(x)$.

$\sin 30 = \frac{1}{2}$
$\sin 30 + 360$

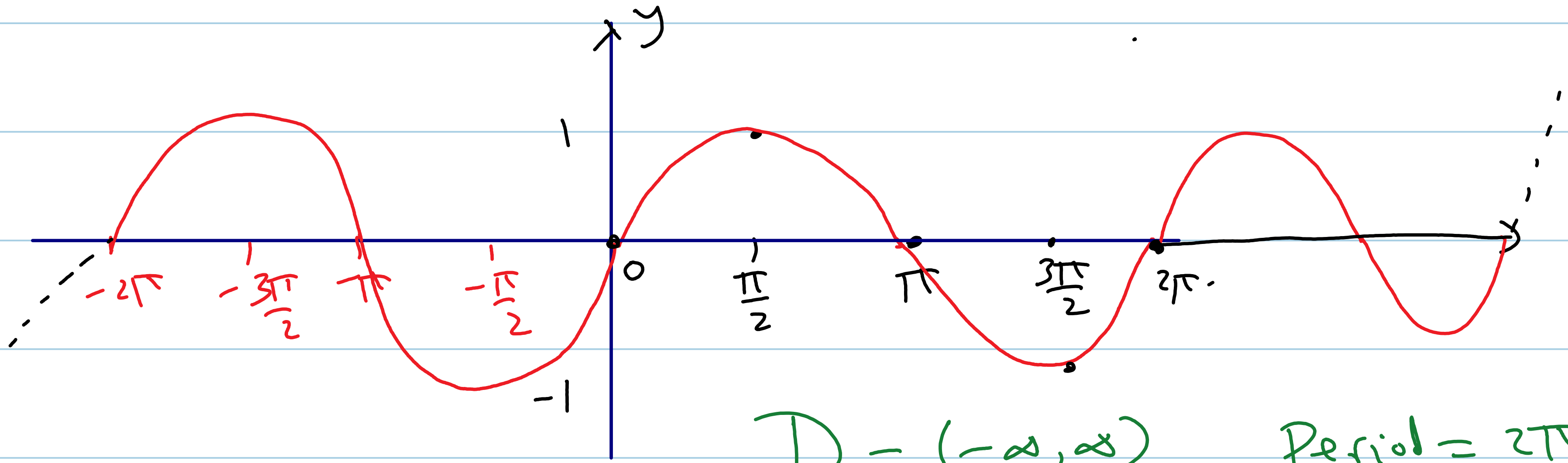
ex. $\sin(x+2\pi) = \sin x \Rightarrow p = 2\pi$ period.
$\cos(x+2\pi) = \cos x \Rightarrow p = \text{period} = 2\pi.$

1. جيب Sine $y = \sin x$ جا = جاس
2. جتا Cosine $y = \cos x$ جتا = جتاس
3. ظل tangent $y = \tan x$ ظا = ظاس
4. قاطع Secant $y = \sec x$
5. قتا Cosecant $y = \csc x$
6. ظتا Cotangent $y = \cot x$

$\tan(x+\pi) = \tan x \Rightarrow \text{period} = \pi.$
$\cot(x+\pi) = \cot x \Rightarrow \text{period} = \pi.$

$\sec(x+2\pi) = \sec x \Rightarrow \text{period} = 2\pi.$
$\csc(x+2\pi) = \csc x \Rightarrow \text{period} = 2\pi$

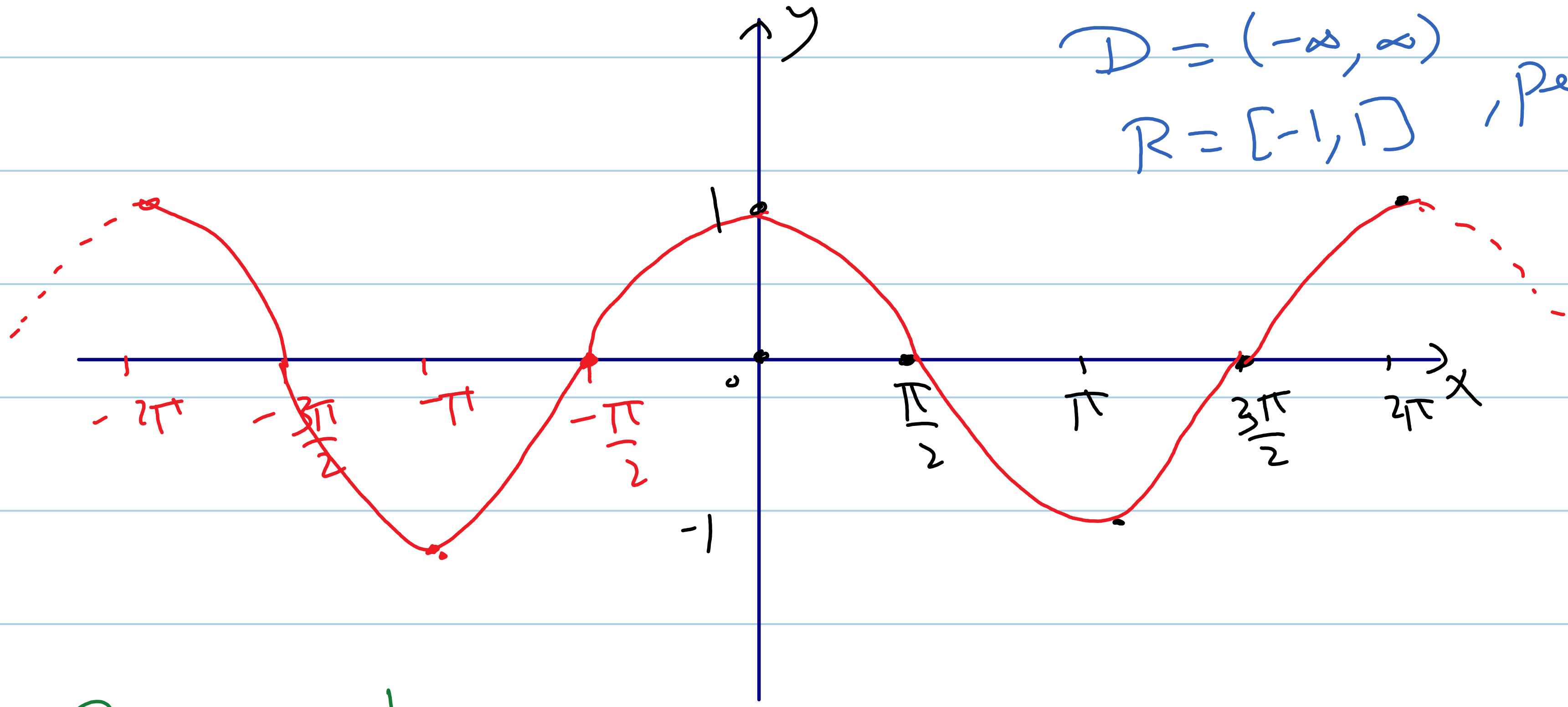
Graphs (1) $y = \sin x$ $x = 0 \Rightarrow \sin 0 = 0$



$D = (-\infty, \infty)$ Period $= 2\pi.$
$R = [-1, 1]$

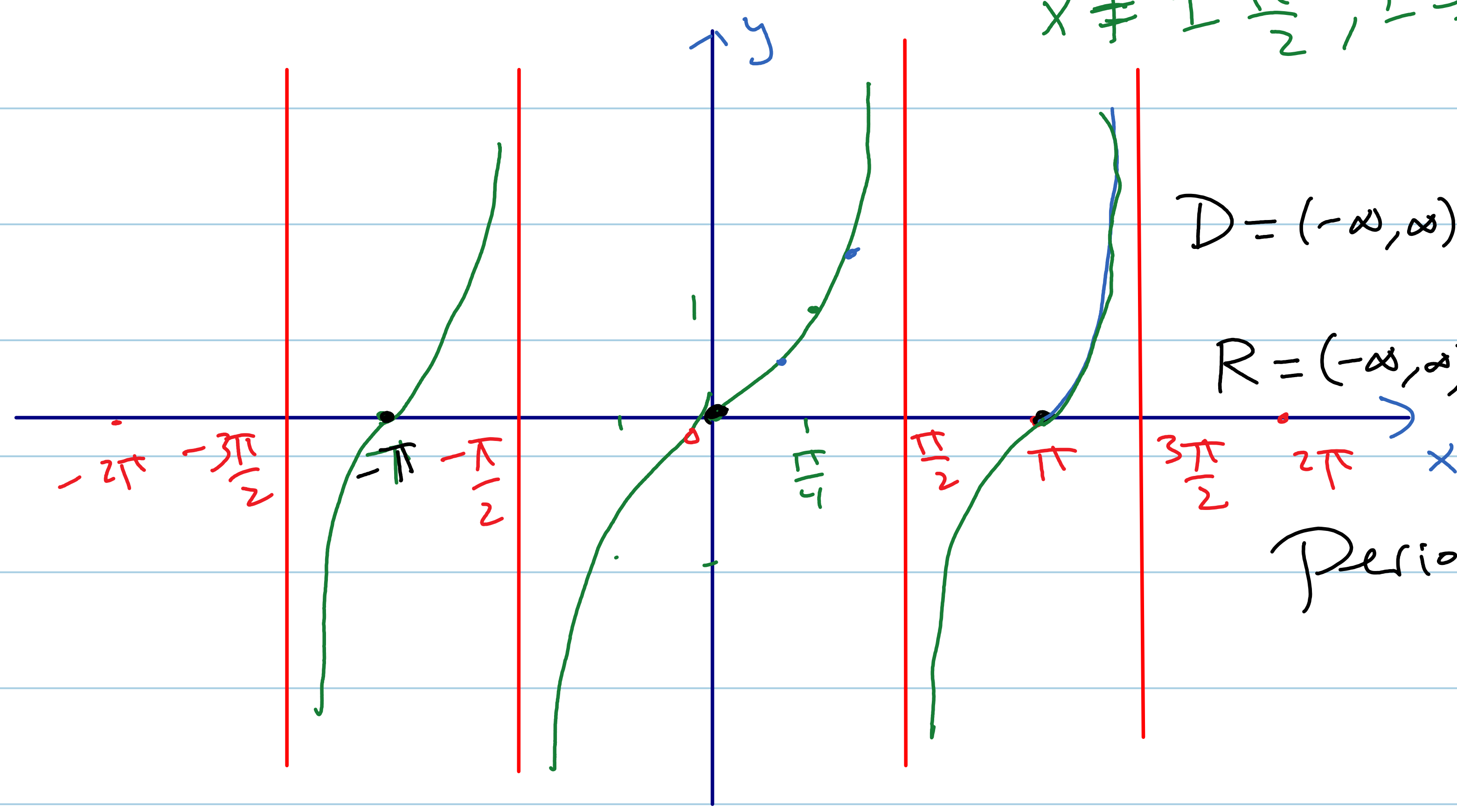
(2) $y = \cos x$

$D = (-\infty, \infty)$
$R = [-1, 1]$, Period $= 2\pi$.



(3) $y = \tan x = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0$

$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$



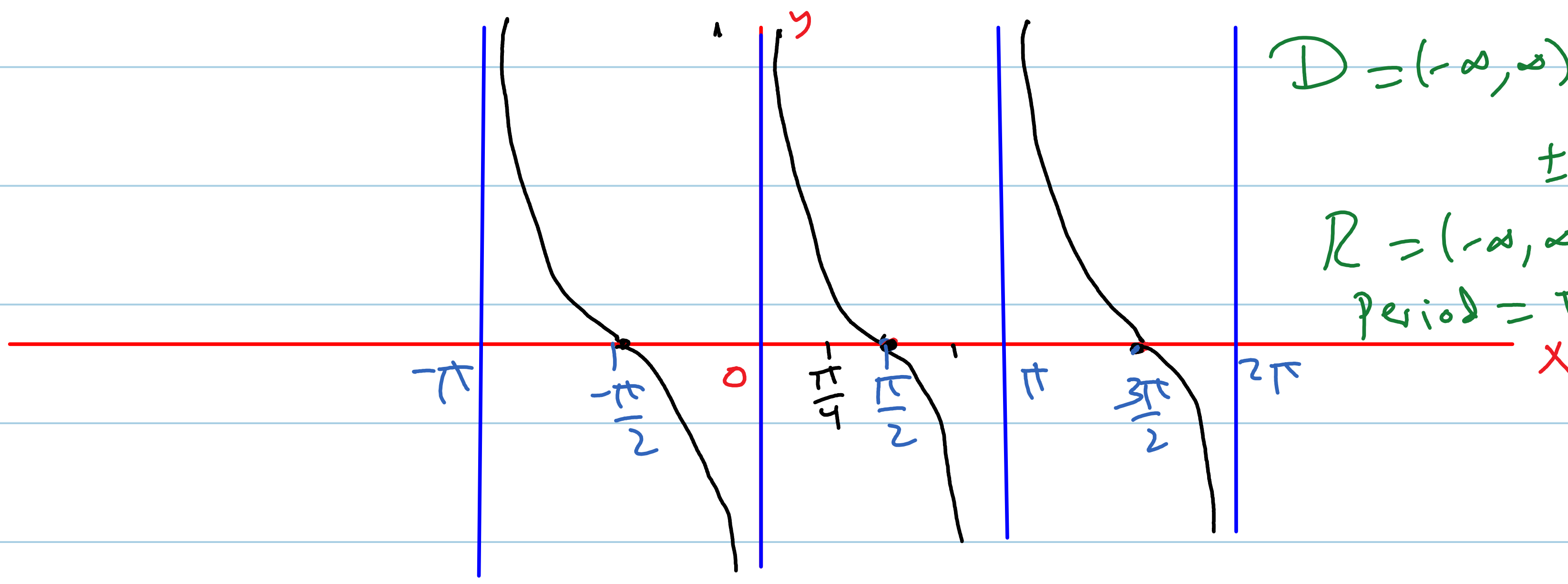
$D = (-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\}$

$R = (-\infty, \infty)$

Period $= \pi$.

(4) $y = \cot x = \dfrac{\cos x}{\sin x}$, $\sin x \neq 0$

$x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$



$D = (-\infty, \infty) \setminus \{0, \pm\pi, \pm 2\pi, \ldots\}$

$R = (-\infty, \infty)$

Period $= \pi$.

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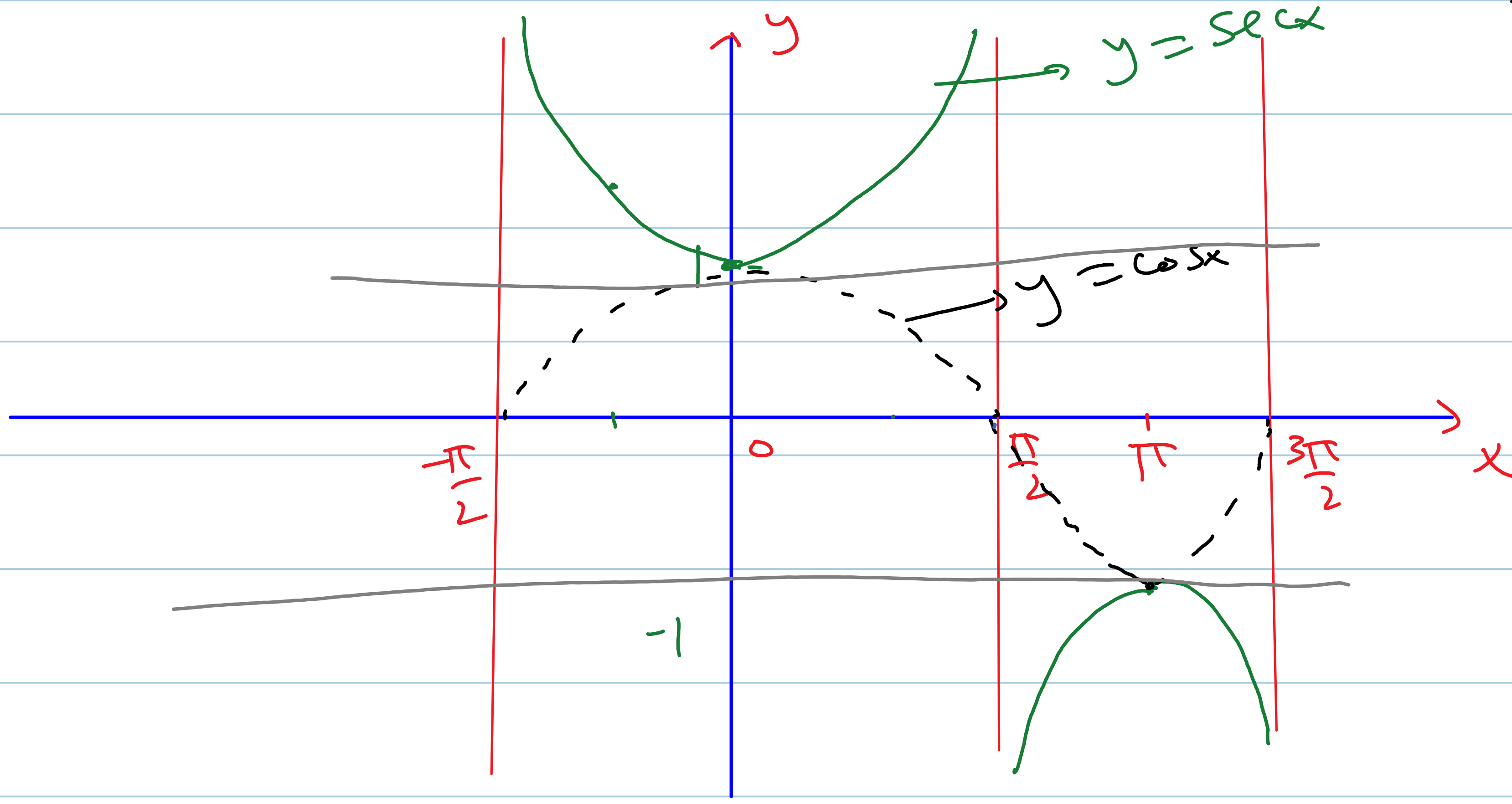
(5) $y = \sec x = \dfrac{1}{\cos x}$, $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

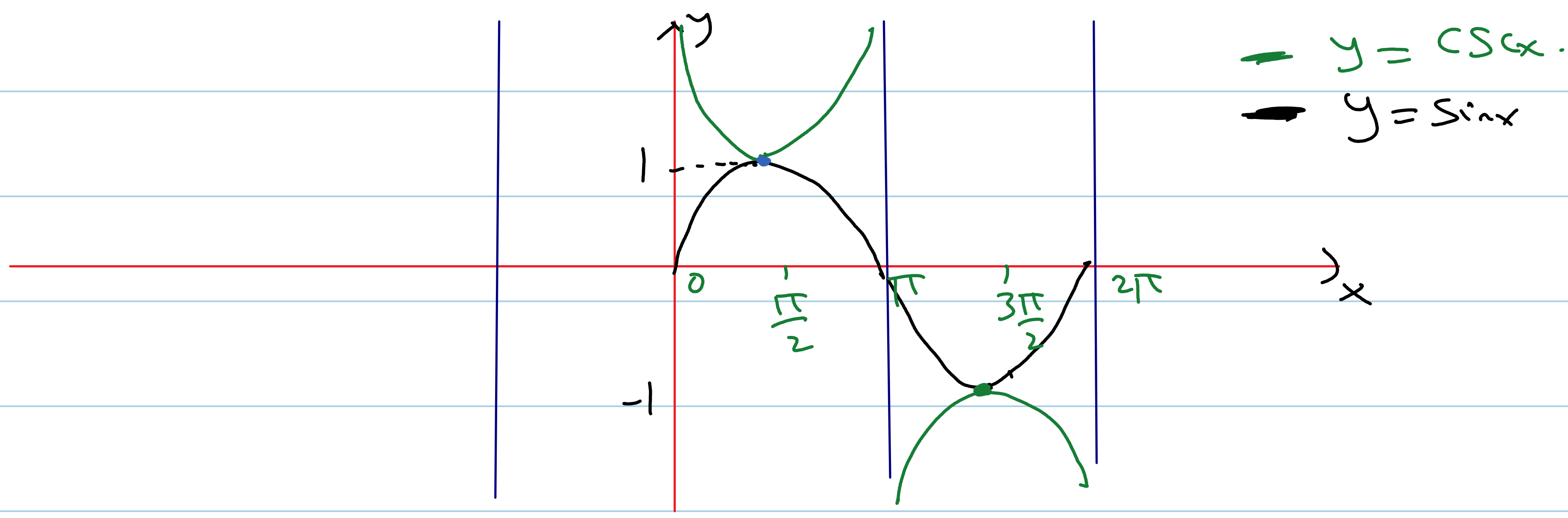


$D = (-\infty, \infty) \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$

$R = (-\infty, -1] \cup [1, \infty)$

period $= 2\pi$.

(6) $y = \csc x = \dfrac{1}{\sin x}$, $x \neq 0, \pm\pi, \pm 2\pi, \ldots$



$D = (-\infty, \infty) \setminus \{0, \pm\pi, \pm 2\pi, \ldots\}$

$R = (-\infty, -1] \cup [1, \infty)$.

Period $= 2\pi$.

(1) $\sin x$ $k=1$, $a=1$

$$\boxed{y = a\sin(kx),\quad x\in\mathbb{R}}$$ OR $y = a\cos(kx)$, $x\in\mathbb{R}$

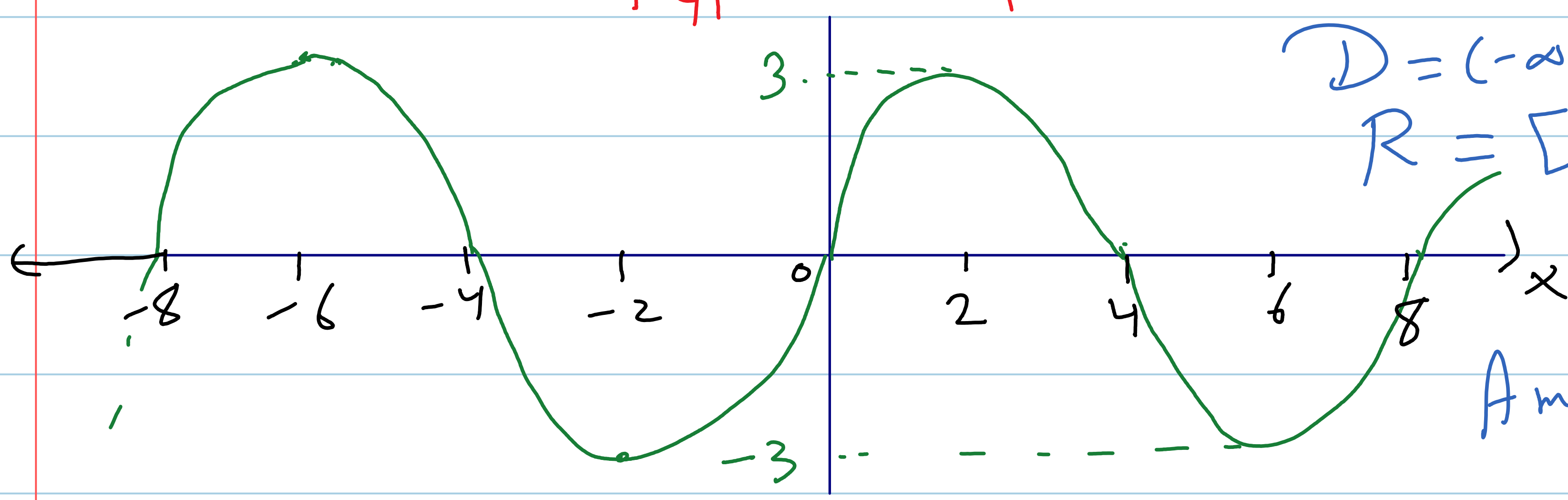
$$\text{Period} = \frac{2\pi}{|k|}$$

$$\text{Amplitude} = |a|$$

$$\text{Range} = [-|a|, |a|].$$

ex. sketch $y = \boxed{3}\sin\left(\frac{\pi}{4}x\right)$ $a=3$, $k=\frac{\pi}{4}$

Sol Period $= \dfrac{2\pi}{\left|\frac{\pi}{4}\right|} = \dfrac{2\pi}{\frac{\pi}{4}} = 2\pi\cdot\dfrac{4}{\pi} = 8$.



$D = (-\infty, \infty)$

$R = [-3, 3]$

Amplitude $= |3| = 3$

Period $= 8$

100. Let

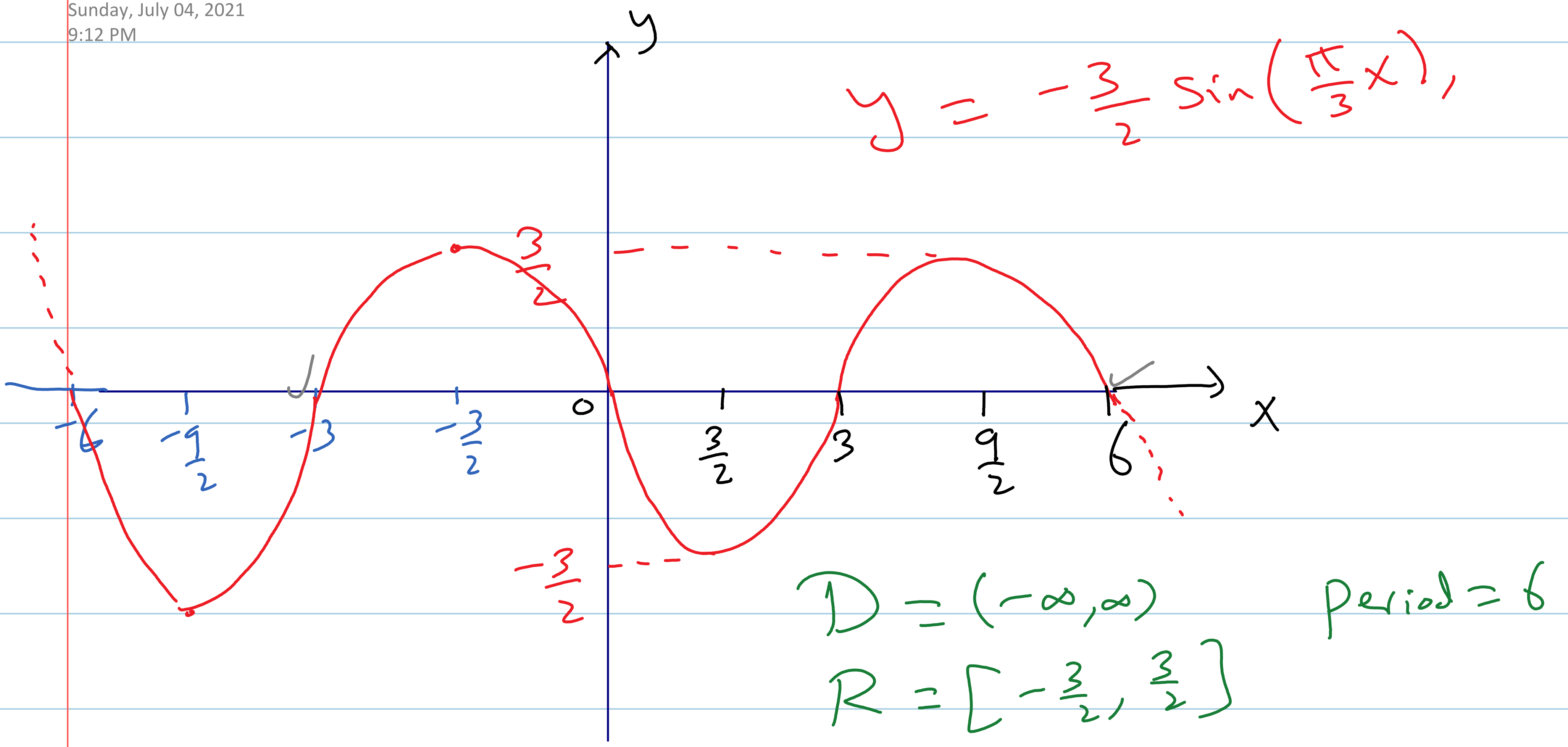
$$f(x) = -\frac{3}{2}\sin\left(\frac{\pi}{3}x\right), \quad x\in\mathbf{R}$$

Find the amplitude and the period of $f(x)$.

Sol. $\boxed{a = -\frac{3}{2}}$, $\boxed{k = \frac{\pi}{3}}$

$$\text{Amplitude} = \left|-\frac{3}{2}\right| = \frac{3}{2}$$

$$\text{Period} = \frac{2\pi}{|k|} = \frac{2\pi}{\left|\frac{\pi}{3}\right|} = 2\pi\cdot\frac{3}{\pi} = 6$$



104. Let

$$f(x) = -\frac{2}{3}\cos\left(\frac{3x}{\pi}\right), \quad x \in \mathbf{R}$$

Find the amplitude and the period of $f(x)$. , Range.

Sol. $a = -\frac{2}{3}$, $k = \frac{3}{\pi}$

$$\text{Period} = \frac{2\pi}{|k|} = \frac{2\pi}{\left|\frac{3}{\pi}\right|} = 2\pi \cdot \frac{\pi}{3} = \frac{2\pi^2}{3}$$

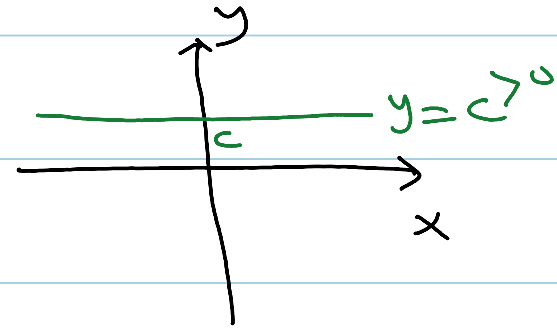
$$\text{Amplitude} = |a| = \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$\text{Range} = \left[-\frac{2}{3}, \frac{2}{3}\right]$$, Domain $= (-\infty, \infty)$.

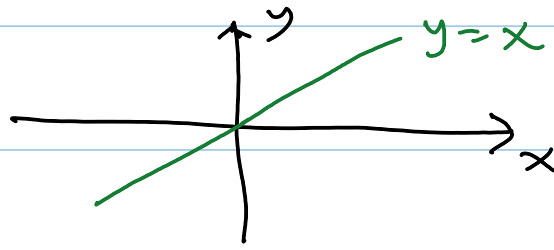
الرسم

1.3 Graphing

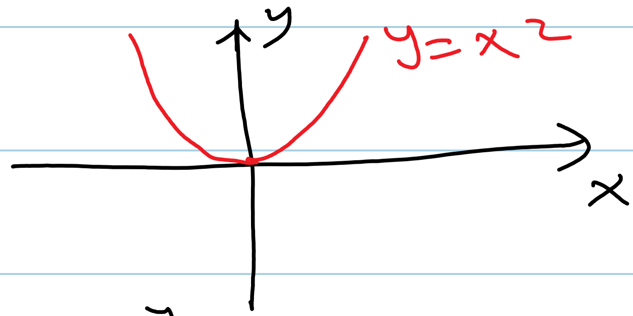
Recall, $y = c$ "constant"



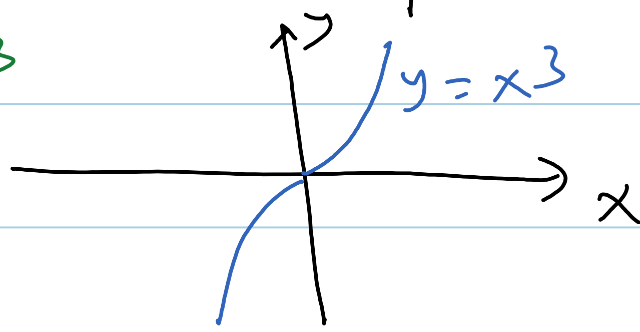
$y = x$



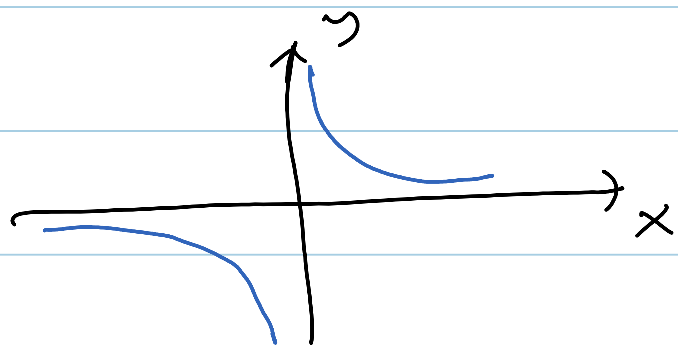
$y = x^2$



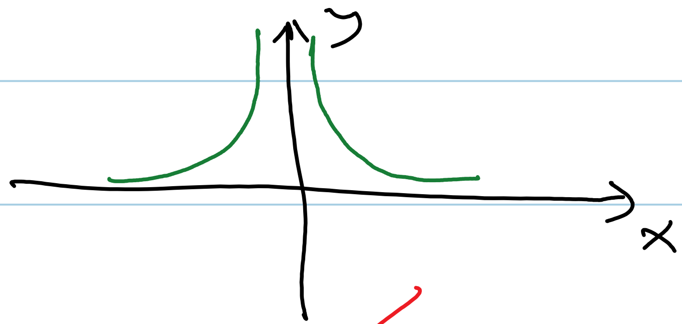
$y = x^3$



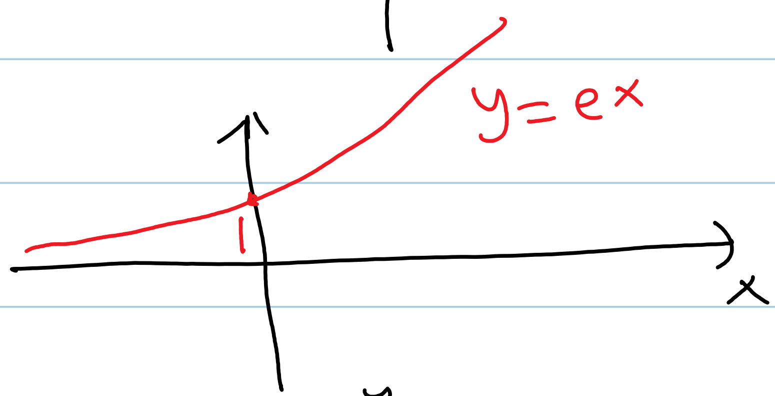
$y = \frac{1}{x}$



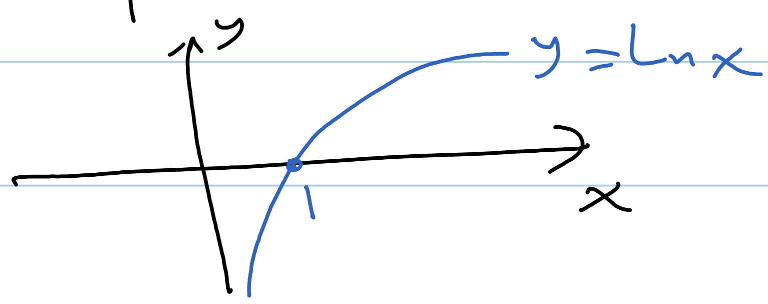
$y = \frac{1}{x^2}$



$y = e^x$



$y = \ln x$



$y = \sin x, \cos x, \tan x, \cot x, \sec x, \csc x$
see 1.2.8

1.3.1 Graphing and Basic Transformations of Functions

Definition The graph of

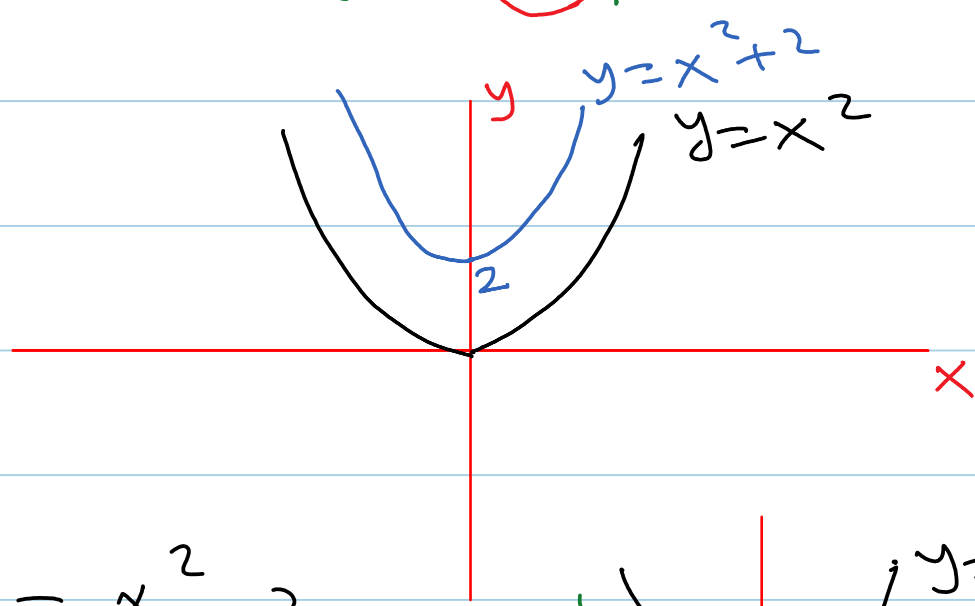
$$\boxed{y = f(x) + a}$$

عمودي

is a **vertical translation** of the graph of $y = f(x)$. If $a > 0$, the graph of $y = f(x)$ is shifted up a units; if $a < 0$, the graph of $y = f(x)$ is shifted down $|a|$ units.

للأعلى   للأسفل

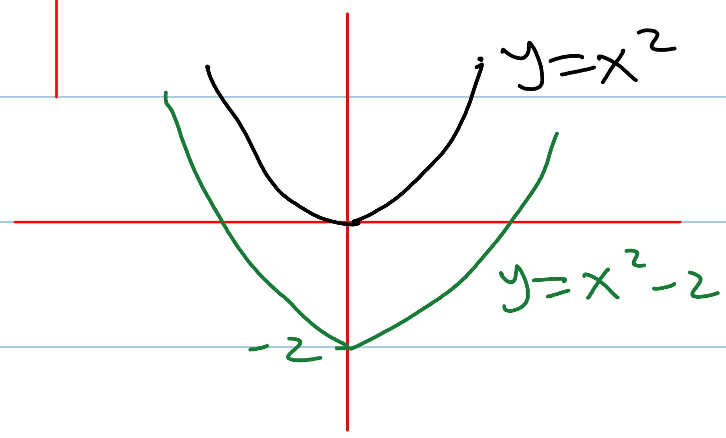
ex. Sketch $y = x^2 + 2$



— $y = x^2$

— $y = x^2 + 2$

ex. $y = x^2 - 2$



Definition The graph of

أفقي

$$y = f(x - c)$$

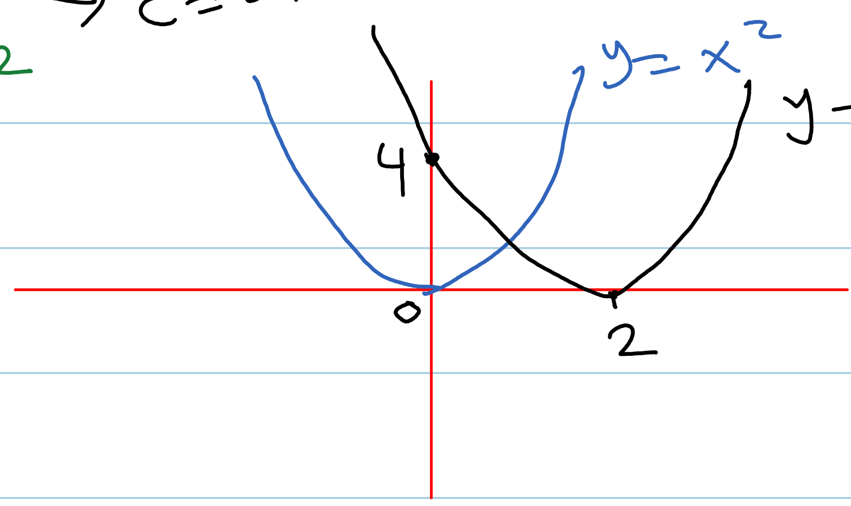
is a **horizontal translation** of the graph of $y = f(x)$. If $c > 0$, the graph of $y = f(x)$ is shifted c units to the right; if $c < 0$, the graph of $y = f(x)$ is shifted $|c|$ units to the left.

لليسار   لليمين

ex. $y = (x-2)^2 \rightarrow c = 2 > 0$ shift 2 units to the right.

$x - 2 = 0$

$x = 2$



$x = 0$

$y = (0-2)^2$

$= 4$

Sunday, July 04, 2021
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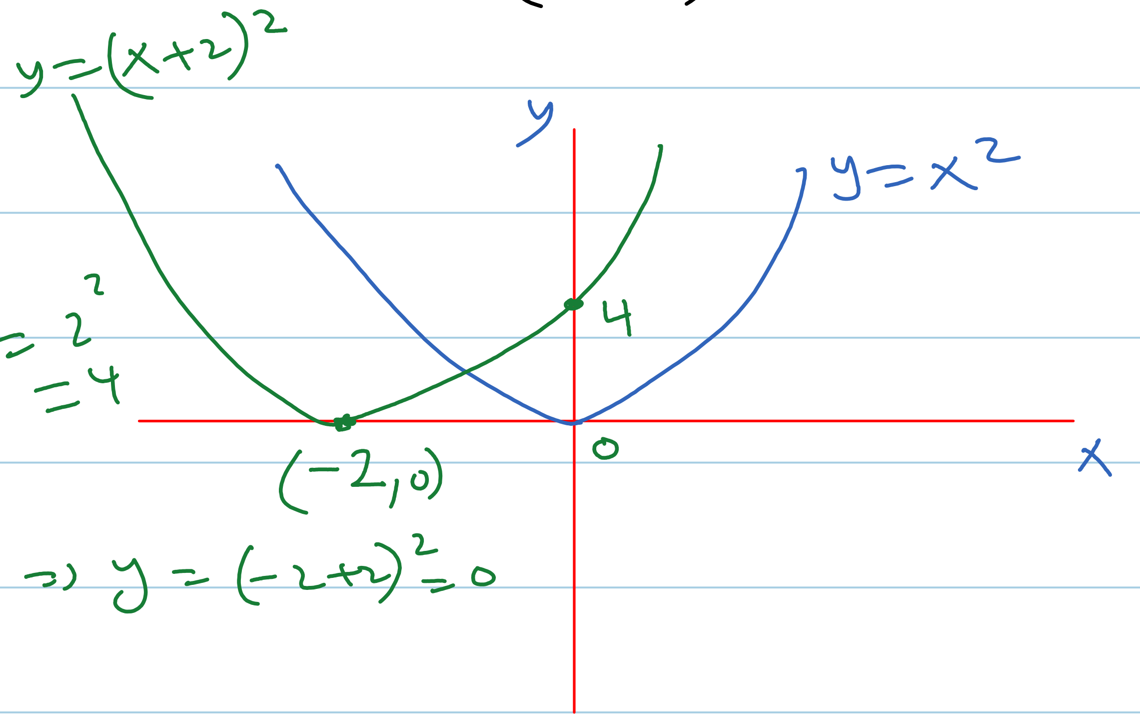
ex. $y=(x+2)^2$ shift 2 units to the left.



$x=0 \Rightarrow y=2^2=4$

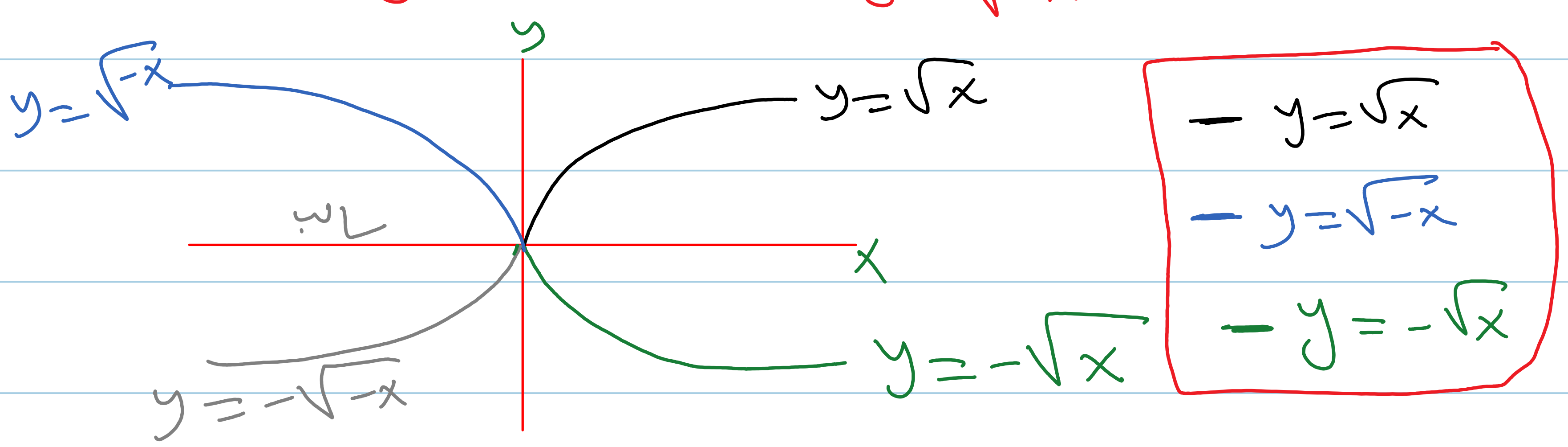
$x=-2 \Rightarrow y=(-2+2)^2=0$

انعكاس

Reflection $y=f(x) \Rightarrow y=-f(x)$ reflection about x-axis

$y=f(-x)$ reflection about y-axis.

ex. sketch $y=-\sqrt{x}$ and $y=\sqrt{-x}$



$y=-\sqrt{-x}$

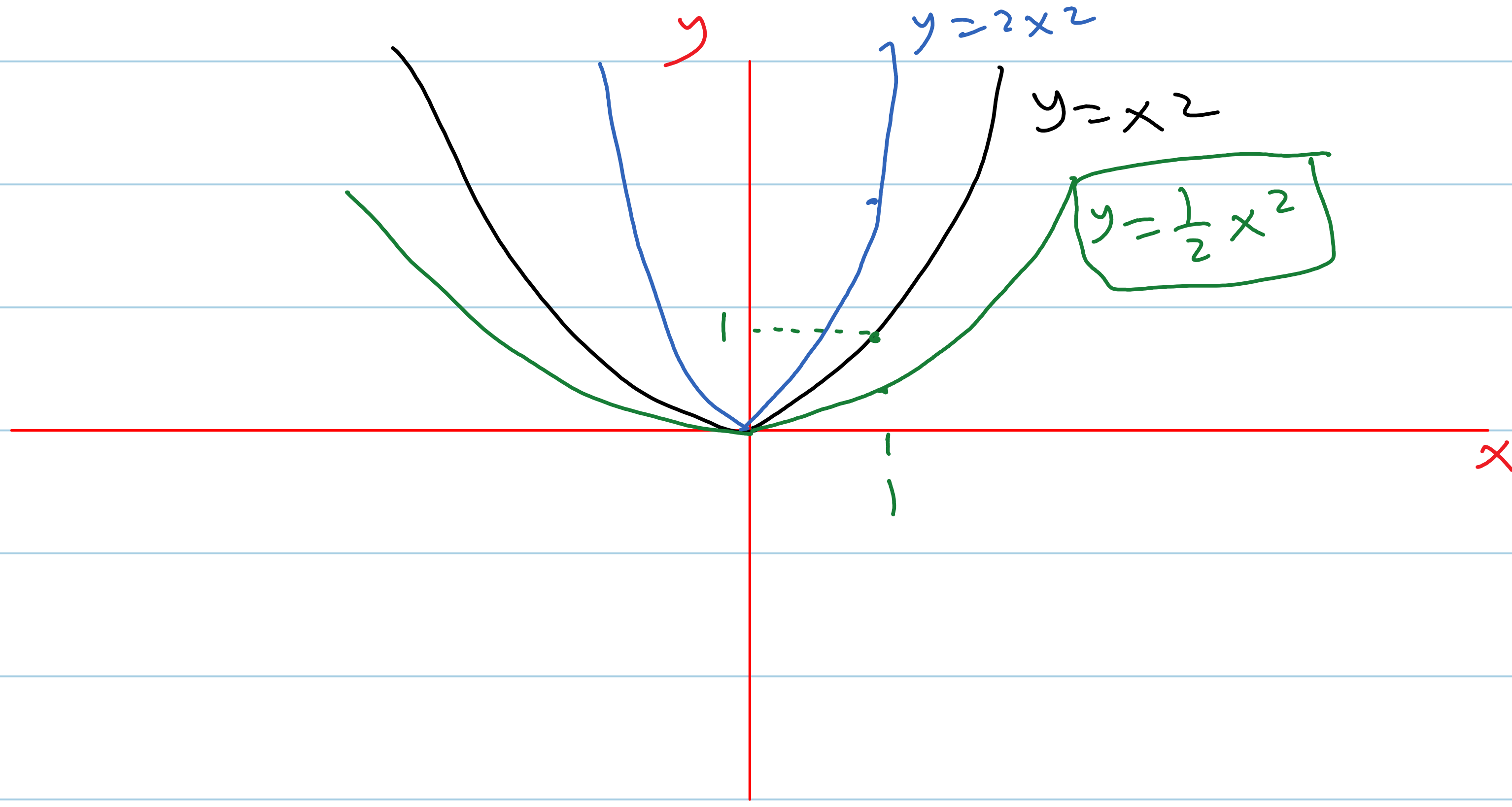
$y=a\,f(x)$, $a>0$ يضغط

$0<a<1$ Compresses the graph of f.

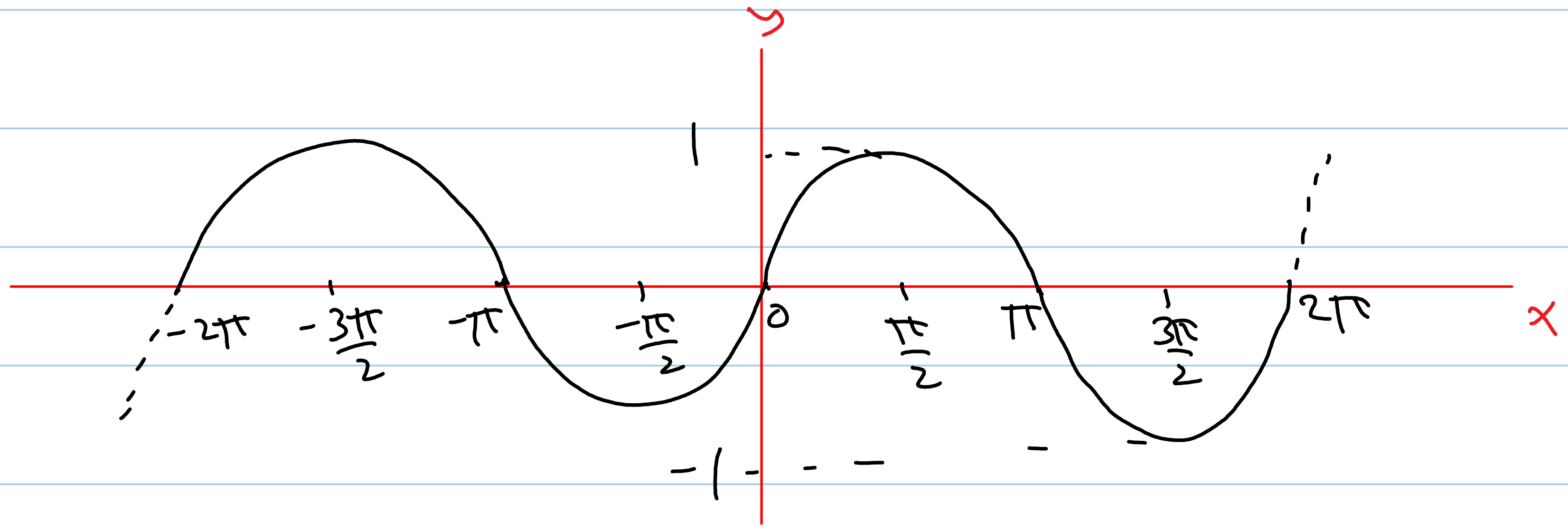
$a>1$ stretches " " "

يتمدد

ex. Sketch $\boxed{y = 2x^2}$, $y = \frac{1}{2}x^2$



Ex. Sketch $\boxed{y = 2\sin\left(x - \frac{\pi}{4}\right)}$, $x \in \mathbb{R}$.



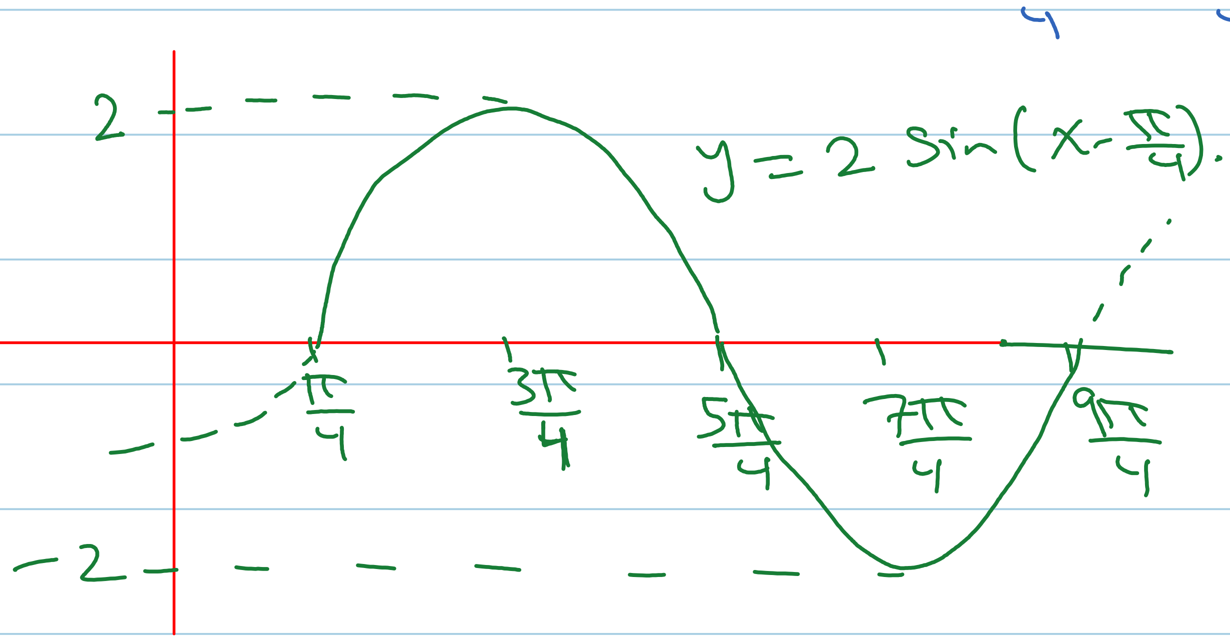
$0 \longrightarrow \frac{\pi}{4}$

$\frac{\pi}{2} \longrightarrow \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

$\pi \longrightarrow \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\frac{3\pi}{2} \longrightarrow \frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}$

$2\pi \longrightarrow 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$



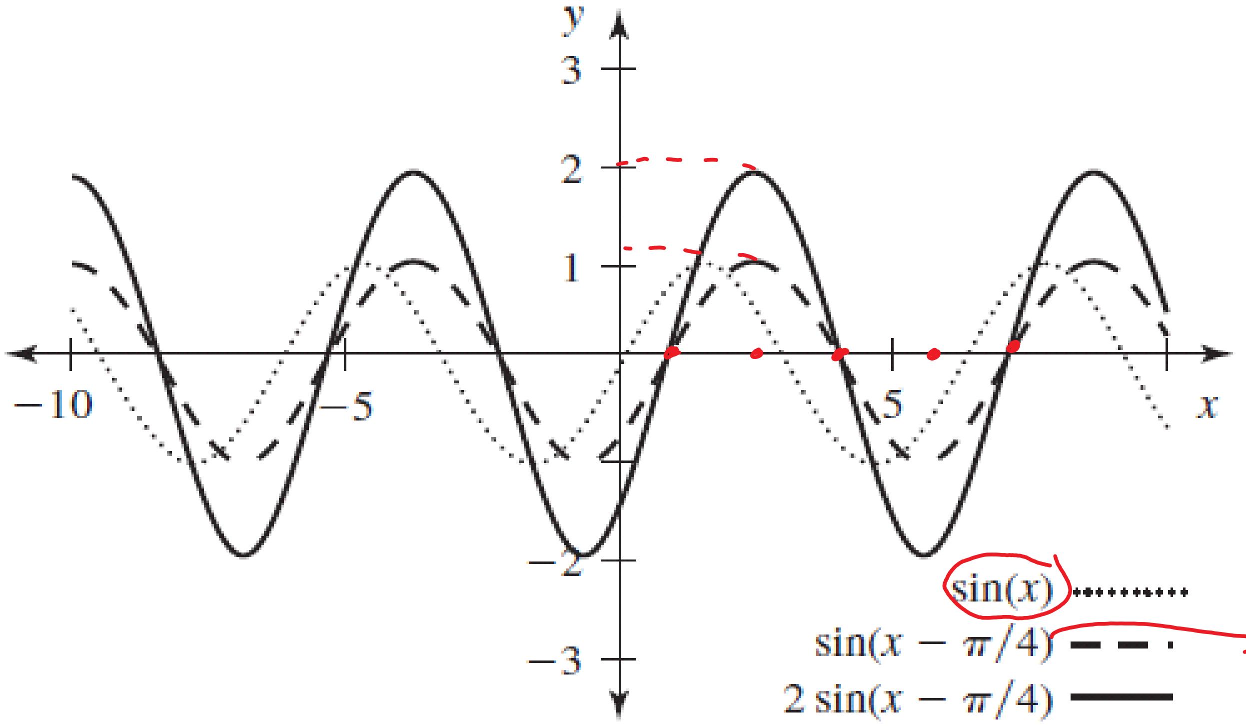


Figure 1.46 The graphs of $y = \sin x$, $y = \sin(x - \frac{\pi}{4})$, and $y = 2\sin(x - \frac{\pi}{4})$.

shift $\frac{\pi}{4}$ units to the right

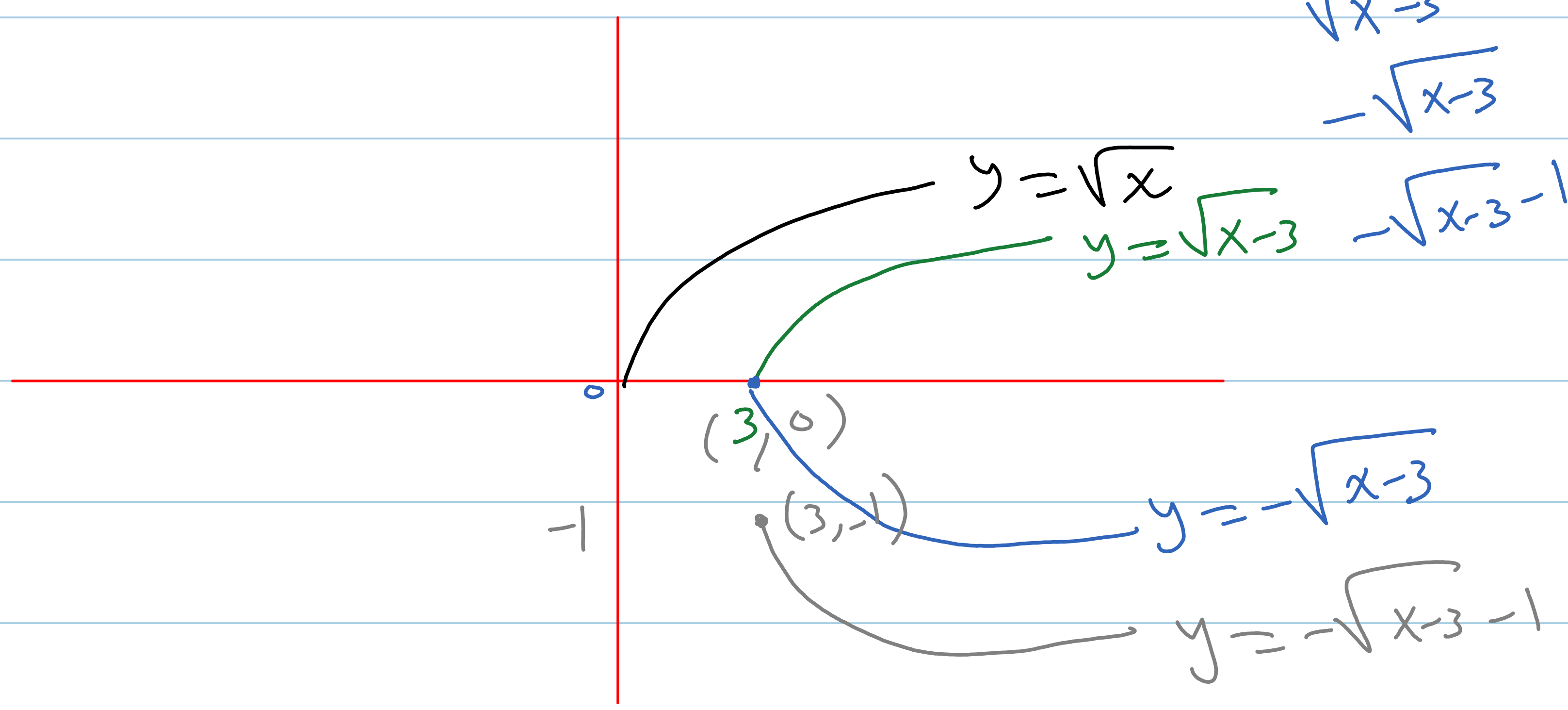
EXAMPLE 2 Explain how the graph of

$$\boxed{y = -\sqrt{x-3} - 1, \quad x \geq 3}$$

can be obtained from the graph of $y = \sqrt{x}$, $x \geq 0$.

$\sqrt{x}$ ✓
$\sqrt{x-3}$
$-\sqrt{x-3}$
$-\sqrt{x-3}-1$

Sol.



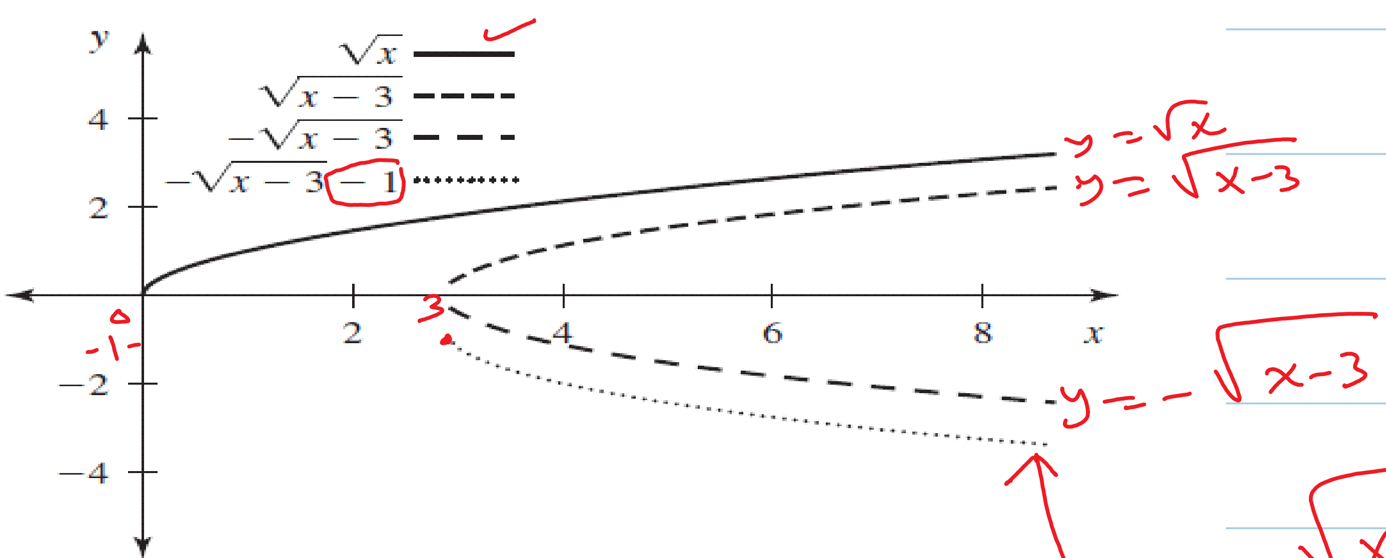


Figure 1.47 The graphs for Example 2.

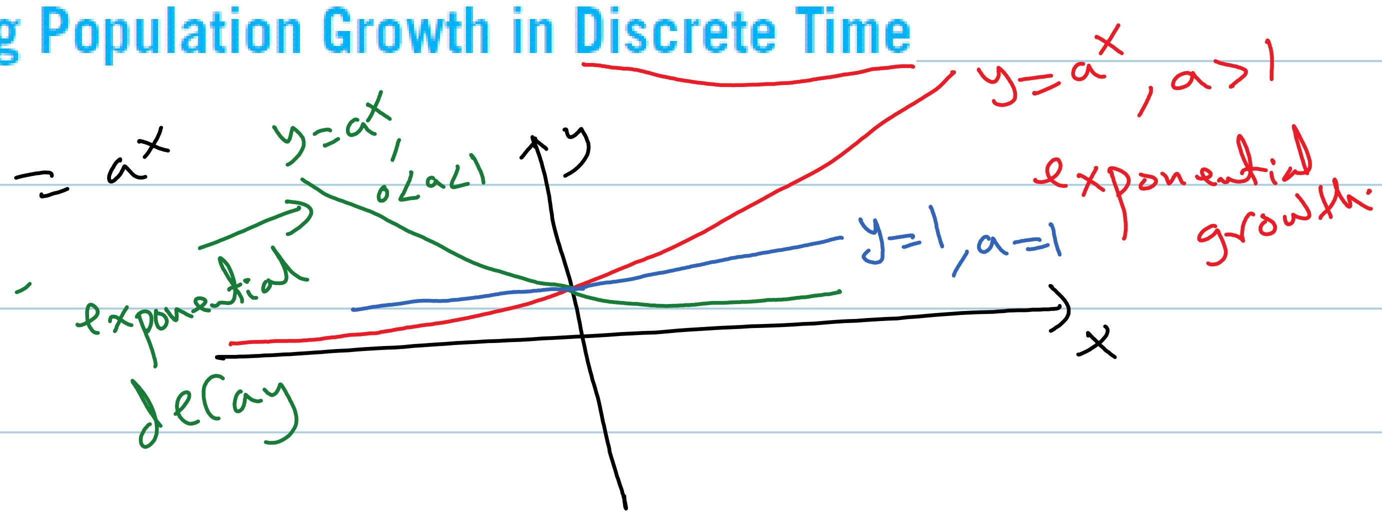
The End of ch1 (1.1, 1.2, 1.3).

2 Discrete-Time Models, Sequences, and Difference Equations

2.1 Exponential Growth and Decay

2.1.1 Modeling Population Growth in Discrete Time

Recall, $y = a^x$



Imagine that we observe bacteria that divide every 20 minutes and that, at the start of the experiment, there was one bacterium. How will the number of bacteria change over time? We call the time when we started the observation time 0. At time 0, there is one bacterium. After 20 minutes, the bacterium splits in two, so there are two bacteria at time 20. Twenty minutes later, each of the bacteria splits again, resulting in four bacteria at time 40, and so on (Figure 2.1).

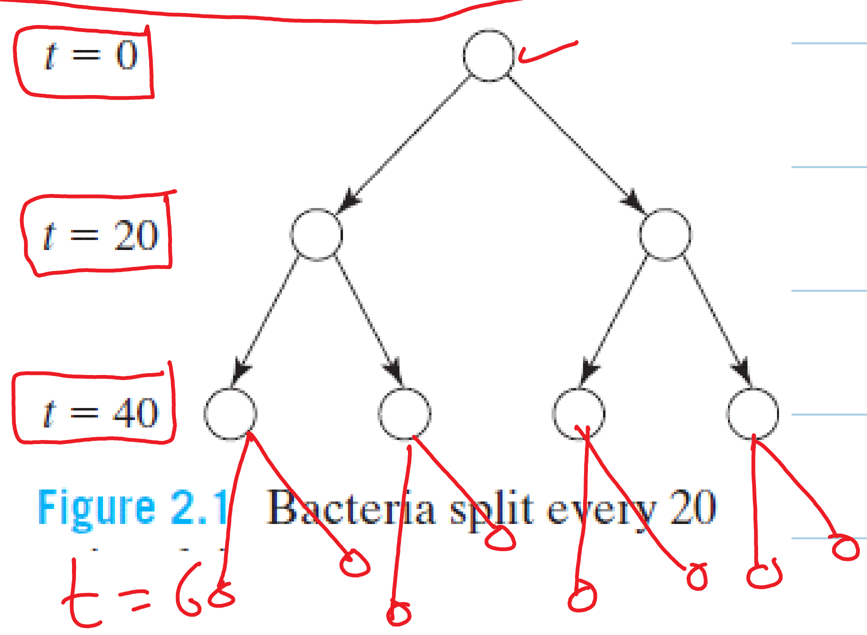


Figure 2.1 Bacteria split every 20

t = Time (min)	0	20	40	60	80	100	....
N(t) = population size	1	2	4	8	16	32	

For simplicity, we say one unit of time equals 20 min.

two unit of time equals 40 min.

three " " " " 60 min.

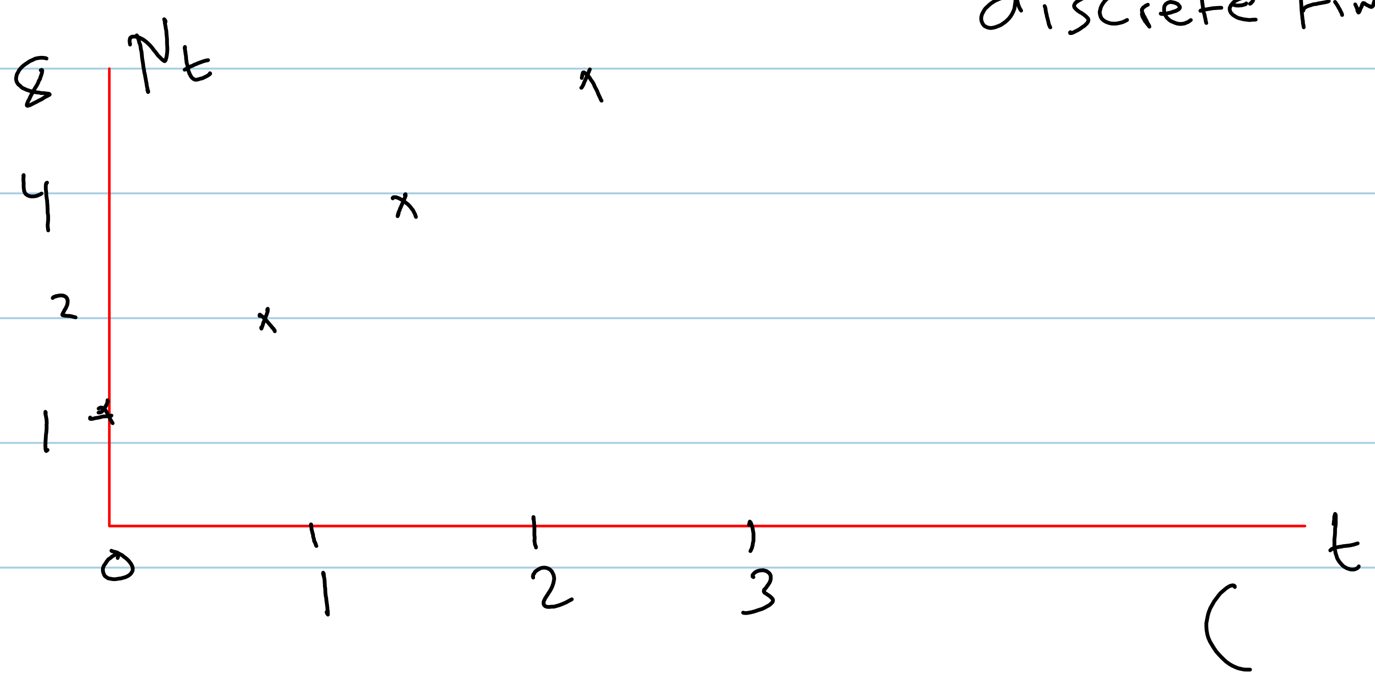
....

$1 \equiv 20\,\text{min}$

Time (20min)	0	1	2	3	4	5
$N(t)$	1	2	2^2	2^3	2^4	2^5

$$N_t = N(t) = 2^t, \quad t = 0, 1, 2, 3, \ldots$$

discrete time



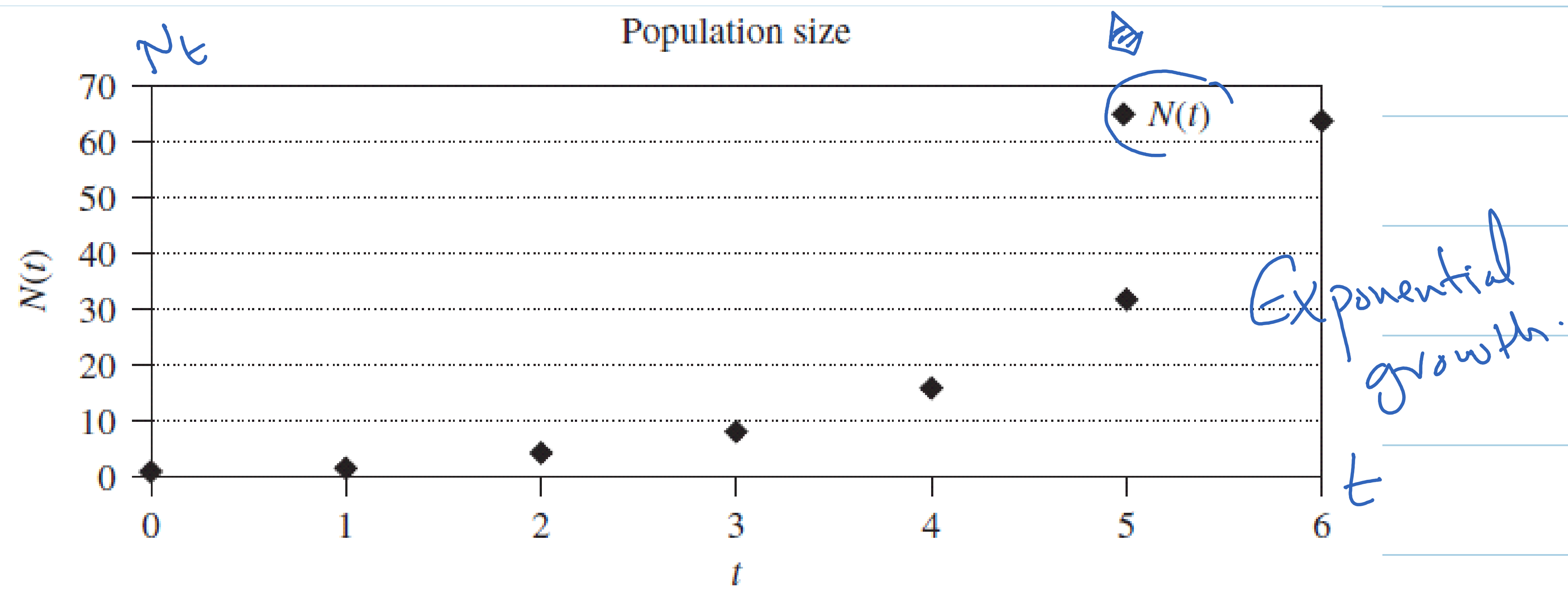


Figure 2.2 The graph of $N(t) = 2^t$ for $t = 0, 1, 2, \ldots, 6$.

$$N_t = N(t) = 2^t, \quad t = 0, 1, 2, \ldots$$

$$N_0 = N(0) = 2^0 = 1, \quad N_1 = 2^1 = 2, \quad N_2 = 2^2 = 4, \ldots$$

Ex. If $N(0) = 100 \Rightarrow N(t) = 100 \cdot (2^t)$,

$t = 0, 1, 2, 3, \ldots$

If $N(0) = 100 \Rightarrow N(1) = 200,\ N(2) = 400$

t	0	1	2	3	4	----
N_t	100	200	400	800	1600	

$$N_t = \underbrace{100}_{N_0}(2^t), \quad t = 0, 1, 2, \ldots$$

In general, $N_t = N_0 R^t$, $t = 0, 1, 2, \ldots$

where N_t: the population size at time t.

N_0: " " " " " 0 $N_0 \geq 0$

R: growth constant. $R > 0$.

If $0 < R < 1$, $N_t \to 0$ as t increases.

$R = 1$, N_t stays the same for all $t = 0, 1, 2, \ldots$

$R > 1$, $N_t \to \infty$ as t increases.

$D = \{0, 1, 2, \ldots\}$ $N_t = a^t$, $t = 0, 1, 2, \ldots$

$f(x) = a^x$, $x \geq 0$

$D = [0, \infty)$

$N_t = 2^t$

$f(x) = 2^x$

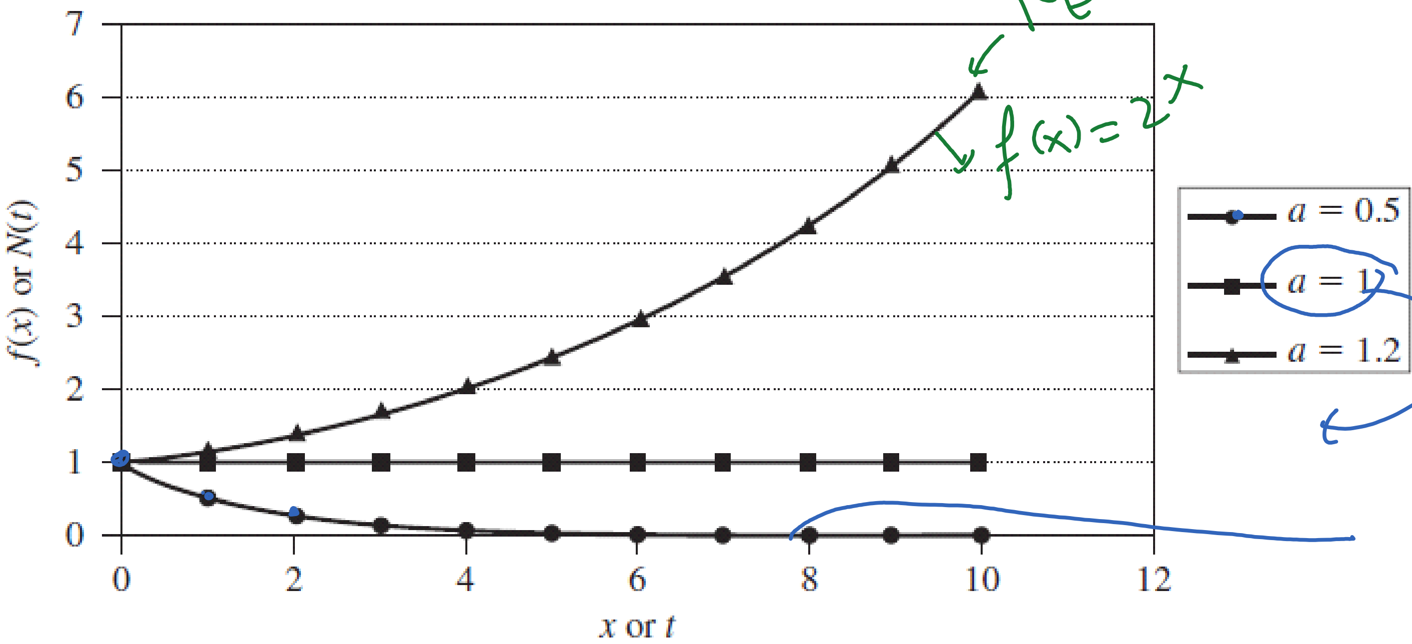


Figure 2.4 The graphs of $f(x) = a^x$, $0 \leq x \leq 10$, and $N(t) = R^t$, $t = 0, 1, 2, \ldots, 10$, for three different values of $a = R$: $a = R = 0.5$, $a = R = 1$, and $a = R = 1.2$.

$(\frac{1}{2})^x = f(x)$

$N_t = (\frac{1}{2})^t$

$N_0 = (\frac{1}{2})^0 = 1$

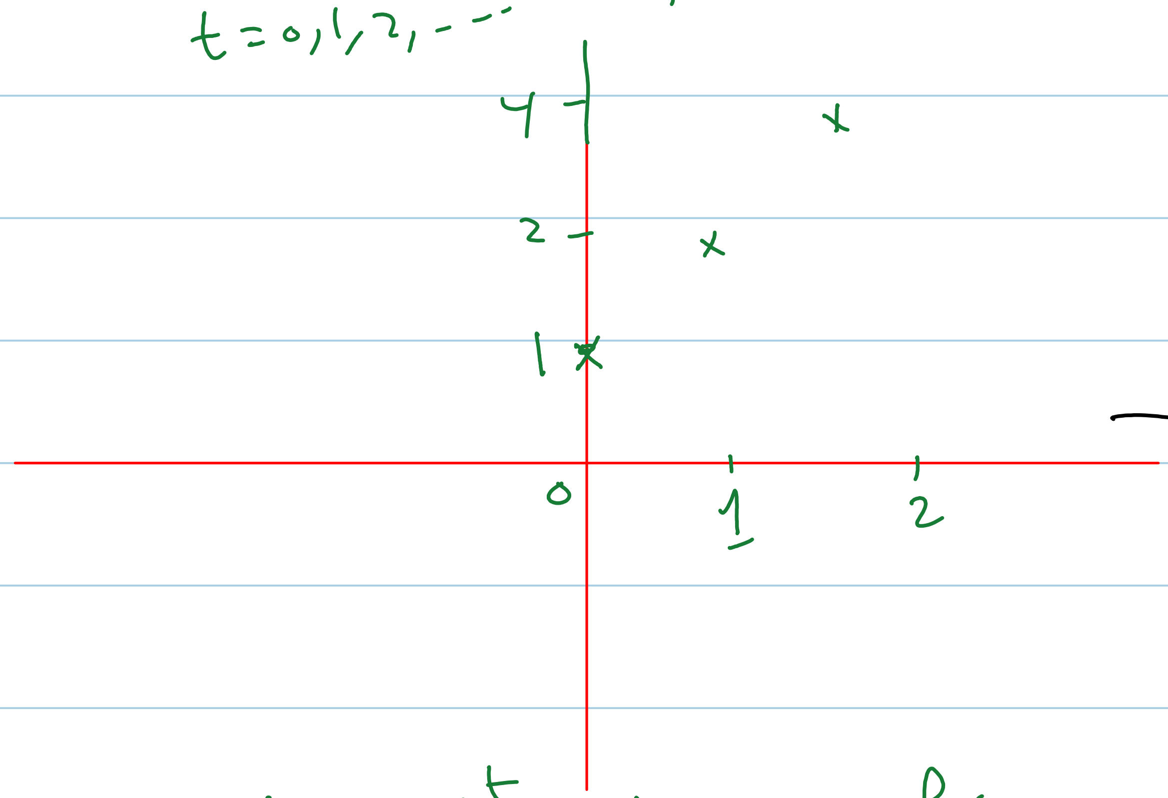
$N_1 = (\frac{1}{2})^1 = \frac{1}{2}$

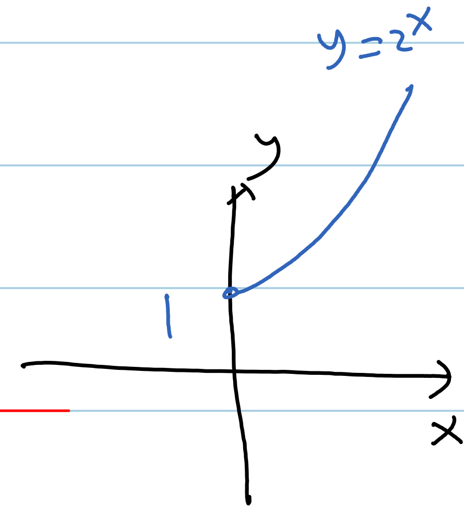
$N_2 = (\frac{1}{2})^2 = \frac{1}{4}$

$t \to \infty$

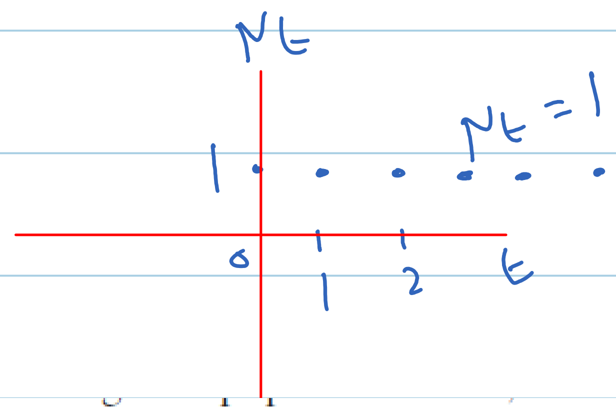
$N_t \to 0$

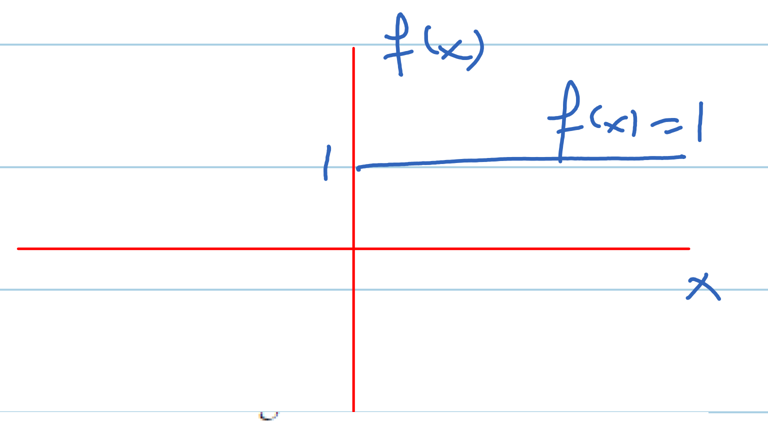
$N_t = 2^t$, $t = 0, 1, 2, \dots$, $f(x) = 2^x$, $x \geq 0$





$R = 1 \Rightarrow N_t = (1)^t = 1$, $f(x) = (1)^x = 1.$





EXAMPLE 1 Suppose a population of cells reproduces every 15 minutes and we measure its size every 30 minutes:

s	0		1		2		3
t	0	1	2	3	4	5	6
t	**Time (min)**	$\boxed{0}$	30	$\boxed{60}$	90	$\boxed{120}$	150	180
N_t	**Population size**	$\boxed{1}$	4	$\boxed{16}$	64	$\boxed{256}$	1024	4096

Write a formula for time $n = 0, 1, 2, \dots$ when (a) one unit of time is 30 minutes, (b) one unit of time is 60 minutes, and (c) one unit of time is 15 minutes.

Sol. (a) $N_t = N_0 R^t$, $N_0 = 1$, $R = 4.$

$$\boxed{N_t = 4^t , \quad t = 0, 1, 2, 3, \dots}$$

(b)

t	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5
s	$\boxed{0}$		$\boxed{1}$		$\boxed{2}$	

$\Rightarrow \boxed{t = 2s}$

$N_s = 4^{2s}$, $s = 0, 1, 2, \dots$

$$\boxed{N_s = 16^s , \quad s = 0, 1, 2, \dots}$$

0 60 120

c) one unit of time is 15 min.

u	0	2	4	6	8	...	
t	0	1	2	3	4	5	6

Time (min)	0	30	60	90	120	150	180
Population size	1	4	16	64	256	1024	4096

$N_t = 4^t, \ t = 0, 1, 2, \dots$

$\therefore \ t = \frac{1}{2}u, \ u = 0, 1, 2, \dots$

$N_u = 4^{\frac{1}{2}u} = (2^2)^{\frac{1}{2}u} = 2^u$

$\therefore \ \boxed{N_u = 2^u, \ u = 0, 1, 2, 3, \dots}$

15 30 45, ...

In Problems 5–10, give a formula for $N(t)$, $t = 0, 1, 2, \dots$, on the basis of the information provided.

9. $N_0 = 2$; population quadruples every 30 minutes; one unit of time is 15 minutes

t	0	30	60	90	...
N_t	2	8.	32	128	

0 1 2 3.

$N_0 = 2$

$N_1 = 4N_0 \leftarrow N_1 = 2(4) = 8$

$N_2 = 4N_1 \leftarrow N_2 = 2(4)^2 = 32$

$N_3 = 4N_2 \leftarrow N_3 = 2(4^3) = 128.$

$N_t = N_0 R^t$

30 60.

$\boxed{N_t = 2(4^t), \ t = 0, 1, 2, \dots}$

t	0	1	2	3	4 ...
s	0	2	4	6	...

$s = 2t$

$\boxed{t = \frac{1}{2}s}$

$N_t = 2\left(4^{\frac{1}{2}s}\right), \ s = 0, 1, 2, \dots$

$= 2\left(2^2\right)^{\frac{1}{2}s} = 2^1 \cdot 2^s = 2^{s+1}$

$N_s = 2^{s+1}$, $s = 0, 1, 2, \ldots$ (0, 15 min., 30 min.)

2.1.2 Recursions

$N_t = N_0 R^t$ explicit form.

تعويض explicit form.

$$\boxed{N_t = 2^t} \Rightarrow N_{100} = 2^{100} = \ldots \quad (2.1.1)$$

Recursion form.

$$\boxed{N_{t+1} = 2N_t}$$ find N_3

$\boxed{t=2} \Rightarrow N_3 = 2N_2$

$N_2 = 2N_1$

$N_1 = 2N_0$

$N_3 = 2N_2$
$= 2(2N_1)$
$= 4N_1$
$= 4(2N_0)$
$N_3 = 8N_0$
$N_3 = N_0(2^t)$

In general, explicit

$$N_t = 2^t \iff \underbrace{N_{t+1} = 2N_t}_{\text{Recursion.}}$$

$N_0 = 1$

$N_1 = 2 = 2N_0$

$N_2 = 2^2 = 4 = 2N_1$

$N_3 = 2^3 = 8 = 2N_2$

explicit, recursion

$$N_t = N_0(R^t) \iff N_{t+1} = \boxed{R}N_t$$

$$\boxed{\frac{N_t}{N_{t+1}} = \frac{1}{R}} \text{ or } \boxed{\frac{N_{t+1}}{N_t} = R}$$

ex. $N_t = 2^t$, $t = 0, 1, 2, \ldots$

t	0	1	2	3	4	...
N_t	1	2	4	8	16	

$\frac{4}{2} = 2 = R \quad \frac{16}{8} = 2 = R$

$$\text{slope} = \frac{N_{t+1} - 0}{N_t - 0} = \frac{N_{t+1}}{N_t} = R$$

(x_1, y_1) (x_2, y_2)

$$\text{slope} = \frac{Dy}{Dx} = \frac{y_2 - y_1}{x_2 - x_1}$$

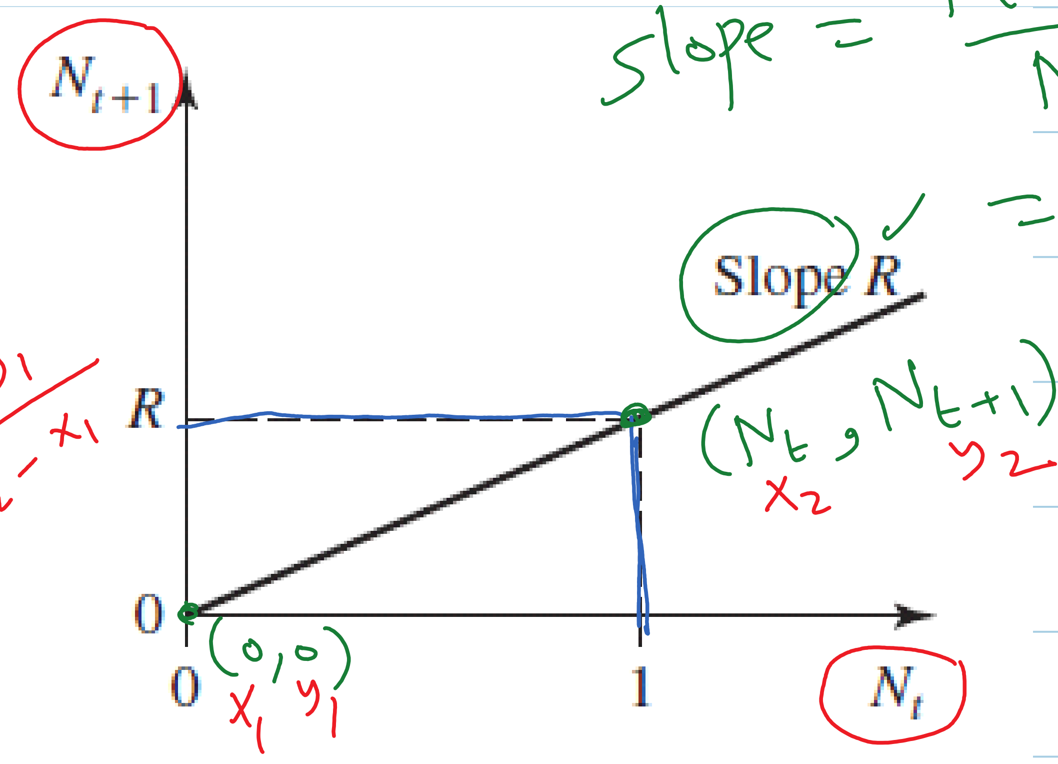


Figure 2.5 The exponential growth recursion $N_{t+1} = RN_t$ when $R > 0$.

CH 2

2.1
- 2.1.1 explicit. $N_t = N_0 R^t$
- 2.1.2 recursion. $N_{t+1} = R N_t$.

2.2
- 2.2.1
- 2.2.2
- 2.2.3

متتاليات

2.2 Sequences

2.2.1
2.2.2
2.2.3

2.2.1 What Are Sequences?

EXAMPLE 1 Let $f : \mathbf{N} \to \mathbf{R}$ $= \{0, 1, 2, 3, \ldots$

$$n \to \boxed{f(n) = \frac{1}{n+1}}$$

Domain $= \mathbb{N} = \{0, 1, 2, 3, \ldots$

Range $= \mathbb{R} = (-\infty, \infty)$

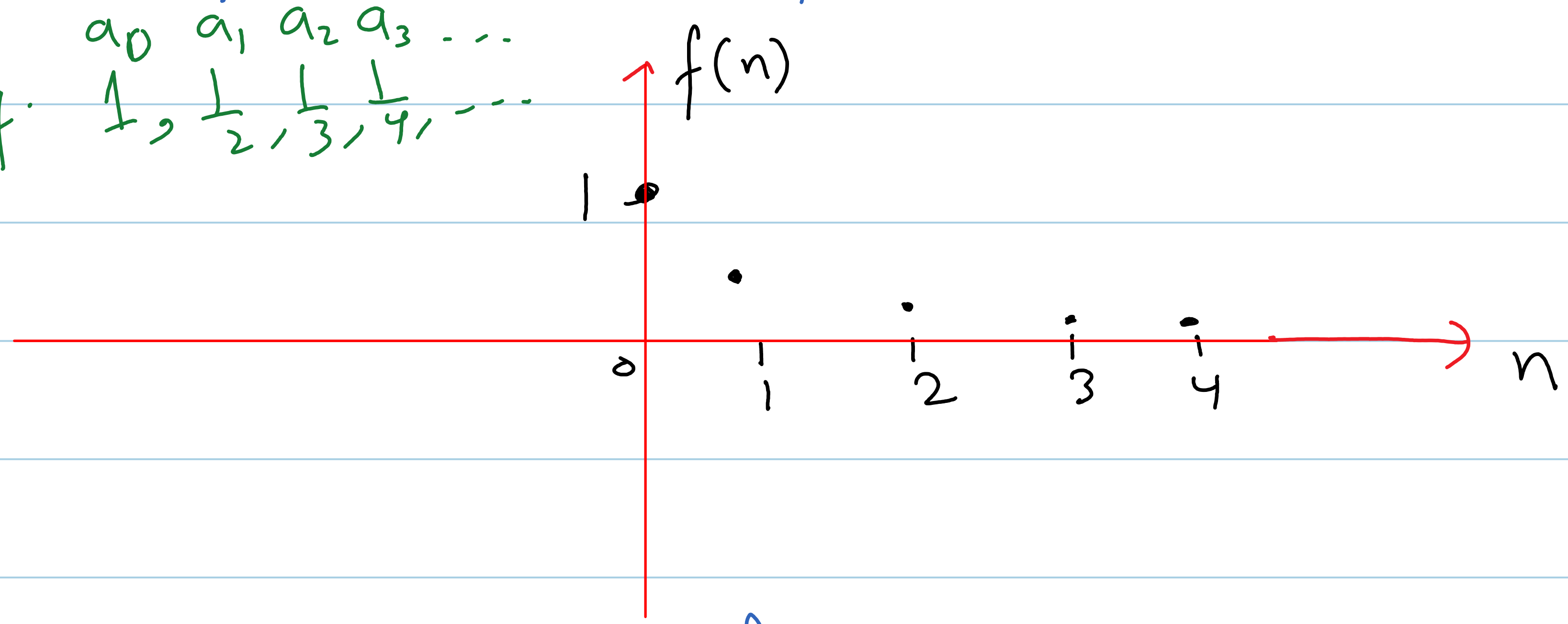
Produce a table for $n = 0, 1, 2 \ldots, 5$ and graph the function.

$f(0) = \frac{1}{0+1} = 1$, $f(1) = \frac{1}{1+1} = \frac{1}{2}$, $f(2) = \frac{1}{3}$

$f(3) = \frac{1}{4}$, $f(4) = \frac{1}{5}$, $f(5) = \frac{1}{6}$.

$a_0 \; a_1 \; a_2 \; a_3 \ldots$

seq. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$



A sequence is a list of numbers

$a_0, a_1, a_2, a_3, \ldots$

in a given order. Each $a_0, a_1, a_2, \ldots$ represents a number (the terms of the sequence)

حدود

(A seq. is a function whose domain is $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$).

$f : \mathbb{N} \longrightarrow \mathbb{R}$

$a_n = f(n)$ nth term.

$a_0 = f(0)$ first term

$a_1 = f(1)$ - second term.

nth term. $a_1 = f(1)$: second term.

ex: The sequence $a_n = (-1)^n, n = 0, 1, 2, \ldots$

$a_0 = (-1)^0 = 1$

$a_1 = (-1)^1 = -1$

$a_2 = (-1)^2 = 1$

$\vdots$

$\{a_n\} = \{\overset{a_0}{1}, \overset{a_1}{-1}, \overset{a_2}{1}, \overset{a_3}{-1}, \ldots\}$

ex: Find a_n for the Seq.

$0, 1, 4, 9, 16, 25, \ldots$

$a_0 \quad a_1 \quad a_2 \quad a_3$

$1 \to 0$
$2 \to 1$
$3 \to 4$

Sol.

$\boxed{a_n = n^2, \quad n = 0, 1, 2, \ldots}$

$0 \to 0$
$1 \to 1$
$2 \to 4$
$3 \to 9$
$\vdots$
$n \to n^2$

or $\boxed{a_n = (n-1)^2 \quad n = 1, 2, 3, \ldots}$

n	$f(n)$
1	0
2	1
3	4
$\vdots$	
n	$(n-1)^2$

OR $a_n = (n-2)^2, n = 2, 3, \ldots$

$\vdots$

Ex: Find a_n for $\overset{a_0}{\frac{1}{1}}, \overset{a_1}{-\frac{1}{4}}, \overset{a_2}{\frac{1}{9}}, \overset{a_3}{-\frac{1}{16}}, \overset{a_4}{\frac{1}{25}}, \ldots$

Sol. $a_n = \dfrac{(-1)^n}{(n+1)^2}, \quad n = 0, 1, 2, 3, \ldots$

OR $a_n = \dfrac{(-1)^{n-1}}{n^2}, \quad n = 1, 2, 3, \ldots$

OR $a_n = \dfrac{(-1)^n}{(n-1)^2}, \quad n = 2, 3, 4, \ldots$

Summary

An explicit description of the seq.

$$\boxed{a_n = f(n), \quad n = 0, 1, 2, \dots}$$

A recursive description of the seq.

$$\boxed{a_{n+1} = g(a_n)}.$$

معطى (Given)

ex. Let $\boxed{a_{n+1} = \frac{3}{a_n}}$, $a_0 = 2$, find a_4.

$\boxed{n=0}$ $\quad a_1 = \frac{3}{a_0} = \frac{3}{2}$

✓ $\boxed{n=1}$ $\quad a_2 = \frac{3}{a_1} = \frac{3}{\left(\frac{3}{2}\right)} = 3\left(\frac{2}{3}\right) = 2.$

$\boxed{n=2}$ $\quad a_3 = \frac{3}{a_2} = \frac{3}{2}$ ✓

$\boxed{n=3}$ $\quad a_4 = \frac{3}{a_3} = \frac{3}{\left(\frac{3}{2}\right)} = 2.$

Remark. In seq. 2.1 (Exponential growth).

$$\overset{a_n}{N_t} = \overset{a_0}{N_0} R^n, \quad t = 0, 1, 2, \dots \text{ (explicit form)}$$

$$\underbrace{N_{t+1}}_{a_{n+1}} = R \underbrace{N_t}_{a_n}, \quad t = 0, 1, 2, \dots$$

$a_{n+1} = R\, a_n$ (recursion form).

2.2.2 Limits

$\lim_{n\to\infty} a_n = ??$

Ex. Let $a_n = \frac{1}{n+1}$, $n = 0, 1, 2, 3, \dots$

find $\lim_{n\to\infty} a_n$.

$a_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

Sol. $\lim_{n\to\infty} \frac{1}{n+1} = 0$. exists. $n\to\infty$, $\frac{1}{n+1} \to 0$

موجودة.

Ex. $a_n = (-1)^n$, $n = 0, 1, 2, \dots$

$= \{1, -1, 1, -1, \dots\}$

find $\lim_{n\to\infty} a_n$

Sol. $a_n = (-1)^n$ has no limit

$\lim_{n\to\infty} (-1)^n$ does not exist (DNE).

Ex. Let $a_n = 2^n$, $n = 0, 1, 2, \dots$

find $\lim_{n\to\infty} a_n$.

Sol. $a_n = \{1, 2, 4, 8, 16, 32, \dots\}$

As $n \to \infty$, $a_n \to \infty$.

$\therefore \lim_{n\to\infty} a_n = +\infty$. (DNE).

ex. $a_n = \dfrac{n}{n+1}$, find $\lim\limits_{n\to\infty} a_n$.

Sol. $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1n}{1n+1}$$

تذكر $\boxed{\lim\limits_{n\to\infty} \frac{1}{n} = 0}$

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Ex. Find $\lim\limits_{n\to\infty} \dfrac{4n^2-1}{1n^2}$

Ans. $$= \lim_{n\to\infty} \frac{\frac{4n^2}{n^2}-\frac{1}{n^2}}{\frac{n^2}{n^2}} = \lim_{n\to\infty} \left(\frac{4-\frac{1}{n^2}}{1}\right) = \frac{4-0}{1} = 4.$$

Ex. $$\lim_{n\to\infty} \frac{n+2}{n^2-4} = \lim_{n\to\infty} \frac{\frac{n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}-\frac{4}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{2}{n^2}}{1-\frac{4}{n^2}} = \frac{0+0}{1-0} = \frac{0}{1} = 0.$$ exists

ex. $$\lim_{n\to\infty} \frac{n^3-4}{n^2+1} = \lim_{n\to\infty} \frac{\frac{n^3}{n^2}-\frac{4}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n-\frac{4}{n^2}}{1+\frac{1}{n^2}}\right) = \frac{\infty-0}{1+0} = \infty$$ DNE

Summary

نسبي Rational

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \begin{cases} 0 \text{ exists}, & \deg(a_n) < \deg(b_n) \\ \text{DNE } \infty \text{ or } -\infty, & \deg(a_n) > \deg(b_n) \\ \dfrac{\text{Coef } a_n}{\text{Coef } b_n} & \deg(a_n) = \deg(b_n) \end{cases}$$

معامل أعلى درجة

معامل // //

ex. $\lim_{n\to\infty} \frac{4-5n^3}{2n^3+1} = -\frac{5}{2}$ exists

ex. $\lim_{n\to\infty} \frac{2n^{2020}+1}{5n^{2021}+4} = 0$

ex. $\lim_{n\to\infty} \frac{n^3}{n+1} = \lim_{n\to\infty} \frac{\frac{n^3}{n}}{\frac{n}{n}+\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{n^2}{1+\frac{1}{n}} = \frac{\infty}{1+0} = \infty$

DNE.

Limit Laws If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and c is a constant, then

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_nb_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ provided } \boxed{\lim_{n\to\infty} b_n \neq 0}$$

ex: $\lim_{n\to\infty} \frac{4n^2+1}{n^2} = \lim_{n\to\infty} \frac{\frac{4n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}}$

$= \lim_{n\to\infty}\left(4+\frac{1}{n^2}\right)$

$= \lim_{n\to\infty} 4 + \lim_{n\to\infty} \frac{1}{n^2}$

$= 4 + 0 = 4$ exists.

ex: $\lim_{n\to\infty} 2^n = +\infty$

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = \lim_{n\to\infty} \frac{1}{2^n} = 0$

$1, 1, 1, \cdots$ $\lim_{n\to\infty} (1)^n = \lim_{n\to\infty} 1 = 1$

شكل عام

If $a_n = a_0 R^n$, $n = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} a_n = \begin{cases} 0, & 0<R<1 \\ \infty, & R>1 \\ a_0, & R=1 \end{cases}$$

ex. $\lim_{n\to\infty}\left[4^n + \left(\frac{1}{4}\right)^n\right] = \infty + 0 = \infty$ DNE.

ex. $\lim_{n\to\infty} 2^{-n} = \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$

2.2.3 Recursions

$a_{n+1} = g(a_n)$, $n = 0, 1, 2, \ldots$

How to find $\lim_{n\to\infty} a_n$ (if exists) ??

EXAMPLE 13 Compute a_n for $n = 1, 2, \ldots, 5$ when

$$\boxed{a_{n+1} = \frac{1}{4}a_n + \frac{3}{4} \quad \text{with } a_0 = 2} \tag{2.7}$$

Find a solution of the recursion, and then take a guess at the limiting behavior of the sequence. $\left(\lim_{n\to\infty} a_n = ??\right)$

Solution: $a_0 = 2$ (given).

$\boxed{n=0} \Rightarrow \boxed{a_1} = \frac{1}{4}a_0 + \frac{3}{4} = \frac{1}{4}(2) + \frac{3}{4} = \boxed{\frac{5}{4}}$

$\boxed{n=1} \Rightarrow \boxed{a_2} = \frac{1}{4}a_1 + \frac{3}{4} = \frac{1}{4}\left(\frac{5}{4}\right) + \frac{3}{4} = \boxed{\frac{17}{16}}$

$\boxed{n=2} \Rightarrow \boxed{a_3} = \frac{1}{4}a_2 + \frac{3}{4} = \frac{1}{4}\left(\frac{17}{16}\right) + \frac{3}{4}$

$= \boxed{\frac{65}{64}} = \frac{65}{4^3}$

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$$a_n = \frac{4^n+1}{4^n}, \quad n = 1, 2, \ldots$$

(explicit form).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4^n+1}{4^n}$$

صح / خطأ

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$= \lim_{n\to\infty} \left(\frac{4^n}{4^n} + \frac{1}{4^n} \right)$$

$$= \lim_{n\to\infty} \left(1 + \left(\frac{1}{4}\right)^n \right)$$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 1 + 0 = 1$$

exists.

ex. Assume that $\lim_{n\to\infty} a_n$ exists

for $a_{n+1} = \frac{1}{4} a_n + \frac{3}{4}, \; a_0 = 2$

find $\lim_{n\to\infty} a_n$.

Sol. Let $\lim_{n\to\infty} a_n = L$ ✓

$a_n \to L$ as $n \to \infty$

$a_{n+1} \to L$ $n+1 \to \infty$

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{4} \lim_{n\to\infty} a_n + \lim_{n\to\infty} \frac{3}{4}$$

L L

$$L = \frac{1}{4} L + \frac{3}{4}$$

$$L - \frac{1}{4} L = \frac{3}{4} \Rightarrow \frac{3}{4} L = \frac{3}{4} \Rightarrow \boxed{L=1}$$

$\lim_{n\to\infty} a_n = 1$.

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EXAMPLE 14 Assume that $\boxed{\lim_{n\to\infty} a_n \text{ exists}}$ for

$$\boxed{a_{n+1} = \sqrt{3a_n}} \quad \text{with } a_0 = 2$$

Find $\lim_{n\to\infty} a_n$.

Sol: let $\lim_{n\to\infty} a_n = L$ (exists)

Then,

$$\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}\sqrt{3a_n}$$

$$\Rightarrow L = \sqrt{3L}$$

$$L^2 = 3L$$

$$L^2 - 3L = 0 \Rightarrow L(L-3) = 0$$

$$\Rightarrow \boxed{L = 0} \quad \text{or} \quad \boxed{L = 3}$$

$$a_{n+1} = \sqrt{3a_n}, \quad \boxed{a_0 = 2}$$

$n=0 \Rightarrow a_1 = \sqrt{3a_0} = \sqrt{3(2)} = \sqrt{6} \approx \boxed{2.45}$

$n=1 \quad a_2 = \sqrt{3a_1} = \sqrt{3\sqrt{6}} \approx \boxed{2.71}$

$n=2 \quad a_3 = \sqrt{3a_2} = \sqrt{3\sqrt{3\sqrt{6}}} \approx \boxed{2.85}$

$\vdots$

$a_n \longrightarrow 3$ as $n \longrightarrow \infty$

$\therefore L = 0$ reject مرفوض

$\therefore \lim_{n\to\infty} a_n = 3$.

ex. Let $a_{n+1} = \frac{3}{a_n}$, $a_0 = \sqrt{3}$

Investigate $\lim_{n\to\infty} a_n = ??$.

Sol. Let $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} a_{n+1} = \frac{3}{\lim_{n\to\infty} a_n}$

$\Rightarrow L = \frac{3}{L}$

$\Rightarrow L^2 = 3$

$\Rightarrow \boxed{L = \sqrt{3}}$ or $\boxed{L = -\sqrt{3}}$ reject.

$\boxed{a_0 = \sqrt{3}}$, $a_1 = \frac{3}{a_0} = \frac{3}{\sqrt{3}} = \boxed{\sqrt{3}}$.

$a_2 = \frac{3}{a_1} = \frac{3}{\sqrt{3}} = \boxed{\sqrt{3}}$

$\vdots$

$a_n = \sqrt{3}$, $n = 0, 1, 2, \ldots$

$\therefore \lim_{n\to\infty} a_n = \sqrt{3}$

ex. $a_{n+1} = \frac{3}{a_n}$, $a_0 = -\sqrt{3}$.

Sol. $\lim_{n\to\infty} a_n = L \Rightarrow L = \frac{3}{L}$

$L^2 = 3 \Rightarrow L = \sqrt{3}$ or $-\sqrt{3}$

reject.

Since $a_0 = -\sqrt{3}$

$a_1 = \frac{3}{a_0} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$

$a_2 = \frac{3}{a_1} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$ $\ldots$ $a_n = -\sqrt{3}$

$\lim_{n\to\infty} a_n = -\sqrt{3}$

ex. $a_{n+1} = \dfrac{3}{a_n}$, $a_0 = 2$

Investigate $\lim\limits_{n\to\infty} a_n$.

Sol. let $\lim\limits_{n\to\infty} a_n = L$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \frac{3}{\lim a_n}$$

$$L = \frac{3}{L} \Rightarrow L = \pm\sqrt{3}.$$

$\boxed{a_0 = 2}$ ✓ given

$\boxed{n=0} \Rightarrow a_1 = \dfrac{3}{a_0} = \dfrac{3}{2}$ ✓

$\boxed{n=1} \Rightarrow a_2 = \dfrac{3}{a_1} = \dfrac{3}{\frac{3}{2}} = 2$ ✓

$\boxed{n=2} \Rightarrow a_3 = \dfrac{3}{a_2} = \dfrac{3}{2}$ ✓

$\vdots$

$\{a_n\} = \{2, \frac{3}{2}, 2, \frac{3}{2}, 2, \frac{3}{2}, \ldots\ldots$

$\Rightarrow \lim\limits_{n\to\infty} a_n$ DNE

الاتصال نهايات

Limits and Continuity 3

3.1 Limits

3.1.1 An Informal Discussion of Limits

$\lim_{x \to c} f(x) = L$ (limit of $f(x)$ as x approaches c is equal to L)

تقترب

(goes)

$f(x) \to L$ as $x \to c$

If $\lim_{x \to c} f(x) = L$ "finite number".

we say the limit exists ($f(x)$ converges to L)

تقترب

If $\lim_{x \to c} f(x)$ DNE, we say $f(x)$ diverges as x goes to c

تتباعد

$\boxed{\lim_{x \to c^+} f(x)}$ x approaches c from the right

$\lim_{x \to c^-} f(x)$ " " " " " left

ex. Find $\lim_{x \to 2} x^2$.

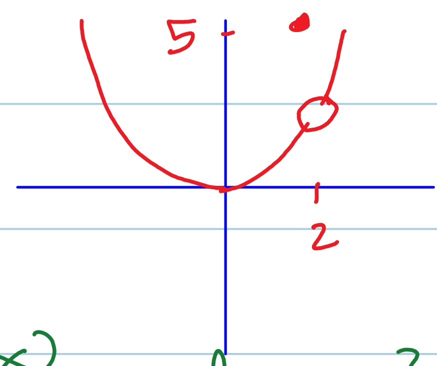
left | | right |
---|---|---|---
x | x^2 | x | x^2
1.9 | 3.61 | 2.1 | 4.41
1.99 | 3.9601 | 2.01 | 4.0401
1.999 | 3.996001 | 2.001 | 4.004001
1.9999 | 3.99960001 | 2.0001 | 4.00040001

left 2 right

$$\lim_{x\to 2^-} x^2 = 4, \quad \lim_{x\to 2^+} x^2 = 4$$

$$\Rightarrow \lim_{x\to 2} x^2 = 4.$$

ex. $g(x) = \begin{cases} x^2, & x \neq 2 \\ 5, & x = 2 \end{cases}$



Then $g(2) = 5$ but $\lim_{x\to 2} g(x) = \lim_{x\to 2} x^2 = 4.$

In general, $\lim_{x\to c} f(x) \neq f(c).$

ex. $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$ $f(x)$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim_{x\to 3} (x+3) = 3+3 = 6$$

exists

but $f(3) = \frac{0}{0}$ undefined غير معرّف

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Ex. Find $\lim_{x\to 0} e^{-|x|}$

Sol. $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$e^{-|x|} = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0. \end{cases}$ $\quad \underset{0}{\overset{e^{x} \quad e^{-x}}{\overline{\qquad\qquad}}}$

$\lim_{x\to 0^+} e^{-|x|} = \lim_{x\to 0^+} e^{-x} = e^{0} = 1$ exists

$\lim_{x\to 0^-} e^{-|x|} = \lim_{x\to 0^-} e^{x} = e^{0} = 1$ exists

$\Rightarrow \lim_{x\to 0} e^{-|x|} = 1$ exists

Ex. Find $\lim_{x\to 0} \frac{|x|}{x}$

Sol. $\frac{|x|}{x} = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{-x}{x}, & x < 0 \\ \text{undefined}, & x = 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ \text{undefined}, & x = 0. \end{cases}$

$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1$ exists.

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} (-1) = -1$ exists

$\Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ doesnot exists since $\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$

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ex. Find $\lim_{x\to 0} \frac{1}{x}$

Sol. $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ DNE

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ DNE.

$\lim_{x\to 0} \frac{1}{x}$ DNE

ex. $\lim_{x\to 0} \frac{1}{x^2}$

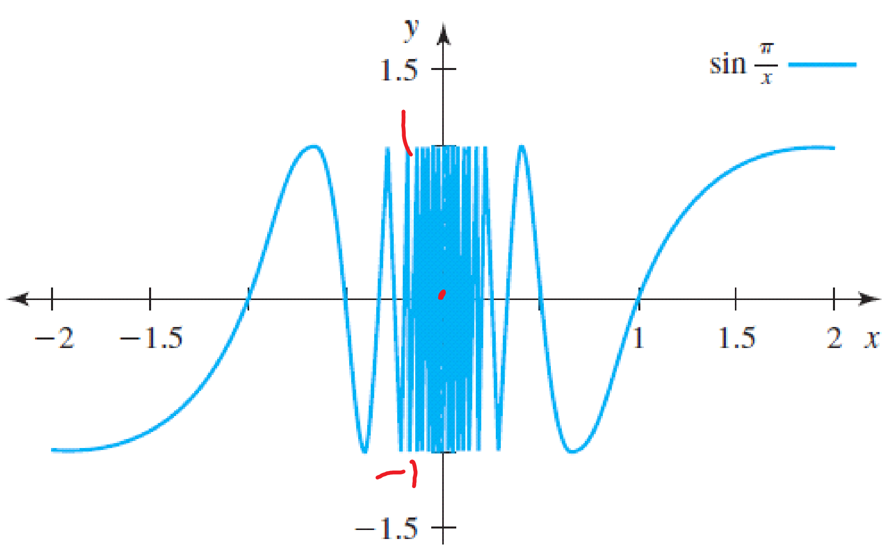
$\lim_{x\to 0^+} \frac{1}{x^2} = +\infty$, $\lim_{x\to 0^-} \frac{1}{x^2} = +\infty$

$\Rightarrow \lim_{x\to 0} \frac{1}{x^2}$ DNE.

ex. $\lim_{x\to 3^+} \frac{-1}{x-3} = -\infty$ DNE.

ex. $\lim_{x\to 3^-} \frac{2-x}{x-3} = +\infty$. DNE.

ex. $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ DNE.



$\sin\left(\frac{\pi}{x}\right)$ oscillates between -1 and 1 ex. Find $\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2}$ $\left(\frac{0}{0}\right)$

المرافقة
conjugate

$$= \lim_{x\to 0} \left[\frac{(\sqrt{x^2+16} - 4)}{x^2} \cdot \frac{(\sqrt{x^2+16} + 4)}{(\sqrt{x^2+16} + 4)} \right]$$

$$= \lim_{x\to 0} \frac{(\sqrt{x^2+16})^2 - (4)^2}{x^2(\sqrt{x^2+16} + 4)}$$

$$= \lim_{x\to 0} \frac{x^2 + \cancel{16} - \cancel{16}}{x^2(\sqrt{x^2+16} + 4)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+16} + 4} = \frac{1}{\sqrt{16} + 4} = \frac{1}{8}$$

exists.

ex. $\lim_{x\to 0} \frac{\frac{1}{x}}{\frac{1}{x} + 1} \cdot \frac{x}{x} = \lim_{x\to 0} \frac{1}{1+x} = \frac{1}{1+0} = 1$.

exists

3.1.2 Limit Laws

Limit Laws Suppose that a is a constant and that

$$\lim_{x\to c} f(x) \quad \text{and} \quad \lim_{x\to c} g(x)$$

exist. Then the following rules hold:

✓ **1.** $\lim_{x\to c} af(x) = a \lim_{x\to c} f(x)$

✓ **2.** $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

✓ **3.** $\lim_{x\to c}[f(x)\cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided that $\boxed{\lim_{x\to c} g(x) \neq 0}$

ex.

$$\lim_{x\to 2} (x^3+4x-1)$$

$$= \lim_{x\to 2} x^3 + 4\lim_{x\to 2} x - \lim_{x\to 2} 1$$

$$= 8 + 4(2) - 1$$

$$= 8 + 8 - 1$$

$$= 15$$ exists.

ex. $\lim_{x\to 4}\left(\frac{x^2+1}{x-3}\right) = \frac{\lim_{x\to 4}(x^2+1)}{\lim_{x\to 4}(x-3)}$

$= \frac{4^2+1}{4-3} = 17$ exists

(1) If $f(x)$ is a polynomial, then

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$$\lim_{x\to c} f(x) = f(c)$$

(2) If $f(x)$ is a rational function

نسبي

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials, and if $\boxed{q(c) \neq 0}$, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} = f(c)$$

(3) $\lim_{x\to c} \sin x = \sin(c)$, $\lim_{x\to c} \cos x = \cos(c)$.

ex. $\lim_{x\to \frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$.

ex. $\lim_{x\to 0} \frac{\cos x}{x^2+1} = \frac{\cos 0}{0^2+1} = \frac{1}{1} = 1$.

ex. $\lim_{x\to 4} \frac{x^2-16}{x-4} \neq \frac{\lim_{x\to 4}(x^2-16)}{\lim_{x\to 4}(x-4)}$

but $\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{x-4}}$

$= 4+4 = 8$.

Rational

ex. $\lim_{x \to -1} \left(\frac{2x^3 - x + 5}{x^2 + 3x + 1} \right)$

$$= \frac{2(-1)^3 - (-1) + 5}{(-1)^2 + 3(-1) + 1} = \frac{-2 + 1 + 5}{1 - 3 + 1}$$

$= -4$ exists.

الاستمرارية

3.2 Continuity

3.2.1 What Is Continuity?

Definition A function f is said to be **continuous** at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

$f(c)$ defined

$\lim_{x \to c} f(x)$ exists

$f(c) = \lim_{x \to c} f(x)$.

Ex. $g(x) = \begin{cases} \frac{x^2 - 9}{x - 3}, & x \neq 3 \\ 6, & x = 3 \end{cases}$. Is f cont. at $x = 3$?

Ans. (1) $g(3) = 6$ defined

(2) $\lim_{x \to 3} g(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$

$= 6$ exists.

(3) $g(3) = \lim_{x \to 3} g(x)$

$\therefore g$ is continuous at $x = 3$.

ex. $$g(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 7 & \text{if } x = 3 \end{cases}$$

Sol. (1) $g(3) = 7$ defined

(2) $\lim_{x\to 3} g(x) = 6$ exists.

(3) $\lim_{x\to 3} g(x) \neq g(3)$.

$\therefore$ g is discontinuous.

Rmk. f is polynomial or sin, cos, ...

$\Rightarrow$ f is continuous on $(-\infty, \infty)$.

ex. where is $f(x) = \dfrac{x^2-16}{x-4}$ cont. ?

Ans. Domain$(f) = (-\infty, \infty) \setminus \{4\}$

$= (-\infty, 4) \cup (4, \infty)$.

$\therefore$ f is Cont. on $(-\infty, 4) \cup (4, \infty)$.

Ex. Let

$$f(x) = \begin{cases} \dfrac{x^2-x-6}{x-3} & \text{if } x \neq 3 \\ a & \text{if } x = 3 \end{cases}$$

and find a so that $f(x)$ is continuous at $x = 3$.

Sol. $\lim_{x\to 3} f(x) = f(3)$ (given) معطى

$$\Rightarrow \lim_{x\to 3} \frac{x^2-x-6}{x-3} = a \qquad \frac{0}{0}$$

$$\Rightarrow \lim_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} = a$$

$$\Rightarrow a = 3+2 = 5.$$

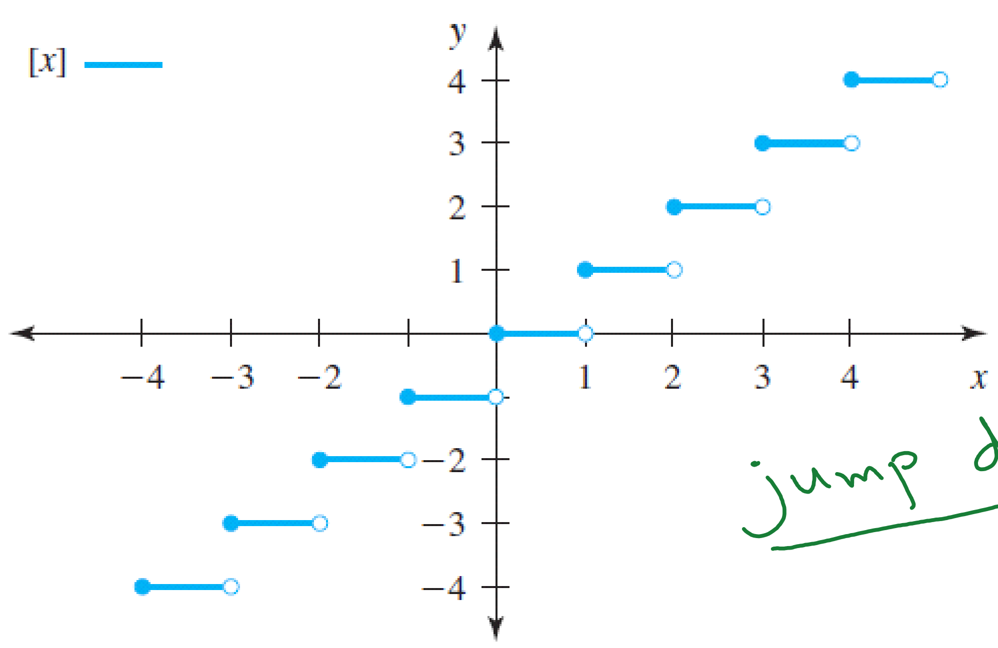
ex. $f(x) = \lfloor x \rfloor = \begin{cases} \vdots & \\ -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ \vdots & \end{cases}$

$$\lim_{x\to 3^+} \lfloor x \rfloor = 3, \quad \lim_{x\to 3^-} \lfloor x \rfloor = 2$$

$$\Rightarrow \lim_{x\to 3} \lfloor x \rfloor \text{ DNE.}$$

In general, $\lim_{x\to n^+} \lfloor x \rfloor = n, \quad \lim_{x\to n^-} \lfloor x \rfloor = n-1$

$\Rightarrow f(x) = \lfloor x \rfloor$ is discont. at $x = 0, \pm 1, \pm 2, \pm 3, \ldots$



jump discontinuity.

Figure 3.12 The floor function $f(x) = \lfloor x \rfloor$.

Definition A function f is said to be continuous from the right at $x = c$ if

مستمر من اليمين

$$\boxed{\lim_{x \to c^+} f(x) = f(c)}$$

and continuous from the left at $x = c$ if

من اليسار

$$\lim_{x \to c^-} f(x) = f(c)$$

ex. $\boxed{\lim_{x \to 2^+} \lfloor x \rfloor = 2}$, $\lim_{x \to 2^-} \lfloor x \rfloor = 1$, $\boxed{\lfloor 2 \rfloor = 2}$

Notice, $\lim_{x \to 2^+} \lfloor x \rfloor = \lfloor 2 \rfloor \Rightarrow y = \lfloor x \rfloor$ is continuous from the right at $x = 2$.

ex. $x > 0$ or $x < 0$

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} \frac{x}{x}, & x > 0 \\ \frac{-x}{x}, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ \boxed{0}, & x = 0. \end{cases}$$

$f(0) = \boxed{0}$, $\lim_{x \to 0^+} f(x) = \boxed{1}$, $\lim_{x \to 0^-} f(x) = \boxed{-1}$

$\Rightarrow \lim_{x \to 0} f(x)$ DNE $\Rightarrow f$ is discont. at $x = 0$.

f is discont. from the right and from the left.

ex. At which point is $f(x) = \frac{1}{(x-4)^2}$ discont.?

Can the discontinuity be removed?

Sol $\lim_{x\to 4} \frac{1}{(x-4)^2} = \infty$ (DNE).

$\therefore f$ is discont. at $x=4$.

(cont. on $(-\infty,4)\cup(4,\infty)$)

the discontinuity cannot be removed.

since $\lim_{x\to 4} \frac{1}{(x-4)^2}$ DNE.

ex: $f(x) = \frac{x^2-9}{x-3} \Rightarrow f$ is discont. at $\boxed{x=3}$

$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \boxed{6}$ exists $= f(3)$

$$F(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 6, & x=3 \end{cases}$$ is cont. at $x=3$.

f

$\lim_{x\to 3} f(x)$

$\therefore$ the discontinuity removed

($x=3$ is removable discontinuity).

ex: $y = \frac{1}{x}$, $x=0$ is not removable discont.

ex: $y = \frac{x}{x^2-x}$, $x=0$ is removable discont.

$\lim_{x\to 0} \frac{x}{x^2-x} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(x-1)} = \frac{1}{-1} = -1$ exists.

$\lim_{x\to 1} \frac{x}{x^2-x} = \lim_{x\to 1} \frac{1}{x-1}$ (DNE)

$\therefore x=1$ is not removable discontinuity.

3.2.2 Combinations of Continuous Functions

Suppose that a is a constant and the functions f and g are continuous at $x = c$. Then the following functions are continuous at $x = c$:

1. $a \cdot f$
2. $f + g$
3. $f \cdot g$
4. $\dfrac{f}{g}$ provided that $g(c) \neq 0$

مغلقة $[a,b]$ $\lim_{x\to a^+} f(x) = f(a)$ $\lim_{x\to b^-} f(x) = f(b)$

We say that a function f is continuous on an interval I if f is continuous for all $x \in I$. Note that if I is a closed interval, then continuity at the left (and, respectively, right) endpoint of the interval means continuous from the right (and, respectively, left).

ex. $f(x) = \sqrt{x}$ is cont. on $[0, \infty)$.

The following functions are continuous wherever they are defined:

بدلالة الجذور

1. polynomial functions
2. rational functions
3. power functions
4. trigonometric functions
5. exponential functions of the form a^x, $a > 0$ and $a \neq 1$
6. logarithmic functions of the form $\log_a x$, $a > 0$ and $a \neq 1$ $x > 0$

EXAMPLE 6 For which values of $x \in \mathbf{R}$ are the following functions continuous? (Domain).

(a) $f(x) = 2x^3 - 3x + 1$ **(b)** $f(x) = \dfrac{x^2 + x + 1}{x - 2}$ **(c)** $f(x) = x^{1/4} = \sqrt[4]{x}$

(d) $f(x) = 3\sin x$ **(e)** $f(x) = \tan x$ **(f)** $f(x) = 3^x$

(g) $2\ln(x + 1)$

(a) f is cont. on $(-\infty, \infty)$ since it is a polynomial.

(b) f is cont. on $\mathbb{R} \setminus \{2\}$
$= (-\infty, 2) \cup (2, \infty)$.

(c) $x \geq 0$
$\therefore f$ is cont. on $[0, \infty)$.

(d) f is cont. on $(-\infty, \infty)$.

(e) $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. is cont. on $\mathbb{R} \setminus \{ \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots \}$

(f) $f(x) = 3^x$

is cont. on $(-\infty, \infty)$.

تذكر

$\ln x^r = r \ln|x|$

(g) $f(x) = 2\ln(x+1)$, $x+1 > 0$

$x + 1 > 0 \Rightarrow x > -1$
$\therefore f$ is cont. on $(-1, \infty)$.

Theorem If $g(x)$ is continuous at $x = c$ with $g(c) = L$ and $f(x)$ is continuous at $x = L$, then $(f \circ g)(x)$ is continuous at $x = c$. In particular,

$$\lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f[g(x)] = f[\lim_{x \to c} g(x)] = f[g(c)] = f(L)$$

cont.

ex. $\lim\limits_{x \to 3} \sin\left(\frac{\pi}{4}(x^2 - 1)\right)$

$= \sin\left(\lim\limits_{x \to 3} \frac{\pi}{4}(x^2-1)\right)$ since $y = \sin x$ is cont. on $(-\infty, \infty)$

$= \sin\left(\frac{\pi}{4}(9-1)\right) = \sin(2\pi) = 0$.

ex. $\lim\limits_{x \to 1} \sqrt{2x^3 - 1} = \sqrt{\lim\limits_{x \to 1}(2x^3-1)} = \sqrt{2(1)^3 - 1} = 1$

ex. $$\lim_{x\to 0} e^{(x-1)} = e^{\lim_{x\to 0}(x-1)}$$ since $y=e^x$ is cont. on $(-\infty,\infty)$

$$= e^{-1}.$$

3.3 Limits at Infinity

$x \longrightarrow \infty$ or $x \longrightarrow -\infty$

ex. $$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

$$= \frac{1}{1+0} = 1.$$

ex. $$\lim_{x\to\infty} \frac{x^2+2x-1}{x^3-3x+1} = \lim_{x\to\infty} \frac{\frac{x^2}{x^3}+\frac{2x}{x^3}-\frac{1}{x^3}}{\frac{x^3}{x^3}-\frac{3x}{x^3}+\frac{1}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}-\frac{1}{x^3}}{1-\frac{3}{x^2}+\frac{1}{x^3}}$$

$$= \frac{0+0-0}{1-0+0} = \frac{0}{1} = 0.$$

ex. $$\lim_{x\to\infty} \frac{2x^3-4x+7}{3x^3+7x^2-1} = \lim_{x\to\infty} \frac{2-\frac{4}{x^2}+\frac{7}{x^3}}{3+\frac{7}{x}-\frac{1}{x^3}}$$

$$= \frac{2-0+0}{3+0-0} = \frac{2}{3}.$$

ex. $$\lim_{x\to\infty} \frac{x^4+2x-5}{x^2-x+2} = \lim_{x\to\infty} \frac{x^2+\frac{2}{x}-\frac{5}{x^2}}{1-\frac{1}{x}+\frac{2}{x^2}}$$

$$= \frac{\infty+0-0}{1-0+0} = \infty \text{ (DNE)}$$

Summary.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L \neq 0 & \text{if } \deg(p) = \deg(q) \\ \text{does not exist} & \text{if } \deg(p) > \deg(q) \end{cases}$$

(∞ or $-\infty$)

EXAMPLE 2 Compute

(a) $\displaystyle\lim_{x\to-\infty} \frac{1 - x + 2x^2}{3x - 5x^2} = \frac{2}{-5}$ (b) $\displaystyle\lim_{x\to\infty} \frac{1 - x^3}{1 + x^5} = 0$

(c) $\displaystyle\lim_{x\to\infty} \frac{2 - x^2}{1 + 2x}$ DNE (d) $\displaystyle\lim_{x\to-\infty} \frac{4 + 3x^2}{1 - 7x}$ DNE

(a) $\displaystyle\lim_{x\to-\infty} \frac{1-x+2x^2}{3x-5x^2} = \frac{2}{-5} = -\frac{2}{5}.$

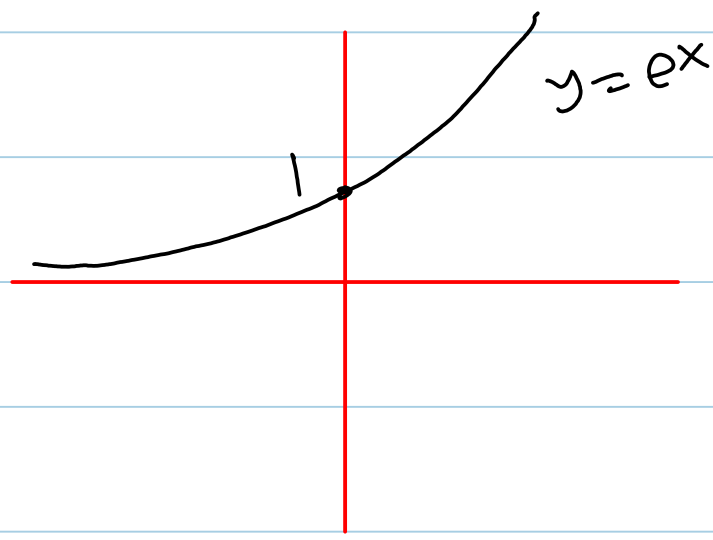
(b) 0

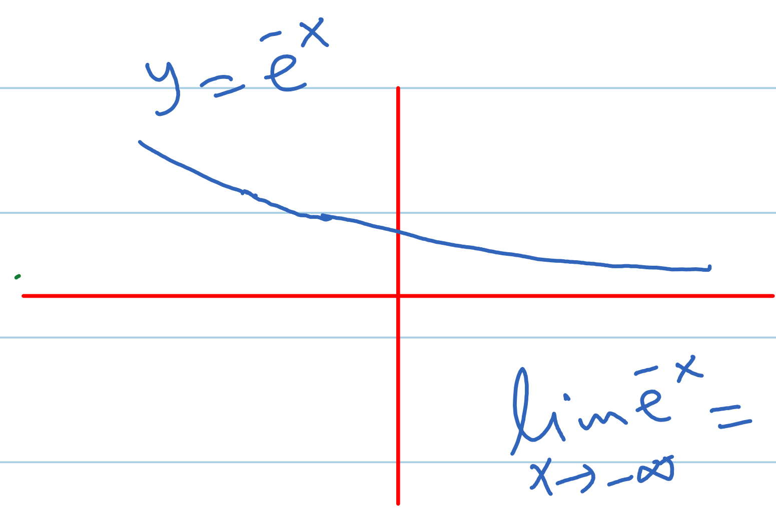
(c) $\displaystyle\lim_{x\to\infty} \frac{2-x^2}{1+2x} = \lim_{x\to\infty} \frac{\frac{2}{x} - \frac{x^2}{x}}{\frac{1}{x} + \frac{2x}{x}}$

$\displaystyle = \lim_{x\to\infty} \left(\frac{\frac{2}{x} - x}{\frac{1}{x} + 2} \right) = \frac{0-\infty}{0+2} = -\infty$ DNE

(d) $\displaystyle\lim_{x\to-\infty} \frac{4+3x^2}{1-7x} = \lim_{x\to-\infty} \frac{\frac{4}{x}+3x}{\frac{1}{x}-7} = \frac{0+-\infty}{0-7}$

$= +\infty$. DNE





$\displaystyle\lim_{x\to-\infty} e^{-x} = \infty$

$\displaystyle\lim_{x\to\infty} e^x = \infty$, $\displaystyle\lim_{x\to-\infty} e^x = 0$, $\displaystyle\lim_{x\to\infty} e^{-x} = 0$

Q14) $\lim_{x\to\infty} \frac{e^{-x}}{1-e^{-x}} = \frac{0}{1-0} = \frac{0}{1} = 0$ exists

Q16) $\lim_{x\to\infty} \frac{e^{x}}{2-e^{x}}$ $\frac{\infty}{-\infty}$ undeterminate form.

$= \lim_{x\to\infty} \left(\frac{e^{x}}{2-e^{x}} \cdot \frac{e^{-x}}{e^{-x}} \right)$

$= \lim_{x\to\infty} \frac{1}{2e^{-x}-1} = \frac{1}{2(0)-1} = \frac{1}{-1} = -1.$

19) $\lim_{x\to\infty} \frac{3e^{2x}}{2e^{2x}-e^{x}}$

$= \lim_{x\to\infty} \frac{3\cancel{e^{2x}}}{\cancel{e^{2x}}[2-e^{-x}]} = \lim_{x\to\infty} \frac{3}{2-e^{-x}}$

$= \frac{3}{2-0} = \frac{3}{2}.$

23) $\lim_{x\to-\infty} \frac{e^{x}}{1+x} \rightarrow e^{x} \cdot \frac{1}{1+x}$

$= \left(\lim_{x\to-\infty} e^{x}\right)\left(\lim_{x\to-\infty} \frac{1}{1+x}\right) = (0)(0) = 0.$

EXAMPLE 3 **Logistic Growth** The logistic curve describes the density of a population over time, where the rate of growth depends on the population size. We will discuss this function in more detail in coming chapters. It suffices here to say that the per capita rate of growth decreases with increasing population size. If $N(t)$ denotes the size of the population at time t, then the logistic curve is given by

$$N(t) = \frac{K}{1+\left(\frac{K}{N(0)}-1\right)e^{-rt}} \quad \text{for } t \geq 0$$, $r > 0$

Find $\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{K}{1+\left(\frac{K}{N(0)}-1\right)e^{-rt}} = \frac{K}{1+0}$

$e^{-rt} \to 0$ $= K.$

25. In Section 1.2.3, Example 6, we introduced the Monod growth function

$$r(N) = a\frac{N}{k+N}, \quad N \geq 0$$, a, k constant

Find $\lim_{N\to\infty} r(N)$.

$$= \lim_{N\to\infty} \frac{aN}{k+N} = \lim_{N\to\infty}\left(\frac{a}{\frac{k}{N}+1}\right) = \frac{a}{0+1} = \boxed{a}.$$

3.4 The Sandwich Theorem and Some Trigonometric Limits

Squeeze thm.

Sandwich Theorem If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval that contains c (except possibly at c) and

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L$$

ex. If $x^2-1 \leq g(x) \leq -\cos x$ for all x.

Find $\lim_{x\to 0} g(x)$.

Sol. $\lim_{x\to 0}(x^2-1) = \boxed{-1}$, $\lim_{x\to 0}(-\cos x) = -\cos 0 = \boxed{-1}$

$\Longrightarrow \lim_{x\to 0} g(x) = -1$ by Sandwich thm.

ex. Find $\lim_{x\to\infty} e^{-x}\cos x$?

$-1 \leq \cos x \leq 1$

$-e^{-x} \leq e^{-x}\cos x \leq e^{-x}$

$\lim_{x\to\infty}(-e^{-x}) = 0$ and $\lim_{x\to\infty} e^{-x} = 0$

by Sandwich thm, $\lim_{x\to\infty} e^{-x}\cos x = 0$.

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ex. Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ $g(x)$.

$-1 \le \sin\left(\frac{1}{x}\right) \le 1, \quad x \ne 0$

$\Rightarrow \quad -x^2 \le \underbrace{x^2 \sin\left(\frac{1}{x}\right)}_{g(x)} \le x^2$

Now, $\lim_{x\to 0}(-x^2) = \boxed{0}$ and $\lim_{x\to 0} x^2 = \boxed{0}$

$\Rightarrow$ by Sandwich thm, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

Trigonometric Limits

حفظ $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ and $\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$ حفظ

proof. $\lim_{x\to 0} \frac{1-\cos x}{x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} \longrightarrow \sin^2 x = (\sin x)(\sin x)$

$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right)$

$= (1)\left(\frac{\sin 0}{1+\cos 0}\right) = (1)\left(\frac{0}{1}\right)$

$= 0$

EXAMPLE 3 Find the following limits.

(a) $\lim_{x\to 0} \frac{\sin 3x}{5x}$ (b) $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$ (c) $\lim_{x\to 0} \frac{\sec x - 1}{x \sec x}$

(a) $\lim_{x\to 0} \frac{\sin 3x}{5x} = \frac{1}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot 3$

$= \frac{3}{5} \lim_{x\to 0} \frac{\sin 3x}{3x} = \frac{3}{5}(1)$

$= \frac{3}{5}.$

(b) $\lim_{x\to 0} \frac{\sin^2 x}{x^2} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = (1)^2 = 1.$

(c) $\lim_{x\to 0} \frac{\sec x - 1}{x \sec x} = \lim_{x\to 0} \frac{\frac{1}{\cos x} - 1}{x \cdot \frac{1}{\cos x}} \cdot \frac{\cos x}{\cos x}$

قاعدة

$= \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$

Q6) $\lim_{x\to 0} \frac{\sin(2x)}{3x} = \frac{2}{3}.$

10) $\lim_{x\to 0} \frac{\sin\left(\frac{-\pi x}{2}\right)}{2x} = \frac{-\frac{\pi}{2}}{\frac{2}{1}} = -\frac{\pi}{2}\left(\frac{1}{2}\right)$

$= -\frac{\pi}{4}.$

14) $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1^2 = 1.$

$\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

18) $\lim_{x\to 0} \frac{1-\cos\left(\frac{x}{2}\right)}{\frac{x}{2}\cdot 2} = \frac{1}{2} \lim_{x\to 0} \frac{1-\cos\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)} = \frac{1}{2}(0)$

$= 0.$

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19. $\lim_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^2}$ $\quad \dfrac{\sin x}{x} \left(\dfrac{1-\cos x}{x}\right)$

$$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1-\cos x}{x} = (1)(0) = 0.$$

20. $\lim_{x\to 0} \dfrac{\csc x - \cot x}{x \csc x}$

$$= \lim_{x\to 0} \left[\frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x \cdot \frac{1}{\sin x}} \cdot \frac{\sin x}{\sin x} \right]$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

3.5 Properties of Continuous Functions

نظرية القيمة الوسطية

IMVT

3.5.1 The Intermediate-Value Theorem

The Intermediate-Value Theorem Suppose that f is continuous on the closed interval $[a, b]$. If L is any real number with $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists at least one number c on the open interval (a, b) such that $f(c) = L$.

(1)

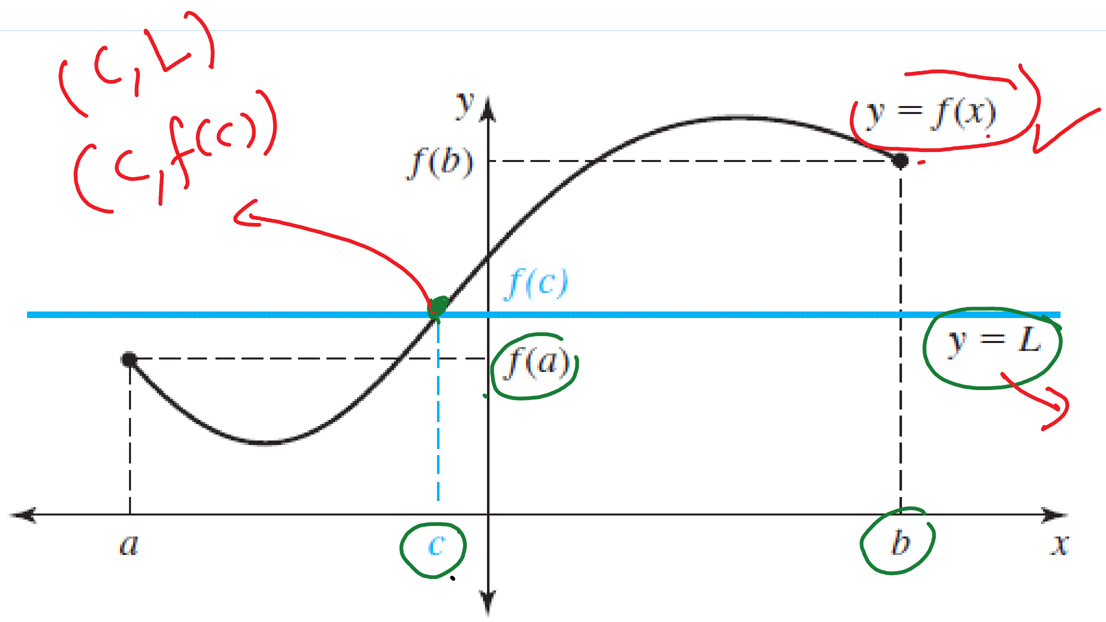

Figure 3.21 The intermediate-value theorem.

L is between f(a) and f(b)

EXAMPLE 1 Let

$[0, \frac{3\pi}{2}]$ (a, b)

$$f(x) = 3 + \sin x \quad \text{for } 0 \le x \le \frac{3\pi}{2}$$

Show that there exists at least one point c in $(0, 3\pi/2)$ such that $\boxed{f(c) = 5/2}$.

$L = \frac{5}{2}$

Sol.

(1) $f(x) = 3 + \sin x$ is continuous on $(-\infty, \infty)$ and hence cont. on $[0, \frac{3\pi}{2}]$.

(2) $f(0) = 3 + \sin 0 = 3 + 0 = \boxed{3}$

$f(\frac{3\pi}{2}) = 3 + \sin\frac{3\pi}{2} = 3 + -1 = \boxed{2}$

$$2 = f(\tfrac{3\pi}{2}) < \tfrac{5}{2} < f(0) = 3$$

by IMVT, there exists at least $c \in (0, \frac{3\pi}{2})$ such that $f(c) = \frac{5}{2}$.

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ex: Show that the equation $x^5 - 7x^2 + 3 = 0$ has a root in $(\overset{a}{0}, \overset{b}{1})$.

$L = 0$

Proof. $f(x) = x^5 - 7x^2 + 3$ (f has a root in $(0,1)$ $f(c) = 0$, $c \in (0,1)$.

L

(1) f is cont. on $(-\infty, \infty)$ and hence on $[0,1]$.

(2) $f(0) = 0 - 0 + 3 = 3 > 0$

$f(1) = 1 - 7 + 3 = -3 < 0$

$\Rightarrow -3 = f(1) < 0 < f(0) = 3$

L

by IMVT, there exists $c \in (0,1)$ such that $f(c) = 0$ L

That is, $x^5 - 7x^2 + 3 = 0$ has a root in $(0,1)$.

Rmk. $f(x)$ has a root in $[a,b]$ if f is cont. on $[a,b]$ and 0 is between $f(a)$ and $f(b)$

That is $f(a)\, f(b) < 0$.

2. Let

$$f(x) = x^3 - 2x + 3, \quad -3 \le x \le -1$$

(b) Use the intermediate-value theorem to conclude that

$$x^3 - 2x + 3 = 0$$

has a solution in $(-3, -1)$.

Sol. • $f(x) = x^3 - 2x + 3$ is cont. on $[-3, -1]$.

• $f(-3) = (-3)^3 - 2(-3) + 3 = -27 + 6 + 3 = -18 < 0$

$f(-1) = (-1)^3 - 2(-1) + 3 = -1 + 2 + 3 = 4 > 0$.

$\Rightarrow \quad f(-3) < 0 < f(-1)$ (L)

by IMVT, f has a root in $(-3,-1)$.

$\Rightarrow \quad x^3 - 2x + 3 = 0$ has a solution in $(-3,-1)$.

3. Let

$$f(x) = \sqrt{x^2+2}, \quad 1 \le x \le 2$$

(a) Graph $y = f(x)$ for $1 \le x \le 2$.

(b) Use the intermediate-value theorem to conclude that

$$\sqrt{x^2+2} = 2$$

has a solution in $(1, 2)$. $\quad f(x) = 2$

(b) $f(x) = \sqrt{x^2+2}$, $[1,2]$, $\boxed{L = 2}$

- f is Cont. on $(-\infty,\infty)$ and hence on $[1,2]$.
- $f(1) = \sqrt{1^2+2} = \sqrt{3}$, $f(2) = \sqrt{2^2+2} = \sqrt{6}$.

$$\sqrt{3} = f(1) < L = 2 < f(2) = \sqrt{6}$$

$\sqrt{4}$

$\therefore$ by IMVT, $\sqrt{x^2+2} = 2$ has a solution in $(1,2)$

5. Use the intermediate-value theorem to show that

$$e^{-x} = x \quad \Rightarrow \quad \underbrace{e^{-x} - x}_{f(x)} = \underbrace{0}_{L}$$

has a solution in $(0, 1)$.

Sol. $\cdot$ $f(x) = e^{-x} - x$ is Cont. on $(-\infty,\infty)$ and hence on $[0,1]$.

- $f(0) = e^{0} - 0 = 1 - 0 = 1 > 0$
- $f(1) = e^{-1} - 1 = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$

$\Big\} \; f(1) < 0 < f(0)$

by IMVT, $e^{-x} = x$ has a solution in $(0,1)$.

4 Differentiation

الاشتقاق

Derivative مشتقة

Differentiate اشتق

Differentiable قابل للاشتقاق

4.1 Formal Definition of the Derivative

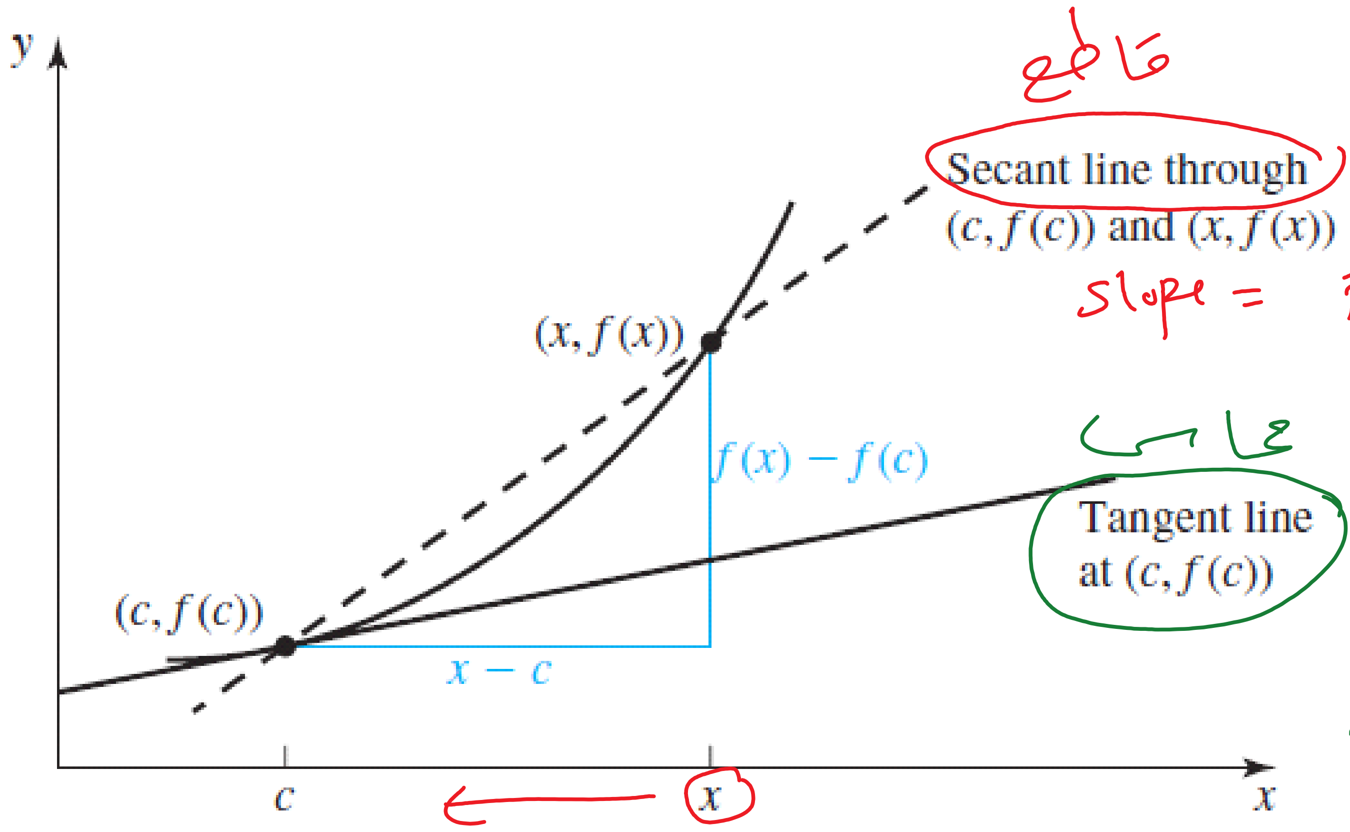
تعريف المشتقة

Definition The derivative of a function f at x, denoted by $\underbrace{f'(x)}_{f \text{ prime of } x}$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

Notation $\quad y' = \dfrac{dy}{dx} = f'(x) = \dfrac{df}{dx} = \dfrac{d}{dx} f(x).$



قاطع

slope $= \dfrac{f(x) - f(c)}{x - c}$

مماس

$f'(c)$ = slope of the tangent line at $x = c$.

Figure 4.6 The derivative $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is the slope of the tangent line at $(c, f(c))$.

4.1.1 Geometric Interpretation and Using the Definition

Definition of the Tangent Line If the derivative of a function f exists at $x = c$, then the tangent line at $x = c$ is the line going through the point $(c, f(c))$ with slope

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \text{slope of the tangent at } x = c.$$

Equation of the Tangent Line If the derivative of a function f exists at $x = c$, then $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$. The equation of the tangent line is given by

$$\boxed{y - \underbrace{f(c)}_{y_1} = \underbrace{f'(c)}_{m}(x - \underbrace{c}_{x_1})}$$

slope $= m = f'(c)$

$(x_1, y_1) = (c, f(c))$

$y - y_1 = m(x - x_1)$

ex. If $f(x) = k$ "Constant" then $f'(x) = 0$

Sol. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h}$ $\frac{0}{h}$ $h \neq 0$

$= \lim_{h \to 0} (0) = 0.$

ex. $f(x) = mx + b \Rightarrow f'(x) = m.$

Proof. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{[m(x+h) + b] - [mx + b]}{h}$

$= \lim_{h \to 0} \frac{\cancel{mx} + mh + \cancel{b} - \cancel{mx} - \cancel{b}}{h}$

$= \lim_{h \to 0} \frac{m\cancel{h}}{\cancel{h}} = \lim_{h \to 0} m = m.$

ex. $y = 1 + 2021x \Rightarrow y' = 2021.$

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EXAMPLE 3 **Using the Definition** Find the derivative of

$$f(x) = \frac{1}{x} \quad \text{for } x \neq 0$$

Sol.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x - (x+h)}{x(x+h)}\right] \qquad \cancel{x} - \cancel{x} - h$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{x(x+h)}\right)$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2},$$

$x \neq 0$.

ex. $f(x) = \sqrt{x}$, find, by def'n, $f'(9)$.

Sol.

$$f'(9) = \lim_{h \to 0} \frac{f(9+h) - f(9)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}\right]$$

$$= \lim_{h \to 0} \frac{\cancel{9} + h - \cancel{9}}{h\left(\sqrt{9+h} + 3\right)}$$

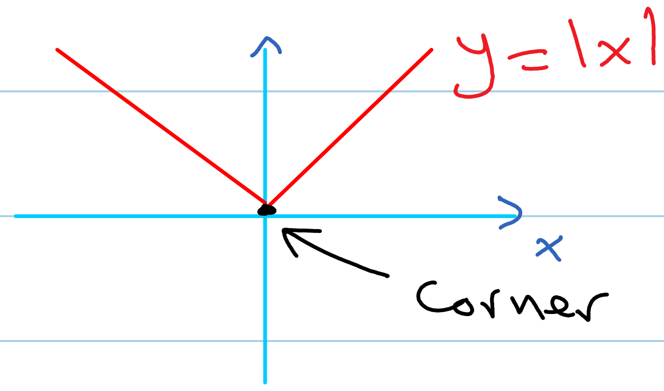
$$= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}.$$

4.1.2 (محذوف)

4.1.3 Differentiability and Continuity

Thm. If f is differentiable at $x=c$, then it is continuous at $x=c$, but the converse is not true

Ex. $y=|x|$



is cont. on $(-\infty,\infty)$

so it is cont. at $x=0$. but it is not diffble at $x=0$. Since.

$$f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

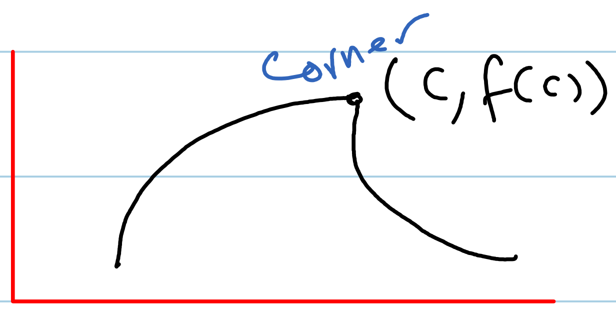
$|0|=0$

$$f'_+(0)=\lim_{\underbrace{h\to 0^+}_{h>0}}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0^+}\frac{h-0}{h}=1$$

$$f'_-(0)=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{-h-0}{h}=-1$$

$\Rightarrow f'_+(0)\neq f'_-(0) \Rightarrow f$ is not diffble at $x=0$.

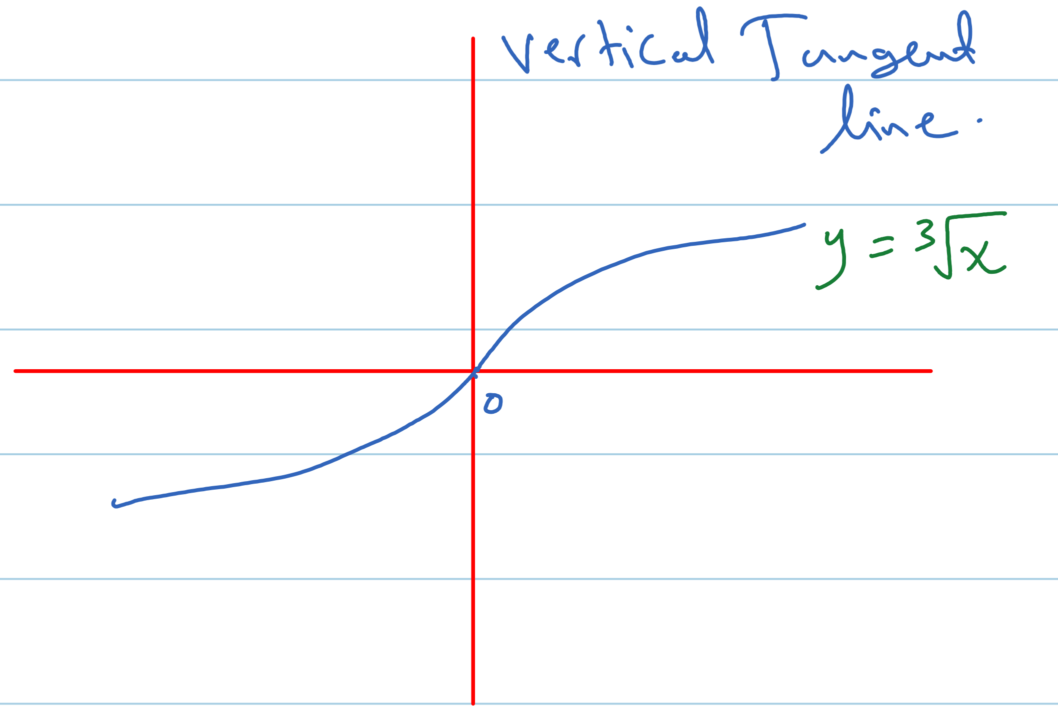


f is not diffble at $x=c$

ex: $f(x) = x^{\frac{1}{3}}$ find $f'(0)$.

Sol:
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h^1}$$
$$= \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = +\infty \text{ (DNE)}$$

$\therefore f'(0)$ DNE $\Rightarrow$ f is not diffble at $x = 0$.



* If f is discontinuous at $x = c$, then it is not diffble at $x = c$

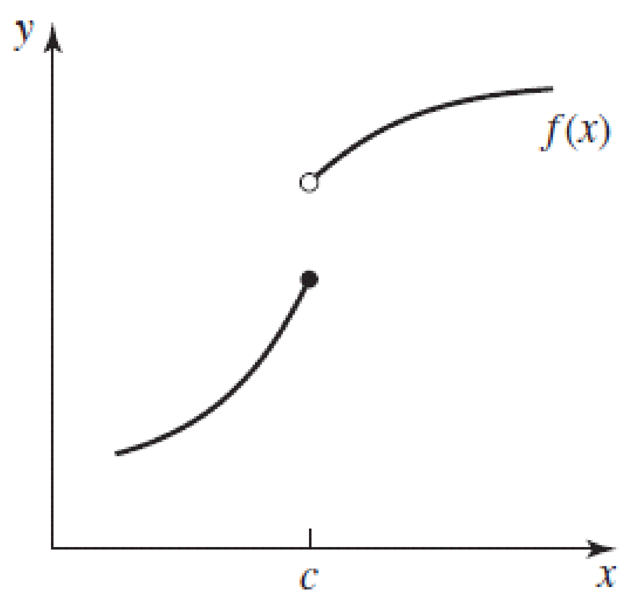


Figure 4.18 The function $y = f(x)$ is not differentiable at $x = c$.

f is not diffble at $x = c$ since f is discont. at $x = c$.

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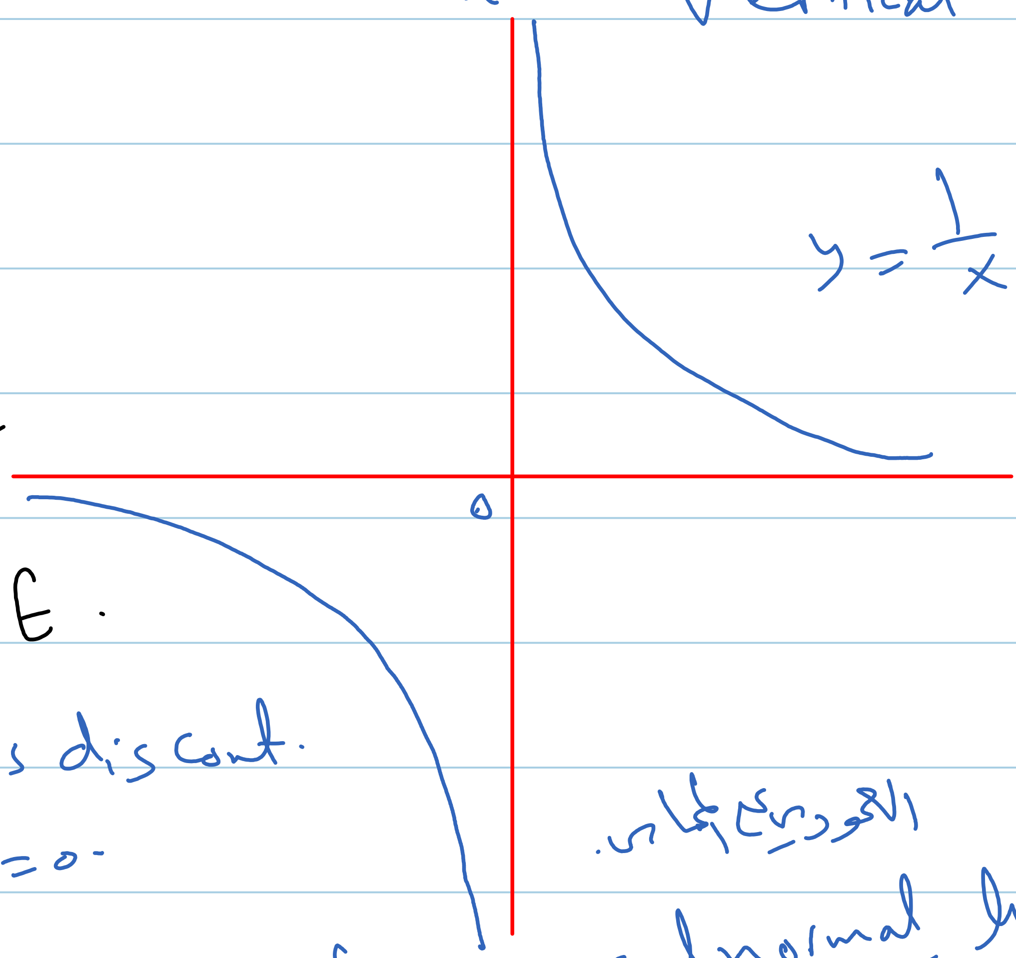
Ex. $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f'(0) = \text{DNE}.$

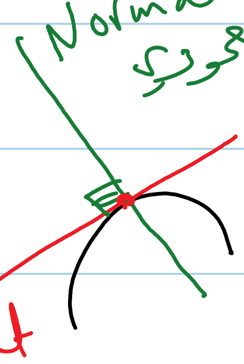
since f is discont. at $x=0$.



العمودي المماس

30. Find the equation of the tangent line (المماس) to the curve $y = x^2 - 3x + 1$ at the point $(2, -1)$. and normal line

$x_1 \quad y_1$



Sol. $f(x) = x^2 - 3x + 1$

$f'(x) = 2x - 3$

Slope of the tangent $= f'(2) = 2(2) - 3 = 1$, slope of the normal line $= \frac{-1}{f'(2)} = -\frac{1}{+1} = -1$

Tangent line. $\boxed{y - y_1 = f'(2)(x - x_1)}$

$y - -1 = 1(x - 2)$

$y + 1 = x - 2 \Rightarrow \boxed{y = x - 3}$ Tangent line

$-1 \qquad -1$

Normal line $y - y_1 = \frac{-1}{f'(2)}(x - x_1)$

$y - -1 = \frac{-1}{1}(x - 2)$

$y + 1 = -x + 2 \Rightarrow \boxed{y = -x + 1}$ Normal line.

$-1 \qquad -1$

35. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$f(a+h)$ $\qquad$ $f(a)$

$$\boxed{\lim_{h\to 0}\frac{2(a+h)^2-2a^2}{h}} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$\boxed{\text{Find } f(x).}$

$$= f'(a) = 4x\Big|_{x=a} = 4a.$$

Ans. $f(x) = 2x^2$

38. The following limit represents the derivative of a function f at the point $(a, f(a))$:

$f(\frac{\pi}{6}+h)$ $\qquad$ $f(\frac{\pi}{6})$

$$\lim_{h\to 0}\frac{\sin(\frac{\pi}{6}+h)-\sin\frac{\pi}{6}}{h}$$

Find f and a. $\quad = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$

Sol. $f(x) = \sin x \quad , \quad a = \frac{\pi}{6}.$

4.2 The Power Rule, the Basic Rules of Differentiation, and the Derivatives of Polynomials

① Power rule $\frac{d}{dx}(x^n) = n x^{n-1}$, $n \in \mathbb{R}$.

② $f(x) = \text{Constant} \Rightarrow f'(x) = 0$.

Ex. (a) $f(x) = x^6 \Rightarrow f'(x) = 6x^5$.

(b) $f(x) = x^{300} \Rightarrow f'(x) = 300\, x^{299}$

(c) $g(t) = t^5 \Rightarrow \frac{dg}{dt} = 5t^4$.

(d) $z = s^3 \quad \frac{dz}{ds} = 3s^2$

(e) $x = y^4 \quad \frac{dx}{dy} = 4y^3$.

(f) $f(x) = \frac{2021 + (1443)^{\frac{1}{5}}}{2\sqrt{6}} \Rightarrow f'(x) = 0$

$\boxed{3}$ $\frac{d}{dx}(a f(x)) = a \frac{df}{dx}$.

④ $\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$.

ex. $y = 2x^4 - 3x^3 + x - 7$

Sol. $y' = 2(4x^3) - 3(3x^2) + 1$.

$= 8x^3 - 9x^2 + 1$.

ex. $\frac{d}{dx}(-5x^7+2x^3-10) = -5(7x^6)+2(3x^2)+0$
$= -35x^6+6x^2$.

ex. $\frac{d}{dt}(t^3-8t^2+3t) = 3t^2-16t+3$.

ex. $\frac{d}{dN}(\ln 5 + N\ln 7) = 0+\ln 7 = \ln 7$.

ex. $\frac{d}{dr}\left(r^2 \sin\frac{\pi}{4} - r^3\cos\frac{\pi}{12} + \sin\frac{\pi}{6}\right)$

$= 2r\sin\frac{\pi}{4} - 3r^2\cos\frac{\pi}{12}$

EXAMPLE 4 Tangent and Normal Lines If $f(x) = 2x^3 - 3x + 1$, find the tangent and normal lines at $(-1, 2)$.

Sol. $f'(x) = 2(3x^2) - 3 = 6x^2-3$.

Slope of the tangent line $= f'(-1) = 6(-1)^2 - 3 = 6-3 = \boxed{3}$.

slope of the normal $= \frac{-1}{f'(-1)} = -\frac{1}{3}$

Tangent line $y-2 = 3(x--1)$

$y-2 = 3x+3 \Rightarrow \boxed{y = 3x+5}$
$+2 \qquad +2$

Normal line $y-2 = -\frac{1}{3}(x+1)$

$y-2 = -\frac{1}{3}x - \frac{1}{3} \Rightarrow \boxed{y = -\frac{1}{3}x + \frac{5}{3}}$
$+2 \qquad +2$

$2-\frac{1}{3}$

4.3 The Product and Quotient Rules, and the Derivatives of Rational and Power Functions

4.3.1 The Product Rule

قاعدة الضرب

$$\boxed{h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)g(x) + f(x)g'(x)}$$

EXAMPLE 1 Differentiate $f(x) = (3x+1)(2x^2-5)$.

Sol. $f'(x) = (3x+1)'(2x^2-5) + (3x+1)(2x^2-5)'$

$= (3)(2x^2-5) + (3x+1)(4x)$

$= \underbrace{6x^2}-15 + \underbrace{12x^2}+4x$

$= 18x^2+4x-15$.

EXAMPLE 2 Differentiate $f(x) = (3x^3-2x)^2$.

Sol. $f(x) = (3x^3-2x)(3x^3-2x)$

$f'(x) = (3x^3-2x)'(3x^3-2x) + (3x^3-2x)(3x^3-2x)'$

$= \underbrace{(9x^2-2)(3x^3-2x)} + \underbrace{(3x^3-2x)(9x^2-2)}$

$= 2(9x^2-2)(3x^3-2x)$.

In general, $(\underbrace{fg}h)' = (fg)(h') + (fg)'h$

$= fgh' + (f'g+fg')h$

$= fgh' + f'gh + fg'h$.

$(fhgk)' = f'hgk + fh'gk + fhg'k + fhgk'.$

EXAMPLE 4 Apply the product rule repeatedly to find the derivative of

$$y = (2x + 1)(x + 1)(3x - 4)$$

Sol. $y' = (2x+1)'(x+1)(3x-4) + (2x+1)(x+1)'(3x-4) + (2x+1)(x+1)(3x-4)'$

$= 2(x+1)(3x-4) + (2x+1)(1)(3x-4) + 3(2x+1)(x+1)$

$= 2(3x^2-4x+3x-4) + (6x^2-8x+3x-4) + 3(2x^2+2x+x+1)$

$\vdots$

$= 18x^2+2x-9.$

4.3.2 The Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}, \quad g \neq 0.$$

EXAMPLE 5 Differentiate $y = \frac{x^3-3x+2}{x^2+1}$. (This function is defined for all $x \in \mathbf{R}$, since $x^2 + 1 \neq 0$.)

$$y' = \frac{(x^2+1)(x^3-3x+2)' - (x^3-3x+2)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(3x^2-3) - (x^3-3x+2)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^4 - \cancel{3x^2} + \cancel{3x^2} - 3 - 2x^4 + 6x^2 - 4x}{(x^2+1)^2}$$

$$= \frac{x^4 + 6x^2 - 4x - 3}{(x^2+1)^2}$$

EXAMPLE 6 **Monod Growth Function** Differentiate the Monod growth function

$$\boxed{f(R) = \frac{aR}{k+R}, \quad R \geq 0}$$

where a and k are positive constants.

ثوابت

$$f'(R) = \frac{(k+R)(aR)' - (aR)(k+R)'}{(k+R)^2}$$

$$= \frac{(k+R)(a) - (aR)(1)}{(k+R)^2}$$

$$= \frac{ak + \cancel{aR} - \cancel{aR}}{(k+R)^2} = \frac{ak}{(k+R)^2}.$$

Power Rule (Negative Integer Exponents) If $f(x) = x^{-n}$, where n is a positive integer, then

$$f'(x) = -nx^{-n-1}$$

ex. $y = \frac{1}{x} = x^{-1} \Rightarrow y' = -x^{-2} = -\frac{1}{x^2}$

ex. $y = \frac{3}{x^4} = 3x^{-4} \Rightarrow y' = 3(-4)x^{-5} = -12x^{-5}$

$$= \frac{-12}{x^5}.$$

Power Rule (General Form) Let $f(x) = x^r$, where r is any real number. Then

$$f'(x) = rx^{r-1}$$

ex. (a) $y = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow y' = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

(b) $y = \sqrt[5]{x} = x^{\frac{1}{5}} \Rightarrow y' = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5}x^{-\frac{4}{5}}.$

$= \frac{1}{5\sqrt[5]{x^4}}$

(c) $g(t) = \frac{1}{\sqrt[3]{t}} = t^{-\frac{1}{3}} \Rightarrow y' = -\frac{1}{3}t^{-\frac{4}{3}} = -\frac{1}{3\sqrt[3]{t^4}}$

(d) $h(s) = s^{\pi} \Rightarrow h'(s) = \pi s^{\pi-1}.$

Ex. $y = \sqrt{x}(x^2-1)$, find y'.

Sol. $y = x^{\frac{1}{2}} \cdot x^2 - x^{\frac{1}{2}} = x^{\frac{5}{2}} - x^{\frac{1}{2}}$

$y' = \frac{5}{2}x^{\frac{5}{2}-1} - \frac{1}{2}x^{\frac{1}{2}-1} = \frac{5}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{5}{2}\sqrt{x^3} - \frac{1}{2\sqrt{x}}.$

$= \frac{5\sqrt{x^3}\cdot\sqrt{x}}{2\sqrt{x}} - \frac{1}{2\sqrt{x}}$

$= \frac{5x^2-1}{2\sqrt{x}}$

Ex. Suppose $\boxed{f(2)=3}$, $\boxed{f'(2)=\frac{1}{4}}$.

Find $\frac{d}{dx}\left(x f(x)\right)$ at $x=2$.

Sol. $\frac{d}{dx}\left(x f(x)\right)\Big|_{x=2} = x f'(x) + f(x)\cdot 1 \Big|_{x=2}$

$= 2f'(2) + f(2)$

$= 2\left(\frac{1}{4}\right) + 3 = \frac{7}{2}$.

Ex. Suppose that f is diffble and $\boxed{y = \frac{f(x)}{x^2}}$, $x \neq 0$

find y'.

Sol. $y' = \frac{x^2 \cdot f'(x) - f(x)\cdot 2x}{(x^2)^2} = \frac{x^2 f'(x) - 2x f(x)}{x^4}$

$= \frac{x f'(x) - 2f(x)}{x^3}$, $x \neq 0$.

35. Suppose that $f(2) = -4$, $g(2) = 3$, $f'(2) = 1$, and $g'(2) = -2$. Find

$$\boxed{(fg)'(2)}$$

Sol. $(fg)'(2) = f(2)g'(2) + f'(2)g(2)$.

$= (-4)(-2) + (1)(3)$

$= 8 + 3 = 11$.

قاعدة السلسلة . المشتقات العليا

4.4 The Chain Rule and Higher Derivatives

4.4.1 The Chain Rule

Chain Rule If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $\boxed{(f \circ g)(x) = f[g(x)]}$ is differentiable at x, and the derivative is given by

$$(f \circ g)'(x) = f'[g(x)]g'(x)$$

Ex1 $y = (3x^2-1)^5$, find y'

Sol. $y' = 5(3x^2-1)^4 \cdot \frac{d}{dx}(3x^2-1)$

$= 5(3x^2-1)^4 \cdot 6x = 30x(3x^2-1)^4$.

ex. 2 $h(x) = (2x+1)^3$, find $h'(x)$

Sol. $h'(x) = 3(2x+1)^2\,(2x+1)'$

$= 3(2x+1)^2(2) = 6(2x+1)^2$.

ex. 3 $y = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$, find y'

$y' = \frac{1}{2}(x^2+1)^{\frac{1}{2}-1}(2x) = x(x^2+1)^{-\frac{1}{2}}$

$= \frac{x}{\sqrt{x^2+1}}$

In general, $y = \sqrt{f(x)} \Rightarrow y' = \frac{f'(x)}{2\sqrt{f(x)}}$, $f(x) > 0$.

ex 4. $f(x) = \sqrt[7]{2x^2+3x} = (2x^2+3x)^{\frac{1}{7}}$

$f'(x) = \frac{1}{7}(2x^2+3x)^{-\frac{6}{7}} \cdot (2x^2+3x)'$

$= \frac{1}{7}(2x^2+3x)^{-\frac{6}{7}}(4x+3).$

$= \dfrac{4x+3}{7(2x^2+3x)^{\frac{6}{7}}}.$

ex 5. $h(x) = \left(\frac{x}{x+1}\right)^2$, find $h'(x)$.

Sol. $h'(x) = 2\left(\frac{x}{x+1}\right)^1 \cdot \left(\frac{x}{x+1}\right)'$

$= \frac{2x}{x+1} \cdot \frac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \frac{2x}{x+1} \cdot \frac{1}{(x+1)^2} = \frac{2x}{(x+1)^3}.$

ex. 6 $h(x) = (ax^2-2)^n$, a constant.

$h'(x) = n(ax^2-2)^{n-1} \cdot (2ax)$

$= 2nax\,(ax^2-2)^{n-1}.$

ex. 7 Suppose $f_{>0}$ is diffble, find $\frac{d}{dx}\left(\frac{1}{\sqrt{f(x)}}\right)$

Sol $\frac{d}{dx}(f(x))^{-\frac{1}{2}} = -\frac{1}{2}(f(x))^{-\frac{3}{2}} \cdot f'(x) = -\dfrac{f'(x)}{2[f(x)]^{3/2}}$

ex.8. $\dfrac{d}{dx}\left(\dfrac{a}{f(x)}\right)$ → a constant, f diffble

$$= a \frac{d}{dx}\left(f(x)\right)^{-1} = a(-1)\left(f(x)\right)^{-2} \cdot f'(x)$$

$$= \frac{-a f'(x)}{\left(f(x)\right)^2}.$$

ex9. Suppose $\boxed{f'(x) = 3x-1}$. Find

$$\frac{d}{dx} f(x^2) \Big|_{x=3}.$$

Sol. $$\frac{d}{dx}\left[f(x^2)\right]\Big|_{x=3} = f'(x^2) \cdot 2x \Big|_{x=3}$$

$$= f'(9) \cdot 6 = \left(3(9)-1\right)(6)$$

$$= (26)(6)$$

$$= 156.$$

ex.10. $h(x) = \left(\underbrace{\sqrt{x^2+1}+1}\right)^2$

$$h'(x) = 2\left(\sqrt{x^2+1}+1\right)^1 \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}$$

$$= \frac{2x\left(\sqrt{x^2+1}+1\right)}{\sqrt{x^2+1}}$$

ex 11. $h(x) = \left(2x^3 - \sqrt{3x^4-2}\right)^3$

$$h'(x) = 3\left(2x^3 - \sqrt{3x^4-2}\right)^2 \cdot \left(6x^2 - \frac{12x^3}{2\sqrt{3x^4-2}}\right)$$

$= \dots$ بسّطها

الاشتقاق الضمني

4.4.2 Implicit Functions and Implicit Differentiation

$$\boxed{y^5x^2 - yx + 2y^2 = \sqrt{x}}$$

y is implicitly a function of x.

$$\boxed{y = x^5 + x^2 + 1}$$

y is explicitly a function of x.

ex 12. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$. $(y(x))^2$

Sol. $\frac{d}{dx}(x^2) + \underbrace{\frac{d}{dx}(y^2)}_{\text{chain Rule}} = \frac{d}{dx}(1)$.

$$2x + 2y\frac{dy}{dx} = 0.$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

ex 13. Find $\frac{dy}{dx}$ if $y^3x^2 - yx + 2y^2 = x$.

Sol. $\frac{d}{dx}(y^3x^2) - \frac{d}{dx}(yx) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(x)$.

$$y^3 \cdot 2x + x^2 \cdot 3y^2 y' - (y \cdot 1 + x \cdot y') + 4yy' = 1$$

$$3x^2y^2y' - xy' + 4yy' = 1 + y - 2xy^3.$$

$$(3x^2y^2 - x + 4y)y' = 1 + y - 2xy^3.$$

$$y' = \frac{1+y-2xy^3}{3y^2x^2-x+4y}.$$

Ex 14. If $\boxed{y^2 = x^3, \; x>0, \; y>0}$. show that $y' = \boxed{\frac{3}{2}\sqrt{x}}$.

Sol.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3).$$

$$2yy' = 3x^2$$

$$\boxed{y' = \frac{3x^2}{2y}}$$

but $\sqrt{y^2} = \sqrt{x^3}$

$|y| = x^{\frac{3}{2}}$

$\boxed{y = x^{\frac{3}{2}}, \; y>0}$

$$= \frac{3x^2}{2x^{3/2}}$$

$$= \frac{3}{2}x^{2-\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

4.4 — 4.4.1, 4.4.2, 4.4.3, 4.4.4

4.4.3 Related Rates

EXAMPLE 15 Find $\frac{dy}{dt}$ when $x^2 + y^3 = 1$ and $\frac{dx}{dt} = 2$ for $x = \sqrt{7/8}$.

$y = y(t)$, $x = x(t)$.

Sol.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^3) = \frac{d}{dt}(1)$$

$$\boxed{2x\frac{dx}{dt} + 3y^2 \frac{dy}{dt} = 0} \;??$$

$x^2 + y^3 = 1$

$\left(\sqrt{\frac{7}{8}}\right)^2 + y^3 = 1$

$\frac{7}{8} + y^3 = 1$

$y^3 = \frac{1}{8}$

$\boxed{y = \frac{1}{2}}$

$$2\sqrt{\frac{7}{8}}(2) + 3\left(\frac{1}{2}\right)^2 \frac{dy}{dt} = 0$$

$$4\sqrt{\frac{7}{8}} = -\frac{3}{4}\frac{dy}{dt}$$

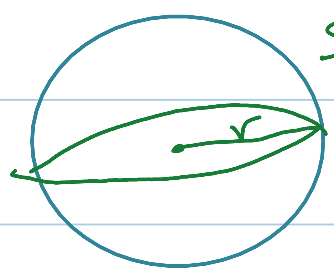
$$\therefore \frac{dy}{dt} = \left(4\sqrt{\frac{7}{8}}\right)\left(\frac{-4}{3}\right) = \frac{-16}{3}\sqrt{\frac{7}{8}} .$$

EXAMPLE 16 **Changing Volume** A spherical balloon is being filled with air. When the radius $r = 6$ cm, the radius is increasing at a rate of 2 cm/s. How fast is the volume changing at this time?

$\frac{dr}{dt} = 2$

Sol.

$$\boxed{V = \frac{4}{3}\pi r^3}$$



$$??\quad \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$$

$$= 4\pi(6)^2(2) = 288\pi \text{ cm}^3/\text{s}.$$

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61. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $x^2 + y^2 = 1$, $\frac{dx}{dt} = 2$ for $x = \frac{1}{2}$, and $y > 0$.

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$
$$\frac{1}{4} + y^2 = 1$$
$$y^2 = \frac{3}{4}$$
$$y = \frac{\sqrt{3}}{2}$$

Sol.
$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(1)$$

??
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2\left(\frac{1}{2}\right)(2) + 2\left(\frac{\sqrt{3}}{2}\right)\frac{dy}{dt} = 0$$

$$2 = -\sqrt{3}\frac{dy}{dt} \Rightarrow \boxed{\frac{dy}{dt} = \frac{-2}{\sqrt{3}}}$$

66. Assume that the radius r and the area $A = \pi r^2$ of a circle are differentiable functions of t. Express dA/dt in terms of dr/dt.

Sol.
$$A = \pi r^2 \Rightarrow \boxed{\frac{dA}{dt} = 2\pi r\frac{dr}{dt}}$$

4.4.4 Higher Derivatives

المشتقات العليا.

$y = f(x)$, $y' = f'(x)$ first derivative

$$\frac{d^2y}{dx^2} = y'' = f''(x) = f^{(2)}(x) \quad \text{second derv.}$$

$$\frac{d^3y}{dx^3} = y''' = f'''(x) = f^{(3)}(x) \quad \text{third derv.}$$

$\vdots$

$$\frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x) \quad \text{nth derivative.}$$

ex: $f(x) = x^5$, find $f^{(n)}(x)$, $n=1,2,\dots$

Sol. $\boxed{f'(x) = 5x^4}$

$\boxed{f''(x) = 20x^3}$, $\boxed{f'''(x) = 60x^2}$

$\boxed{f^{(4)}(x) = 120x}$

$\boxed{f^{(5)}(x) = 120}$, $\boxed{f^{(6)}(x) = 0}$

$\vdots$

$\boxed{f^{(n)}(x) = 0}$, $n = 6, 7, 8, \dots\dots$

ex. $y = \sqrt{x}$ find $y^{(2)}$.

Sol. $y' = \dfrac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow y'' = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$= -\frac{1}{4}x^{-\frac{3}{2}}$

ex. Find $\dfrac{d^3y}{dx^3}$ if $\boxed{x^2+y^2=1}$

Sol. $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$

$2x + 2y\,y' = 0 \Rightarrow \boxed{y' = -\dfrac{x}{y}}$

$$y'' = \frac{y\cdot(-1) - (-x)\cdot y'}{y^2}$$

$$= \frac{-y + x\cdot\frac{-x}{y}}{y^2}$$

من المعادلة الأصلية

$$= \frac{-(y^2+x^2)}{y^3} = -\frac{1}{y^3}$$

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$$y'' = -y^{-3}$$

$$y''' = -(-3)\,y^{-4}\,y'$$

$$= 3\,y^{-4}\cdot\left(\frac{-x}{y}\right) = \frac{-3x}{y^5}.$$

موقعه

EXAMPLE 21 **Acceleration** Assume that the position of a car moving along a straight line is given by

$$\boxed{s(t) = 3t^3 - 2t + 1}$$

Find the car's velocity and acceleration.

Sol. $V(t) = s'(t) = 3(3t^2) - 2 = 9t^2 - 2.$

$a(t) = s''(t) = 9(2t) = 18t.$

jerk $j(t) = s'''(t) = 18.$

4.5 Derivatives of Trigonometric Functions

احفظ

(1) $\dfrac{d}{dx}\sin x = \cos x$ ✓

(2) $\dfrac{d}{dx}\cos x = -\sin x$ ✓

(3) $\dfrac{d}{dx}\tan x = \sec^2 x$

(4) $\dfrac{d}{dx}\cot x = -\csc^2 x$

(5) $\dfrac{d}{dx}\sec x = \sec x \tan x$

(6) $\dfrac{d}{dx}\csc x = -\csc x \cot x$

Pf.
$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

(5)
$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}[\cos x]^{-1}$$
$$= -1(\cos x)^{-2}\cdot(-\sin x)$$
$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x.$$

EXAMPLE 1 Find the derivative of $f(x) = -4\sin x + \cos\frac{\pi}{6}$.

Constant

Sol.
$$f'(x) = -4\cos x + 0 = -4\cos x.$$

EXAMPLE 2 Find the derivative of $y = \cos(x^2 + 1)$.

$$y' = -\sin(x^2+1)\cdot(x^2+1)' = -\sin(x^2+1)(2x)$$
$$= -2x\sin(x^2+1).$$

EXAMPLE 3 Find the derivative of $y = x^2 \sin(3x) - \cos(5x)$.

Sol. $y' = (x^2)(\cos(3x) \cdot 3) + (\sin 3x)(2x) + \sin(5x).5$

$= 3x^2 \cos(3x) + 2x \sin(3x) + 5 \sin(5x).$

EXAMPLE 4 Compare the derivatives of

(a) $\tan x^2$ **(b)** $\tan^2 x$

Sol. (a) $y = \tan x^2 = \tan(x^2)$

$y' = \sec^2(x^2) \cdot 2x = 2x \sec^2(x^2).$

(b) $y = \tan^2 x = (\tan x)^2$

$y' = 2(\tan x)^1 \cdot \sec^2 x = 2 \tan x \cdot \sec^2 x.$

EXAMPLE 5 **Repeated Application of the Chain Rule** Find the derivative of $f(x) = \sec \sqrt{x^2+1}$.

Sol. $f(x) = \sec(\sqrt{x^2+1})$.

$f'(x) = \sec(\sqrt{x^2+1}) \tan(\sqrt{x^2+1}) \cdot (\sqrt{x^2+1})'$

$= \dfrac{x}{\sqrt{x^2+1}} \sec(\sqrt{x^2+1}) \tan(\sqrt{x^2+1}).$

29. $f(x) = \sqrt{\sin(2x^2-1)}$

$f'(x) = \dfrac{\cos(2x^2-1) \cdot 4x}{2\sqrt{\sin(2x^2-1)}} = \dfrac{2x \cos(2x^2-1)}{\sqrt{\sin(2x^2-1)}}.$

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47. $g(x) = \dfrac{1}{\csc^3(1-5x^2)} = \sin^3(1-5x^2).$

$$= \left(\sin(1-5x^2)\right)^3$$

$$g'(x) = 3\left(\sin(1-5x^2)\right)^2 \cdot \cos(1-5x^2)\cdot(-10x).$$

$$= -30x \sin^2(1-5x^2)\cos(1-5x^2).$$

57. $f(x) = \dfrac{\sec x^2}{\sec^2 x} = \sec(x^2)\cdot(\cos x)^2$

$$y' = \left(\sec(x^2)\right)\left(2(\cos x)^1(-\sin x)\right)$$

$$+ (\cos x)^2\left(\sec x^2 \tan x^2\right)\cdot 2x.$$

$= \text{- - - . .}$

60. Find the points on the curve $y = \cos^2 x$ that have a horizontal tangent.

تذكر ان

$$\boxed{\sin(2A) = 2\sin A\cos A}$$

Sol. $y = (\cos x)^2$

$$y' = 2(\cos x)(-\sin x) = 0.$$

$$\Rightarrow -\sin(2x) = 0$$

$$\Rightarrow \boxed{\sin(2x) = 0}$$

$\boxed{\sin A = 0}$
iff $A = n\pi$
$n = 0, \pm 1, \pm 2, \ldots$

$\boxed{y = \frac{1}{x}}$

$y^{(n)} = ??$

$2x = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$

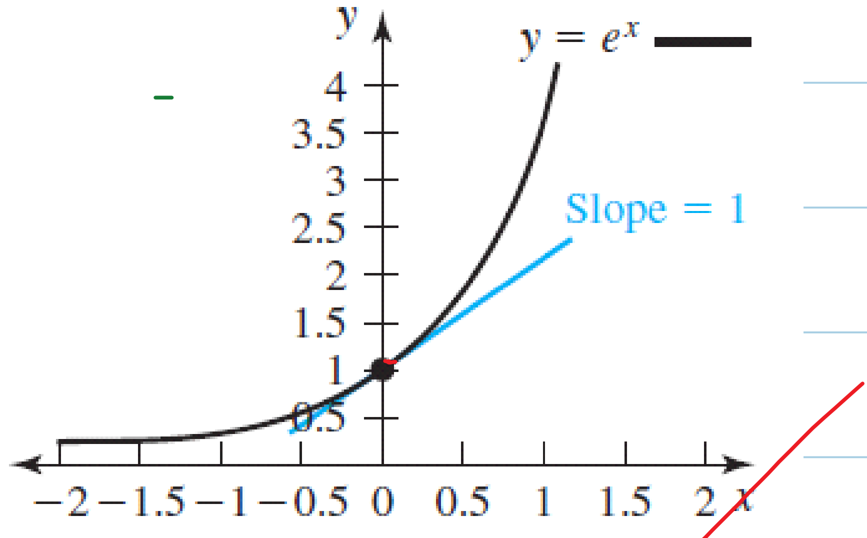
$\boxed{x = \frac{n\pi}{2}}, \quad n = 0, \pm 1, \pm 2, \ldots$

Points: $\left(\frac{n\pi}{2}, \cos^2\left(\frac{n\pi}{2}\right)\right), \quad n = 0, \pm 1, \pm 2, \ldots$

4.6 Derivatives of Exponential Functions

$$\boxed{\lim_{h\to 0}\frac{e^h-1}{h}=1} \quad \left(\tfrac{0}{0}\right) \quad \to \ln e = 1$$

h	0.1	0.01	0.001	0.0001
$\frac{e^h-1}{h}$	1.0517	1.0050	1.00050	1.000050



$$\lim_{h\to 0}\frac{a^h-1}{h}=\boxed{\ln a}$$

$$\boxed{\frac{d}{dx}e^x=e^x}$$

$$\boxed{\frac{d}{dx}a^x=(\ln a)a^x}$$

$$\boxed{\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}}$$

$$\boxed{\frac{d}{dx}\left[a^{g(x)}\right]=\underbrace{a^{g(x)}}_{\text{نفسه}}\cdot\underbrace{g'(x)}_{\text{مشتقة الأس}}\cdot\underbrace{\ln a}_{\text{Ln الأساس}}} \quad (*)$$

$$\frac{d}{dx}(a^x)=\left[e^{\ln(a^x)}\right]'$$

$$\boxed{e^{\ln A}=A}, \quad A>0.$$

$$=\left(e^{\overset{g(x)}{x\ln a}}\right)'=e^{x\ln a}\cdot(x\ln a)'$$

$$=e^{\ln(a^x)}\cdot(\ln a)=a^x(\ln a)$$

$$\Rightarrow \frac{d}{dx}(a^x)=a^x(\ln a)$$

EXAMPLE 1 Find the derivative of $f(x)=e^{-x^2/2}$.

$$f'(x)=e^{-\frac{x^2}{2}}\cdot\left(-\frac{x^2}{2}\right)'\cdot\ln e$$

$$=e^{-\frac{x^2}{2}}\left(-\frac{\cancel{2}x}{\cancel{2}}\right)\cdot 1=-x\,e^{-\frac{x^2}{2}}.$$

Ex2 $y=3^{\sqrt{x}} \Rightarrow y'=3^{\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x}}\right)(\ln 3) = \frac{\ln 3}{2}\,\frac{3^{\sqrt{x}}}{\sqrt{x}}.$

ex3. $y = e^{\sin(\sqrt{x})}$

$$y' = e^{\sin(\sqrt{x})} \cdot (\sin\sqrt{x})' \cdot (\ln e)$$

$$= e^{\sin(\sqrt{x})} \cdot \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

ex4. Find $\lim_{h\to 0} \frac{3^{2h} - 1}{h-0.}$ $\quad f(h)$, $f(0)$ $\quad 1 = 3^{2(0)} = f(0)$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = f'(0), \quad f(x) = 3^{2x}$$

$$f'(x) = 3^{2x} \cdot (2) \ln 3$$

$$= 3^{0} (2) \ln 3$$

$$= 2\ln 3 = \ln 3^2 = \ln 9.$$

Q53) $\lim_{h\to 0} \frac{e^{2h} - 1}{h-0} = f'(0)$ where $f(x) = e^{2x}$ $\quad$ ($f(h)$, $f(0)$)

$$= 2e^{0} = 2(1) \qquad f'(x) = e^{2x} \cdot (2) \ln e$$

$$= 2. \qquad = 2e^{2x}..$$

Q54) $\lim_{h\to 0} \frac{e^{5h} - 1}{3h} = \frac{1}{3} \lim_{h\to 0} \frac{f(h)-f(0)}{h-0}$ $\quad$ ($f(h)$, $f(0)$)

$$= \frac{1}{3} f'(0), \text{ where } f(x) = e^{5x}.$$

$$f'(x) = 5e^{5x}$$

$$= \frac{1}{3}(5e^{0})$$

$$= \frac{5}{3}.$$

(20) $f(x) = e^{\cos(1-2x^3)}$

$$f'(x) = e^{\cos(1-2x^3)} \cdot (-\sin(1-2x^3)) \cdot (-6x^2).$$

(48) $h(t) = 6^{\sqrt{6t^6-6}}$ $\Rightarrow h'(t) = 6^{\sqrt{6t^6-6}} \cdot \left(\frac{36t^5}{2\sqrt{6t^6-6}}\right) \cdot (\ln 6)$

4.7.1 4.7.2 4.7.3

4.7 Derivatives of Inverse Functions, Logarithmic Functions, and the Inverse Tangent Function

4.7.1 Derivatives of Inverse Functions

Derivative of an Inverse Function If $f(x)$ is one to one and differentiable with inverse function $f^{-1}(x)$ and $f'[f^{-1}(x)] \neq 0$, then $f^{-1}(x)$ is differentiable and

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'[f^{-1}(x)]} \tag{4.12}$$

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{\boxed{x=b}} = \frac{1}{\left.\frac{df}{dx}\right|_{\boxed{x=f^{-1}(b)}}}$$

$y = f(x)$

$b = f(x)$

$\vdots$

$x = ??$

Ex ①. $\boxed{f(x) = \frac{x}{1+x}}$, $x \geq 0$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=\frac{1}{3}}$.

Sol.

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{\boxed{x=\frac{1}{3}}} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}\left(\frac{1}{3}\right)}}}$$

$f^{-1}\left(\frac{1}{3}\right) = ??$

$\frac{1}{3} = \frac{x}{1+x}$

$3x = 1 + x$

$2x = 1$

$x = \frac{1}{2} = f^{-1}\left(\frac{1}{3}\right)$

$$= \frac{1}{f'\left(\frac{1}{2}\right)}$$

$$f'(x) = \frac{(1+x)(1) - x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$f'\left(\tfrac{1}{2}\right) = \frac{1}{\left(1+\frac{1}{2}\right)^2} = \frac{1}{9/4} = \frac{4}{9}.$$

$$\therefore \left.\frac{df^{-1}}{dx}\right|_{x=\frac{1}{3}} = \frac{1}{f'\left(\frac{1}{2}\right)} = \frac{9}{4}.$$

ex ② . $\boxed{f(x) = 2x + e^x}$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=1}$

Sol. $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \dfrac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(1)}}$

$f^{-1}(1) = ??$

$\boxed{1 = 2x + e^x}$

by inspection
تخمين

$x = 0 = f^{-1}(1)$

$= \dfrac{1}{f'(0)}$

$= \left.\dfrac{1}{2+e^x}\right|_{x=0}$

$= \dfrac{1}{2+1} = \dfrac{1}{3}$

ex ③ $f(x) = \tan x \;,\; -\frac{\pi}{2} < x < \frac{\pi}{2}$

Find $\left.\frac{df^{-1}}{dx}\right|_{x=1}$.

Sol. $\boxed{f'(x) = \sec^2 x}$.

$f^{-1}(1) = ??$ $\quad \tan x = 1$

$x = \frac{\pi}{4} = f^{-1}(1)$

$\left.\frac{df^{-1}}{dx}\right|_{x=1} = \dfrac{1}{f'(\frac{\pi}{4})} = \dfrac{1}{\sec^2(\frac{\pi}{4})} = \dfrac{1}{2}$

16. Let

$$f(x) = x^2 + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{Find } \frac{d}{dx} f^{-1}(x)\Big|_{x=\frac{\pi^2}{16}+1}$$

Sol. $f'(x) = 2x + \sec^2 x$. $\quad f^{-1}\left(\frac{\pi^2}{16}+1\right) = ??$

$\frac{\pi^2}{16} + 1 = x^2 + \tan x$.

$\left(\frac{\pi}{4}\right)^2 + 1 = x^2 + \tan x$

$x = \frac{\pi}{4}$ بالتخمين

$\left.\frac{df^{-1}}{dx}\right|_{x=\frac{\pi^2}{16}+1} = \dfrac{1}{f'(\frac{\pi}{4})}$

$= \dfrac{1}{2\frac{\pi}{4} + \sec^2(\frac{\pi}{4})}$

$$= \frac{1}{\frac{\pi}{2}+2} = \frac{2}{\pi+4}.$$

If $f(x) = \tan x \Rightarrow \boxed{\frac{df^{-1}}{dx} = \frac{d}{dx}\left(\underbrace{\tan^{-1} x}_{\arctan x}\right) = \frac{1}{1+x^2}}$

$x \in \mathbb{R}$

If $f(x) = \sin x \Rightarrow \boxed{\frac{df^{-1}}{dx} = \frac{d}{dx}\underbrace{\sin^{-1} x}_{\arcsin x} = \frac{1}{\sqrt{1-x^2}}}$

$-1 < x < 1$

ex. $y = \sin^{-1}(x^3)$

$$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot (x^3)' = \frac{3x^2}{\sqrt{1-x^6}}.$$

ex. $y = e^{\tan^{-1} x} \Rightarrow y' = e^{\tan^{-1} x} \cdot \left(\frac{1}{1+x^2}\right)$

ex. $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

4.7.2 The Derivative of the Logarithmic Function

$f(x) = e^x$, $f'(x) = e^x$ and $f^{-1}(x) = \ln x$.

$$\frac{d}{dx}\ln x = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}}$$

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{d}{dx}\left(\frac{\ln g(x)}{\ln a}\right)$$

$$= \frac{1}{\ln a}\frac{d}{dx}(\ln g(x))$$

In general,

$$\boxed{\frac{d}{dx}\log_a(g(x)) = \frac{1}{\ln a}\,\frac{g'(x)}{g(x)}}$$

ex. $y = \ln(3x) \Rightarrow y' = \frac{(3x)'}{3x} = \frac{3}{3x} = \frac{1}{x}$.

ex. $y = \ln(x^2+1) \Rightarrow y' = \frac{2x}{x^2+1}$

ex. $y = \ln(\sin x) \Rightarrow y' = \frac{\cos x}{\sin x} = \cot x$.

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ex. $y = \log(2x^3-1) = \log_{10}(2x^3-1)$

$$y' = \frac{1}{\ln 10} \cdot \frac{(2x^3-1)'}{2x^3-1} = \frac{1}{\ln 10} \cdot \frac{6x^2}{2x^3-1}.$$

Q35) $f(x) = \ln\left(\sqrt{x^2+1}\right) = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+1).$

$$f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}.$$

Q54) $f(x) = \log_{10}(2x^2-1)$

$$f'(x) = \frac{1}{\ln 10} \cdot \frac{4x}{2x^2-1}.$$

59) $f(u) = \log_3(3+u^4)$

$$f'(u) = \frac{1}{\ln 3} \cdot \frac{4u^3}{3+u^4}.$$

ex. $f(x) = \log_2\left[\sin^{-1}(x^2)\right]$

تذكر

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{1}{\ln 2} \cdot \frac{\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x}{\sin^{-1}(x^2)}$$

$$= \frac{2x}{(\ln 2)\sin^{-1}(x^2)\sqrt{1-x^4}}$$

Rmk. $\sin^{-1}x \neq (\sin x)^{-1}$

$\sin^{-1}x$ is sine inverse.

$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$

In general, $f^{-1}(x) \neq \frac{1}{f(x)}$

f^{-1}: the inverse of f.

4.7.3 Logarithmic Differentiation

ex: $y = x^x$ find $\frac{dy}{dx}$.

Sol: We take logarithms to both sides:

$$\ln y = \ln(x^x) = x \ln x.$$

$$\boxed{\ln y = x . \ln x}$$ implicit form.

$$\frac{y'}{y} = x \cdot \frac{1}{x} + (\ln x)(1).$$

$$y' = y(1 + \ln x)$$

$$\boxed{y' = x^x(1 + \ln x)}$$

ex: $y = (\sin x)^x$. Find $\frac{dy}{dx}$

$$\ln y = x \ln(\sin x).$$

$$\frac{1}{y} y' = x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) . 1$$

$$y' = y \left[x \cot x + \ln(\sin x) \right].$$

$$= (\sin x)^x \left(x \cot x + \ln(\sin x) \right).$$

ex. $y = x^r, r \in \mathbb{R}$. show $y' = r x^{r-1}$.

Sol. $\ln y = r \ln x$

$$\frac{y'}{y} = r \cdot \frac{1}{x} \Rightarrow y' = y \cdot r x^{-1}$$
$$= x^r \cdot r x^{-1}$$
$$= r x^{r-1}.$$

ex. $y = a^x \Rightarrow y' = a^x \ln a$.

Pf. $\ln y = x \ln a$

$$\frac{y'}{y} = \ln a \Rightarrow y' = (\ln a) y$$
$$= (\ln a) a^x.$$

ex. $$y = \frac{e^x x^{\frac{3}{2}} \sqrt{1+x}}{(x^2+3)^4 (3x-2)^3}$$

Sol. $$\ln y = \ln\left(e^x \cdot x^{\frac{3}{2}} \cdot \sqrt{1+x}\right) - \ln\left((x^2+3)^4 \cdot (3x-2)^3\right)$$

$$\ln y = \ln e^x + \ln x^{\frac{3}{2}} + \ln (1+x)^{\frac{1}{2}}$$
$$- \left[\ln (x^2+3)^4 + \ln (3x-2)^3\right]$$

$$\ln y = x + \frac{3}{2} \ln x + \frac{1}{2} \ln (x+1) - 4 \ln (x^2+3)$$
$$- 3 \ln (3x-2).$$

→ Now Differentiate:

$$\frac{y'}{y} = 1 + \frac{3}{2} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x+1} - 4 \frac{2x}{x^2+3} - 3 \cdot \frac{3}{3x-2}$$

$$\frac{dy}{dx} = \left(1 + \frac{3}{2x} + \frac{1}{2(1+x)} - \frac{8x}{x^2+3} - \frac{9}{3x-2}\right) \frac{e^x x^{3/2}\sqrt{1+x}}{(x^2+3)^4(3x-2)^3}$$

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$$y = 2 \cdot x^x$$

$$\ln y = \ln 2 + \ln(x^x)$$

$$\ln y = \ln 2 + x \cdot \ln x .$$

$$\frac{y'}{y} = 0 + x \cdot \frac{1}{x} + (\ln x)(1)$$

$$y' = 2x^x(1 + \ln x).$$

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$$f(x) = x^{\frac{3}{x}}$$

$$\ln f(x) = \frac{3}{x}\ln x = \frac{3\ln x}{x}$$

$$\frac{f'(x)}{f(x)} = \frac{x \cdot \frac{3}{x} - (3\ln x)(1)}{x^2}$$

$$f'(x) = \left(\frac{3 - 3\ln x}{x^2}\right) \cdot x^{\frac{3}{x}}$$

$$= 3(1 - \ln x)\, x^{\frac{3}{x} - 2}.$$

72

$$y = (x^x)^x = x^{x^2} \Rightarrow \ln y = x \ln(x^x) = x^2 \ln x.$$

$$\frac{y'}{y} = x^2 \cdot \frac{1}{x} + (\ln x)(2x)$$

$$y' = (x^x)^x \left[x + 2x \ln x \right]$$

$$= x^{x^2+1} \left(1 + 2\ln x \right)$$

76. Differentiate

$$y = \frac{e^{x-1}\sin^2 x}{(x^2+5)^{2x}}$$

$$\ln y = \ln(e^{x-1}) + \ln(\sin^2 x) - \ln(x^2+5)^{2x}$$

$$\ln y = x-1 + 2\ln(\sin x) - 2x \cdot \ln(x^2+5)$$

$$\frac{y'}{y} = 1 + 2\frac{\cos x}{\sin x} - \left[2x \cdot \frac{2x}{x^2+5} + \ln(x^2+5)(2) \right]$$

$$y' = \left(1 + 2\cot x - \frac{4x^2}{x^2+5} - 2\ln(x^2+5) \right) \frac{e^{x-1}\sin^2 x}{(x^2+5)^{2x}} .$$

<u>The End of ch4.</u>

(5.1 – 5.5)

5 Applications of Differentiation

5.1 Extrema and the Mean-Value Theorem

القصوى نظرية القيمة المتوسطة

5.1.1 The Extreme-Value Theorem

Definition Let f be a function defined on the set D that contains the number c. Then

عظمى مطلقة

f has a **global** (or **absolute**) **maximum** at $x = c$ if

$$f(c) \geq f(x) \quad \text{for all } x \in D$$

and

صغرى مطلقة

f has a **global** (or **absolute**) **minimum** at $x = c$ if

$$f(c) \leq f(x) \quad \text{for all } x \in D$$

The Extreme-Value Theorem If f is continuous on a closed interval $[a, b]$, $-\infty < a < b < \infty$, then f has a global maximum and a global minimum on $[a, b]$.

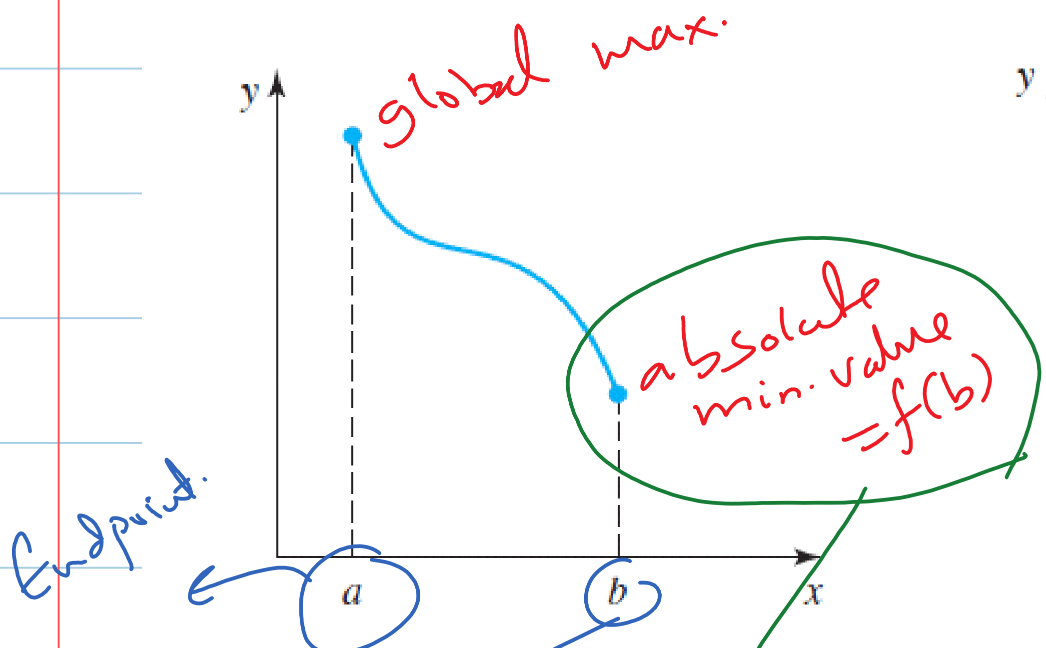

Figure 5.2 Extreme values at the endpoint.

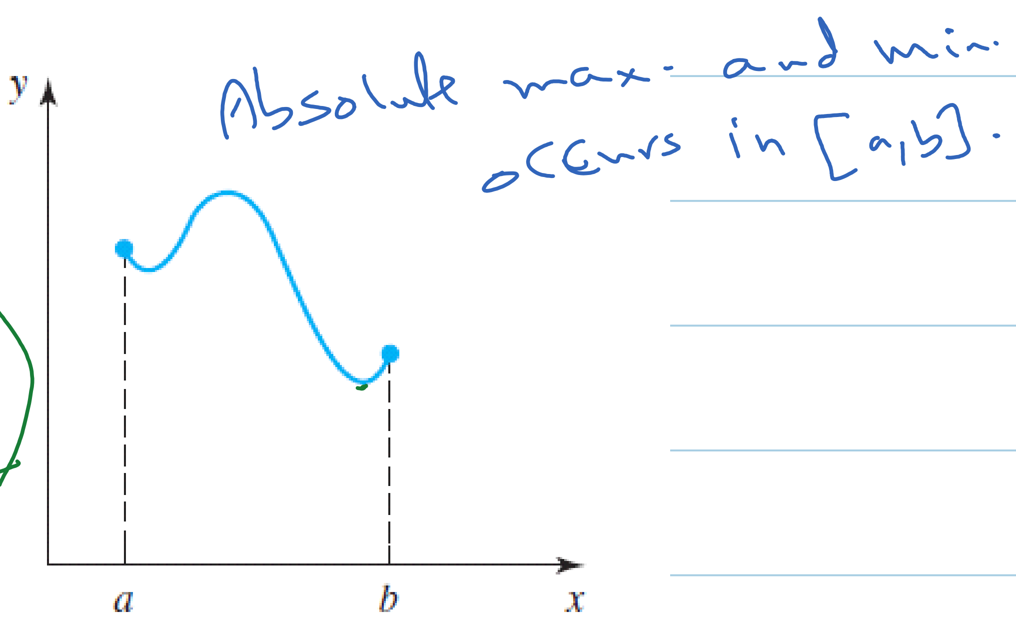

Figure 5.3 Extreme values in the interior.

f has an absolute min. at $x = b$.

Sunday, July 04, 2021
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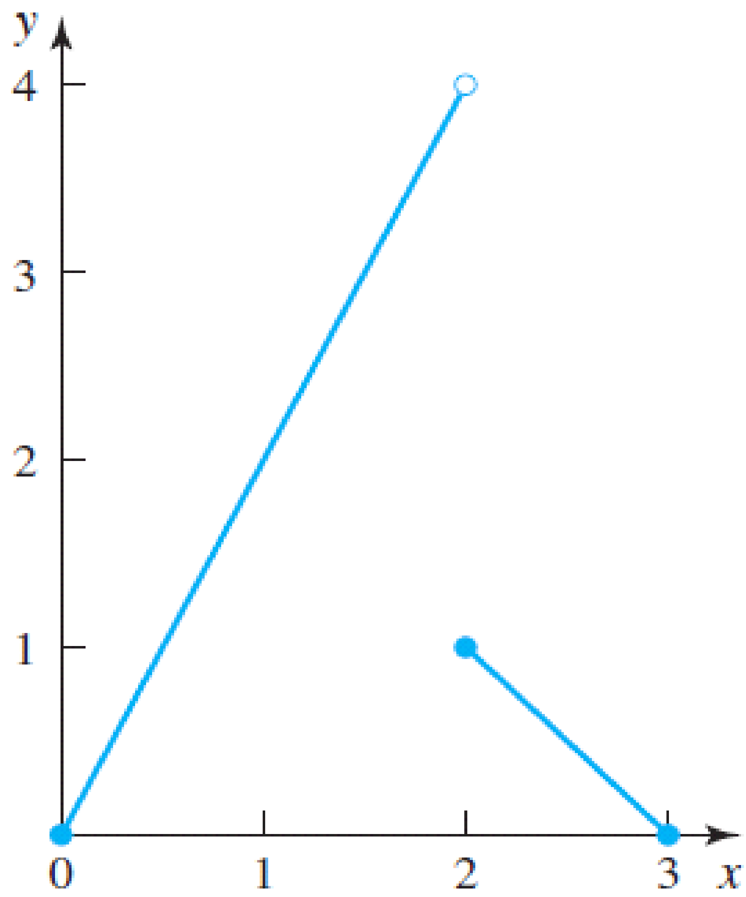
EXAMPLE 2 Let

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < 2 \\ 3 - x & \text{if } 2 \le x \le 3 \end{cases}$$



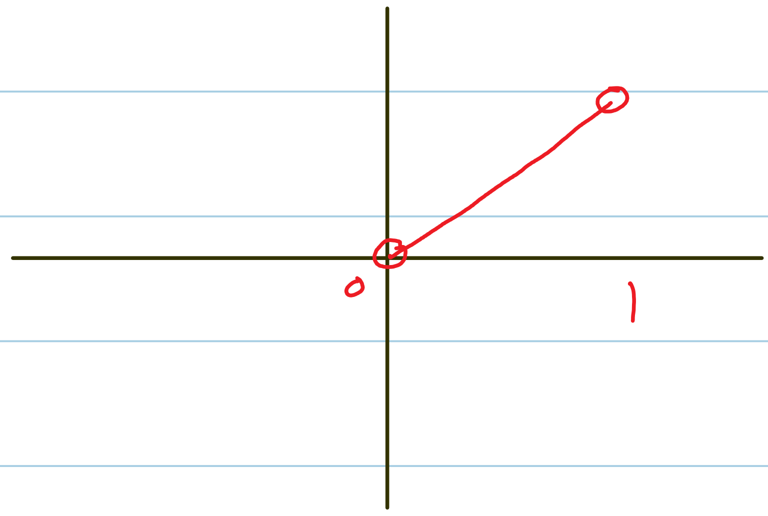
f has absolute min. at $\boxed{x=0}$, $\boxed{x=3}$

Absolute min. value $= f(0) = f(3) = 0$.

f has no global max. values.

EXAMPLE 3 Let

$$\boxed{f(x) = x \qquad \text{for } 0 < x < 1}$$



f is Cont. on (0,1)

(0,1) is not closed

f has neither absolute max. nor absolute min.

relative

5.1.2 Local Extrema

القصوى المحلية.

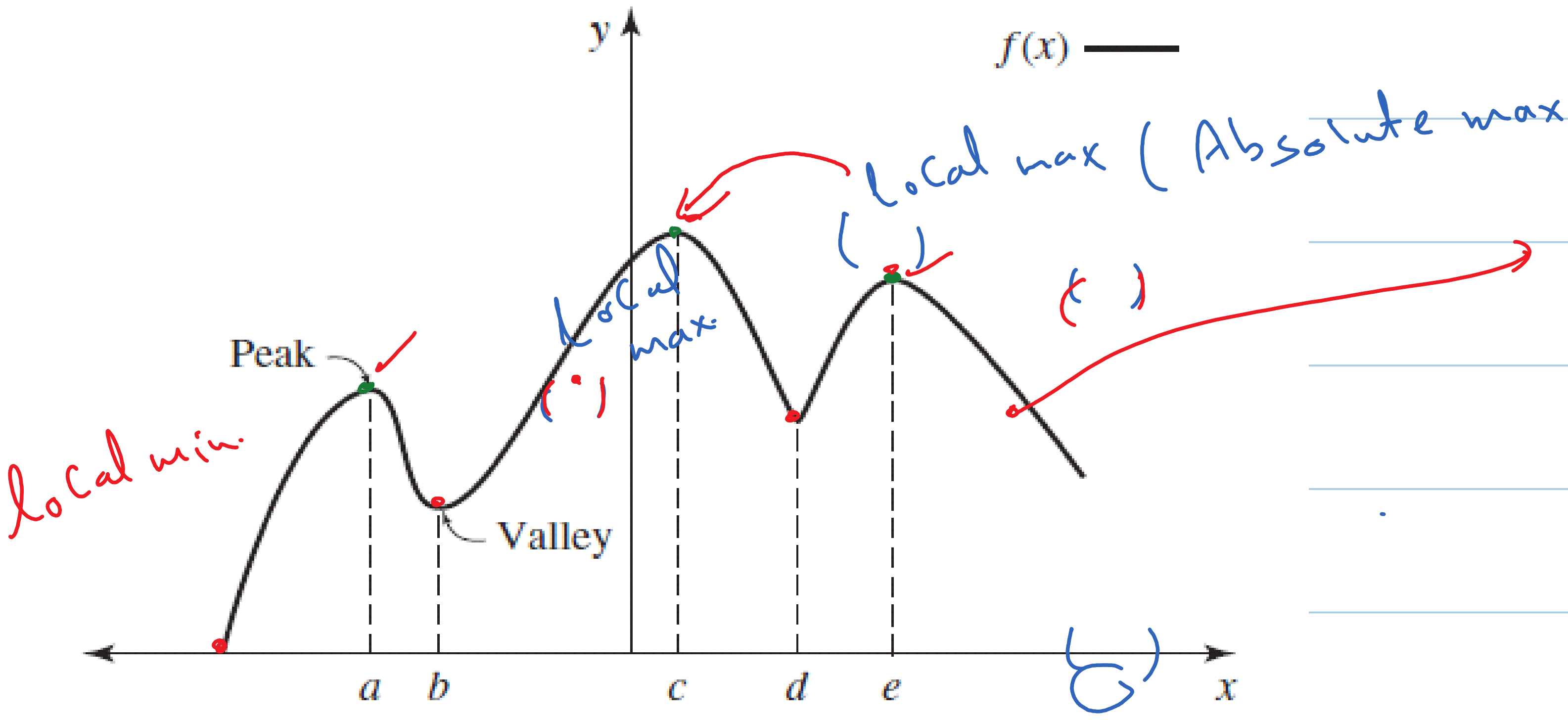
A function f defined on a set D has a **local** (or **relative**) **maximum** at a point c if there exists a $\delta > 0$ such that

around c

$$\boxed{f(c) \ge f(x)} \qquad \text{for all } x \in (c - \delta, c + \delta) \cap D$$

A function f defined on a set D has a **local** (or **relative**) **minimum** at a point c if there exists a $\delta > 0$ such that

$$f(c) \le f(x) \qquad \text{for all } x \in (c - \delta, c + \delta) \cap D$$



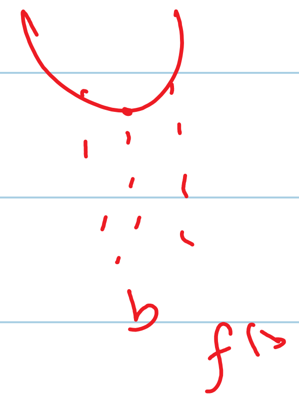
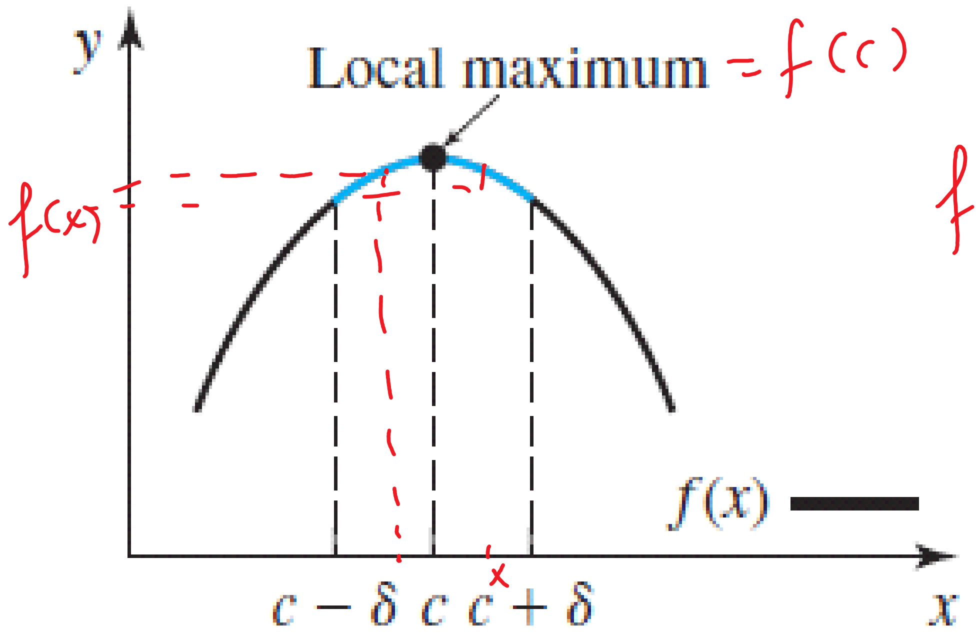


Figure 5.6 The function $y = f(x)$ has valleys at $x = b$ and d and peaks at $x = a, c$, and e.



$f(c) \geq f(x)$
$\forall x \in (c-\delta, c+\delta)$.

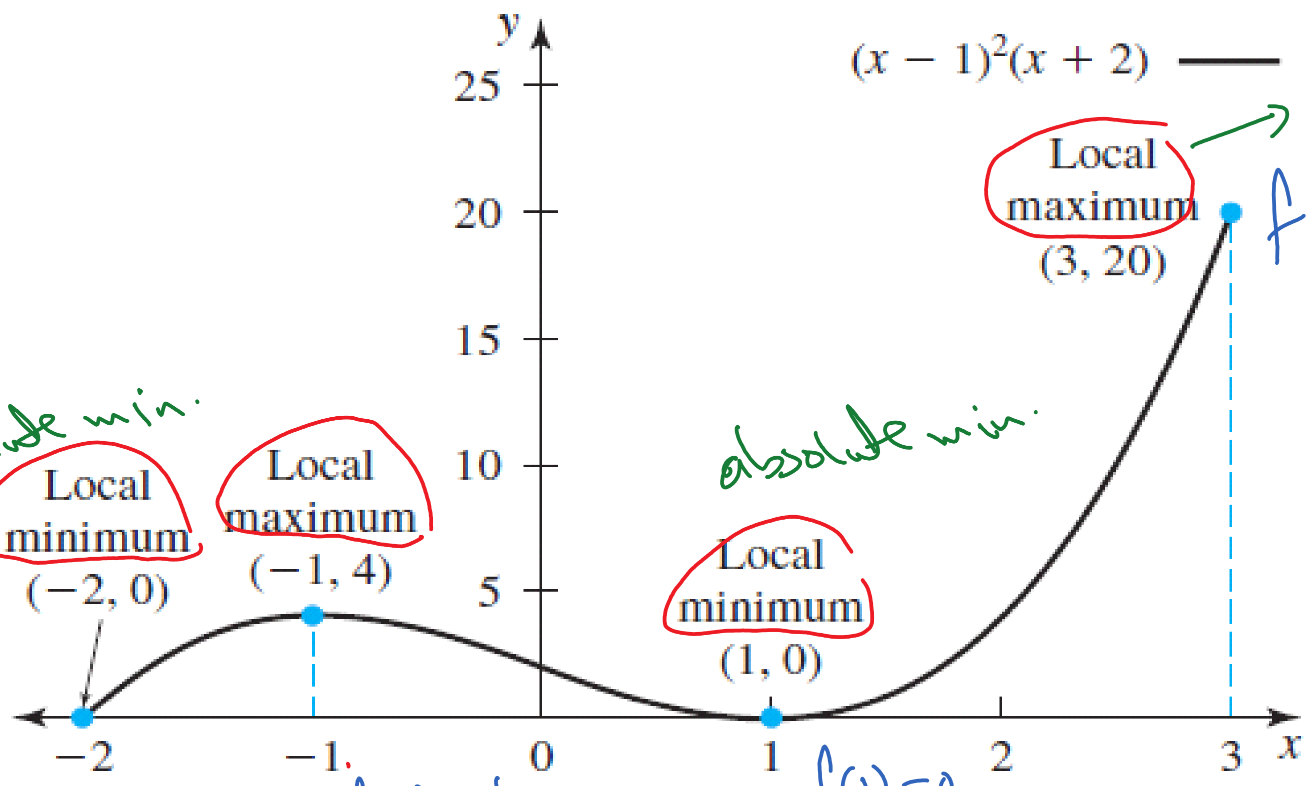
EXAMPLE 4 Let

Cont. on $[-2, 3]$.

$$f(x) = (x-1)^2(x+2) \quad \text{for } \boxed{-2 \le x \le 3}$$

(a) Use the graph of $f(x)$ to find all local extrema.

(b) Find the global extrema.



and absolute max.

$f(3) = 20$

$f(3) = (3-1)^2(3+2)$

$= 4(5) = 20$

absolute min.

absolute min.

$f(-2) = 0$ $\quad f(-1) = 4$ $\quad f(1) = 0$

(a) Local max. $(3, 20)$ and $(-1, 4)$.

$f(3)$ $\quad f(-1)$.

Local min. $(-2, 0)$ and $(1, 0)$

$f(-2)$ $\quad f(1)$

(b) Global max. $(3, 20)$.

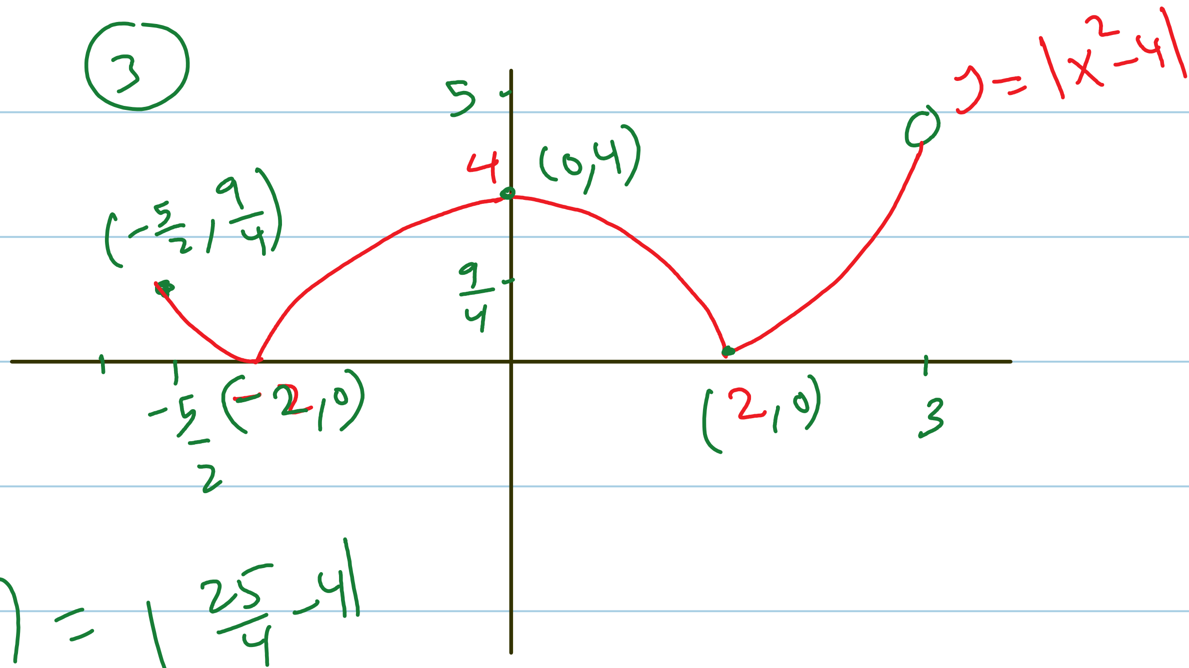
Global min. $(-2, 0)$ and $(1, 0)$.

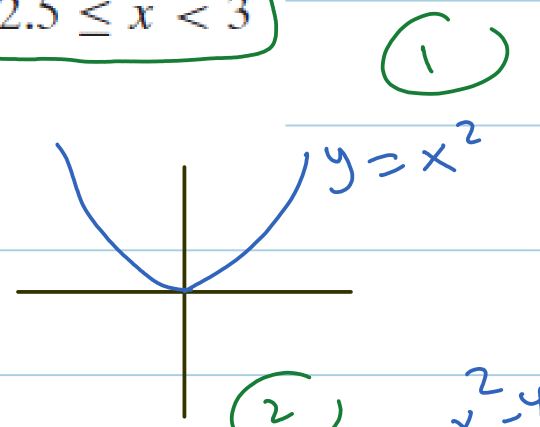
EXAMPLE 5 Let

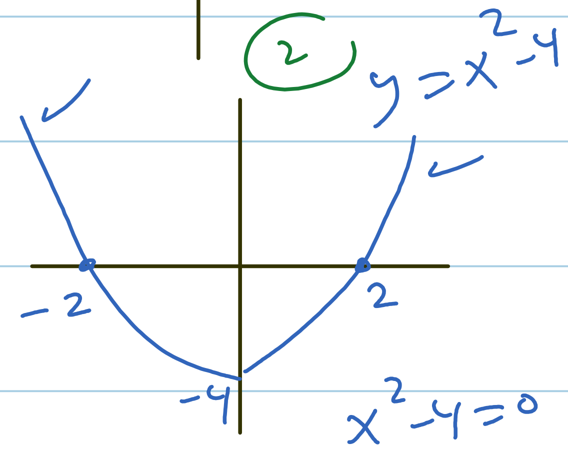
$$f(x) = |x^2 - 4| \quad \text{for} \quad -2.5 \le x < 3$$

Find all local and global extrema.

Solution.







$x^2-4=0$

$x=\pm 2$

$f\left(-\frac{5}{2}\right) = \left|\frac{25}{4} - 4\right|$

$= \frac{9}{4}$

Local max. $\left(-\frac{5}{2}, \frac{9}{4}\right)$ and $(0,4)$.

Local min. $(-2,0)$ and $(2,0)$.

Global max. None

Global min. $(-2,0)$ and $(2,0)$.

Fermat's Theorem If f has a local extremum at an interior point c and $f'(c)$ exists, then $f'(c) = 0$.

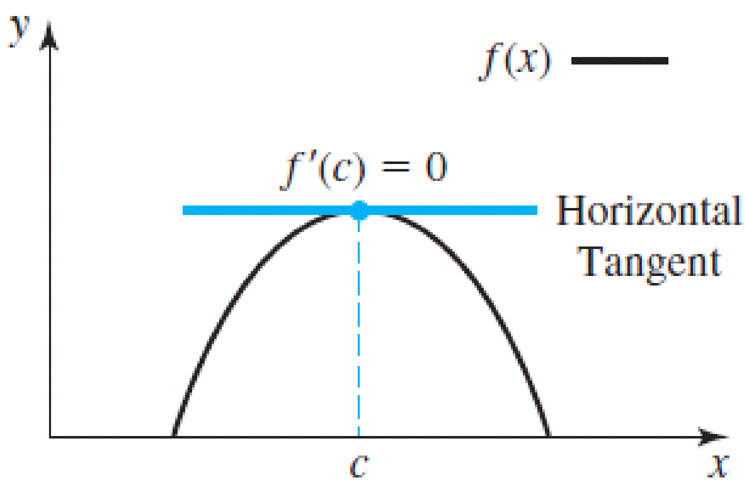


Figure 5.13 Fermat's theorem.

If $f'(c) \neq 0$, f has no extreme value at $x=c$.

If $f'(c) = 0 \Longrightarrow$ Can not tell.

EXAMPLE 6 Explain why $\boxed{y = \tan x}$ does not have a local extremum at $x = 0$.

Sol. $f'(x) = \sec^2 x$

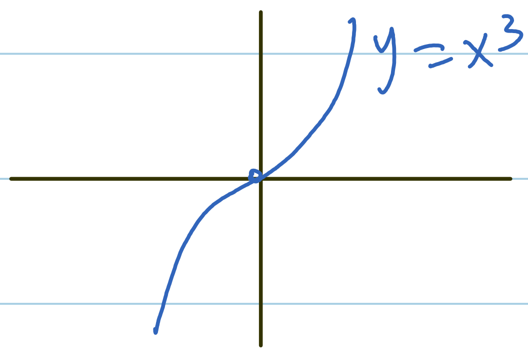
$f'(0) = \sec^2(0) = (1)^2 = 1 \neq 0$

Since $f'(0) \neq 0 \Rightarrow f$ does not have extreme value at $x=0$.

Ex. $y = x^3$ at $x=0$.

$y' = 3x^2$, $y'(0) = 3(0)^2 = 0$.

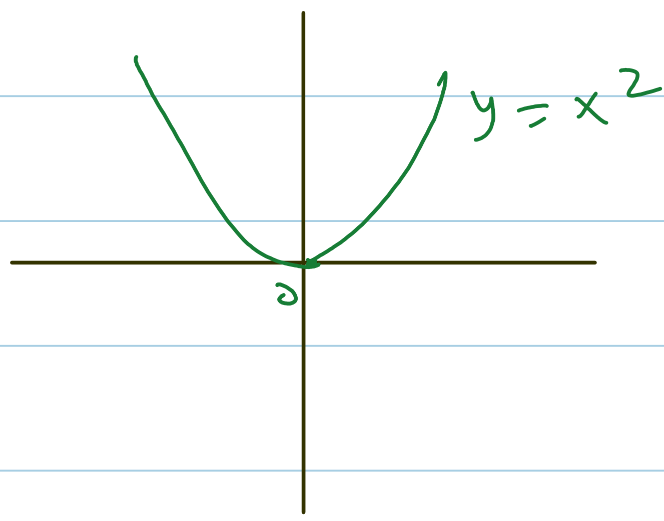
We cannot tell anything by Fermat's thm.



There is NO extreme values at $x=0$.

ex. $y = x^2$ at $x=0$: $y' = 2x$

$y'(0) = 2(0) = 0$.



has absolute min. at $x=0$.

5.1.3 The Mean-Value Theorem

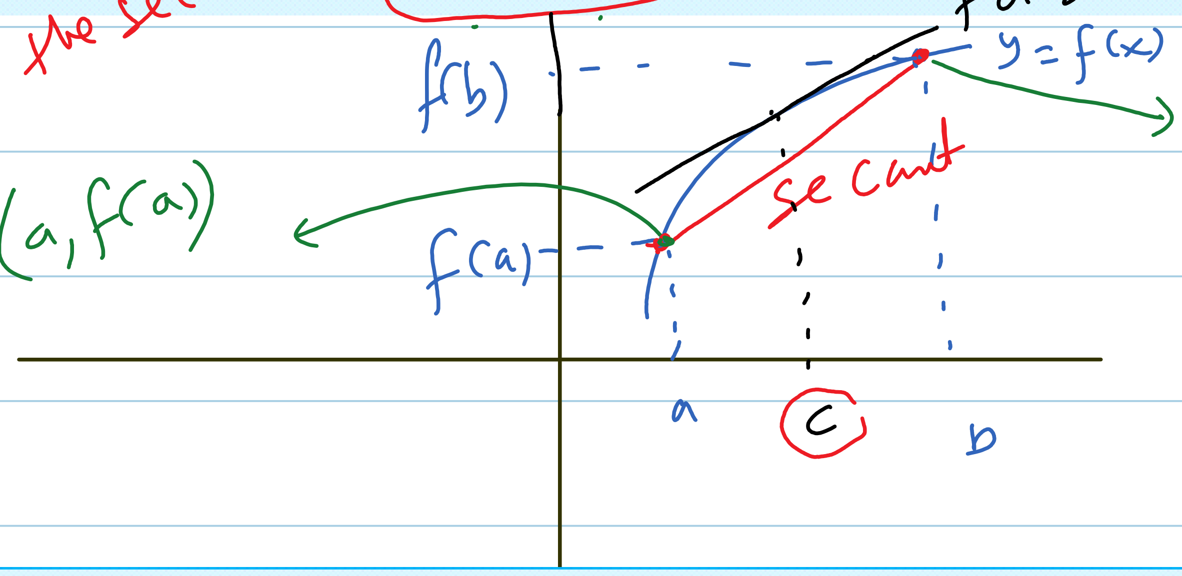
hypothesis

The Mean-Value Theorem (MVT) If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number $c \in (a, b)$ such that

slope of the tangent

conclusion slope of the secant

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



Rolle's Theorem If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

ex. $f(x) = \frac{1}{25}x^3$, $[0, 10]$

show that there is $c \in (0, 10)$ such that $f'(c) = 4$. and find c.

Sol. f is cont. on $[0, 10]$. and diffble on $(0, 10)$ since it is a polynomial.

By MVT, there is $c \in (0, 10)$ such that

$$f'(c) = \frac{f(10) - f(0)}{10 - 0} = \frac{\frac{1}{25}(10)^3 - 0}{10} = 4.$$

$$f(x) = \frac{1}{25}x^3 \Rightarrow f'(x) = \frac{3}{25}x^2$$

$$f'(c) = 4 \Rightarrow \frac{3}{25}c^2 = 4$$

$$c^2 = \frac{100}{3}$$

$c = \pm \frac{10}{\sqrt{3}}$

$c = \frac{10}{\sqrt{3}} \in (0,10)$, $c = -\frac{10}{\sqrt{3}}$ reject $\notin (0,10)$.

EXAMPLE 8 **Population Growth** Denote the population size at time t by $N(t)$, and assume that $N(t)$ is continuous on the interval $[0, 10]$ and differentiable on the interval $(0, 10)$ with $N(0) = 100$ and $|dN/dt| \le 3$ for all $t \in (0, 10)$. What can you say about $N(10)$?

$|N'(t)| \le 3$.

Sol: By MVT, there is $c \in (0,10)$:

$$3 \geq |N'(c)| = \left|\frac{N(10) - N(0)}{10 - 0}\right| = \left|\frac{N(10) - 100}{10}\right|$$

$$\left|\frac{N(10) - 100}{10}\right| \le 3$$

$|x| < a$
$-a < x < a$

$$-3 \le \frac{N(10) - 100}{10} \le 3$$

$$-30 \le N(10) - 100 \le 30$$
$+100 \quad +100 \quad +100$

$$70 \le N(10) \le 130$$

Corollary 2 If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

EXAMPLE 10 Assume that f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, with $f(0) = 2$ and $f'(x) = 0$ for all $x \in (-1, 1)$. Find $f(x)$.

Sol: $f(x) = \text{constant} = c$

$2 = f(0) = c \Rightarrow \boxed{c = 2}$ $\therefore f(x) = 2,\ x \in [-1,1]$

cont. on [0,2] and diff'ble on (0,2)

35. Suppose $f(x) = x^2, x \in [0, 2]$.
(a) Find the slope of the secant line connecting the points (0, 0) and (2, 4).

x_1, y_1 x_2, y_2

(b) Find a number $c \in (0, 2)$ such that $f'(c)$ is equal to the slope of the secant line you computed in (a), and explain why such a number must exist in (0, 2).

Sol. (a) $\text{Slope} = \frac{4-0}{2-0} = 2$

(b) By MVT, there is $c \in (0,2)$:

$f(x) = x^2$

$f'(x) = 2x$

$$f'(c) = \frac{f(2) - f(0)}{2-0}$$

$$2c = \frac{4-0}{2-0} = 2$$

$$\boxed{c = 1} \in (0,2).$$

40. Let $f(x) = 1/(1 + x^2)$. Use the MVT to find an interval that contains a number c such that $f'(c) = 0$.

Sol. $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f'(c) = \frac{-2c}{(1+c^2)^2} = 0$

$\Rightarrow -2c = 0$

$\boxed{c = 0} \in (-1,1)$

مثلاً

Interval : $[-1, 1]$

46. Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Show that if $f(a) < f(b)$, then f' is positive at some point between a and b.

$[a,b]$

By MVT, there is $c \in (a,b)$: $f'(c) = \frac{f(b) - f(a)}{b-a} > 0$

positive

positive

53. Suppose that f is differentiable for all $x \in \mathbf{R}$ with $\boxed{f(2) = 3}$ and $\boxed{f'(x) = 0}$ for all $x \in \mathbf{R}$. Find $f(x)$.

Sol. $f(x) = \text{Constant} = C$

$C = f(2) = 3 \Rightarrow \boxed{C = 3}$

$\therefore f(x) = 3$, for all $x \in \mathbb{R}$.

Discussion on 5.1.1 and 5.1.2

$\underbrace{\boxed{1, 3, 5, 9}}_{5.1.1}, \underbrace{\boxed{14, 19, 28, 31}}_{5.1.2}$

1. $f(x) = 2x - 1, 0 \leq x \leq 1$

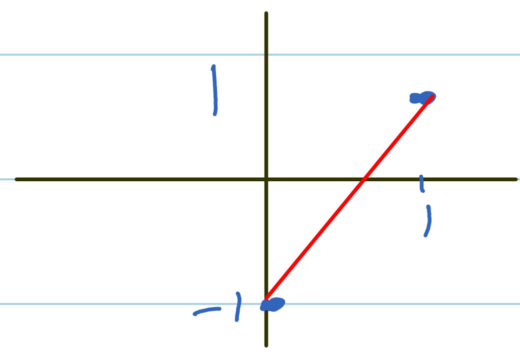
$y = 2x - 1$

$x = 0 \Rightarrow y = -1$

$x = 1 \Rightarrow y = 1$

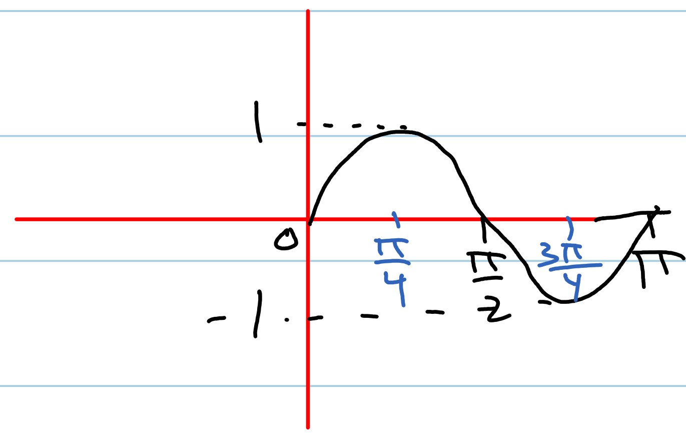
$f(0) = -1$: $(0, -1)$ abs. min.

$f(1) = 1$ $(1, 1)$ abs. max.



3. $f(x) = \sin(2x), 0 \leq x \leq \pi$

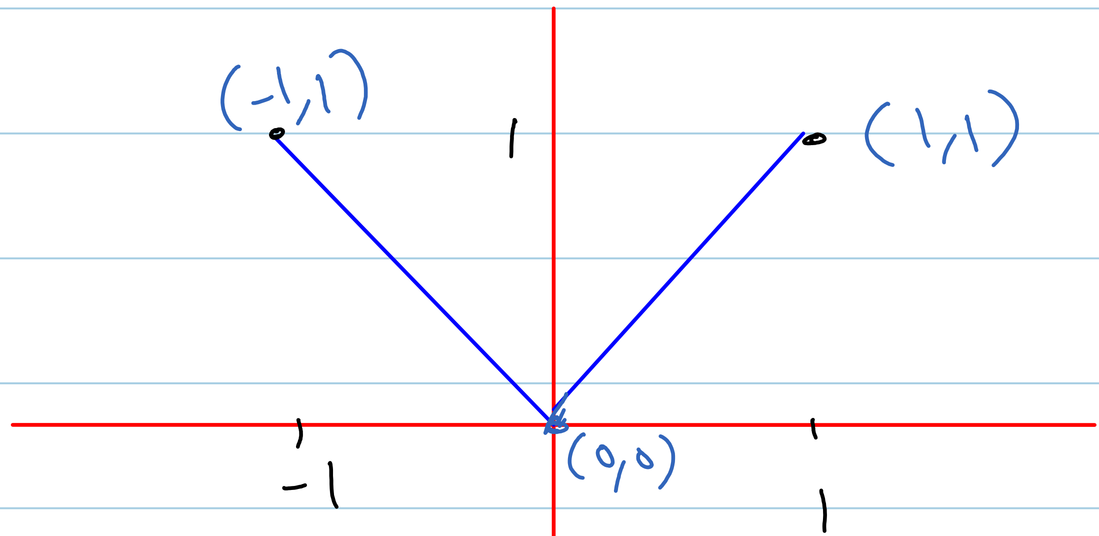
Period $= \dfrac{2\pi}{|2|} = \pi$



Absolute max. $\left(\frac{\pi}{4}, 1\right)$

" min. $\left(\frac{3\pi}{4}, -1\right)$

(5) $f(x) = |x|$, $[-1, 1]$



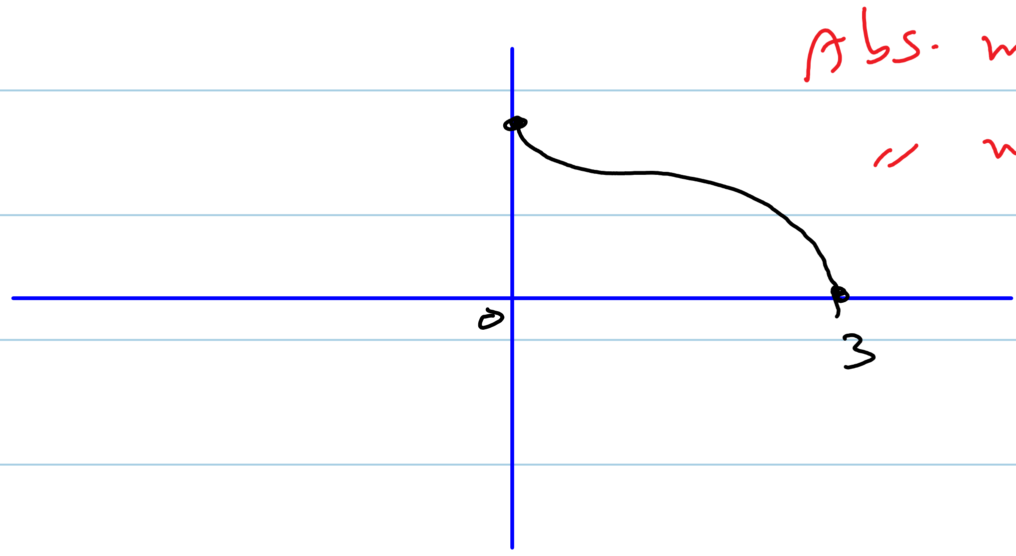
Absolute max. (-1, 1)
and (1, 1)
Abs. min. (0, 0).

9. Sketch the graph of a function that is continuous on the closed interval [0, 3] and has a global maximum at the left endpoint and a global minimum at the right endpoint.

$x = 0$

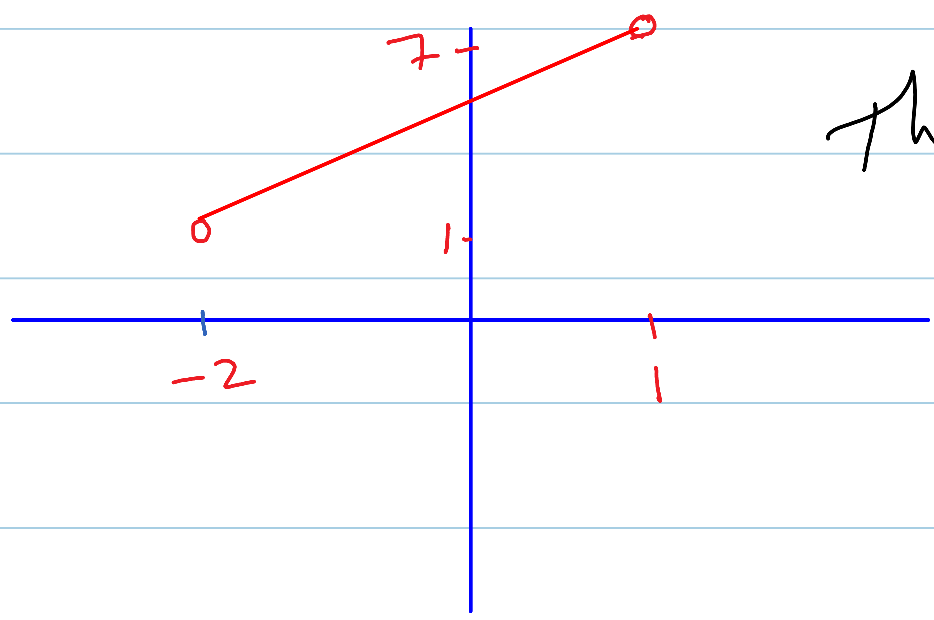
$x = 3$

Abs. max. at $x = 0$
" min. at $x = 3$.



5.1.2

14. $f(x) = 5 + 2x, x \in (-2, 1)$

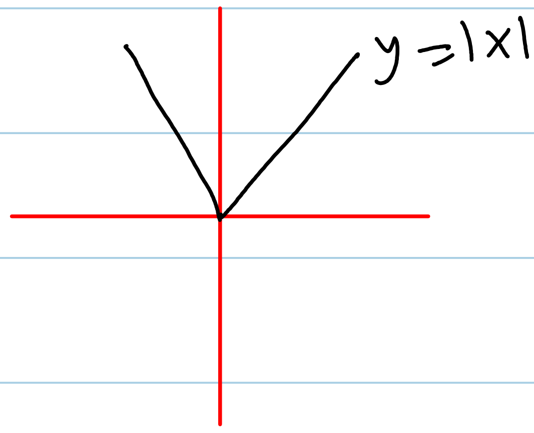


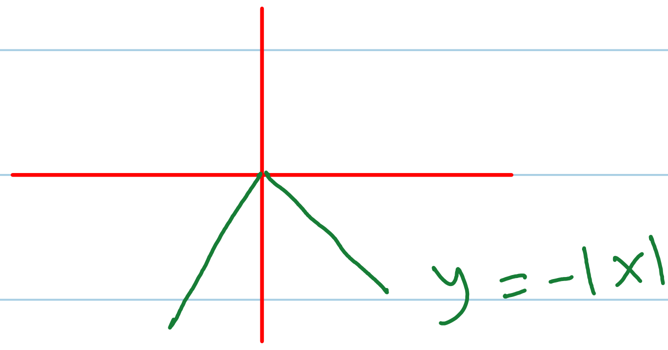
There is No extreme values. (local or global).

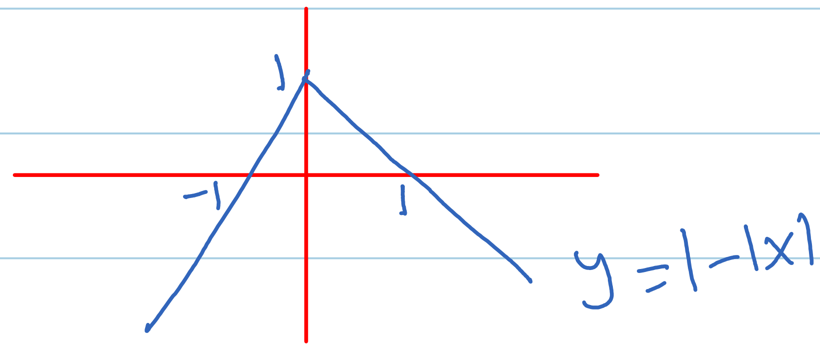
31. Graph

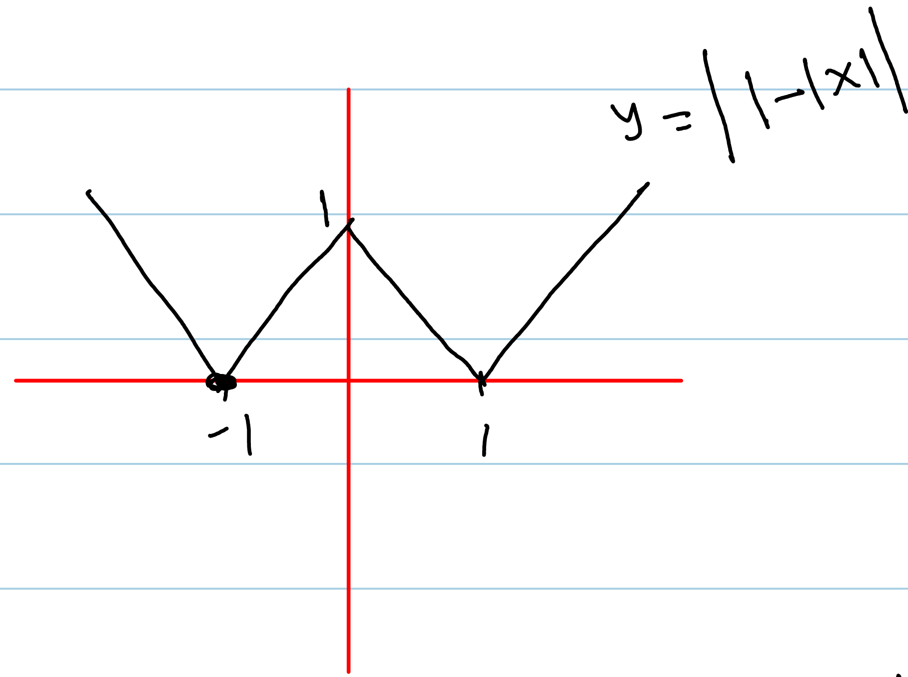
$$f(x) = \big|1 - |x|\big|, \quad -1 \le x \le 2$$

and determine all local and global extrema on $[-1, 2]$.



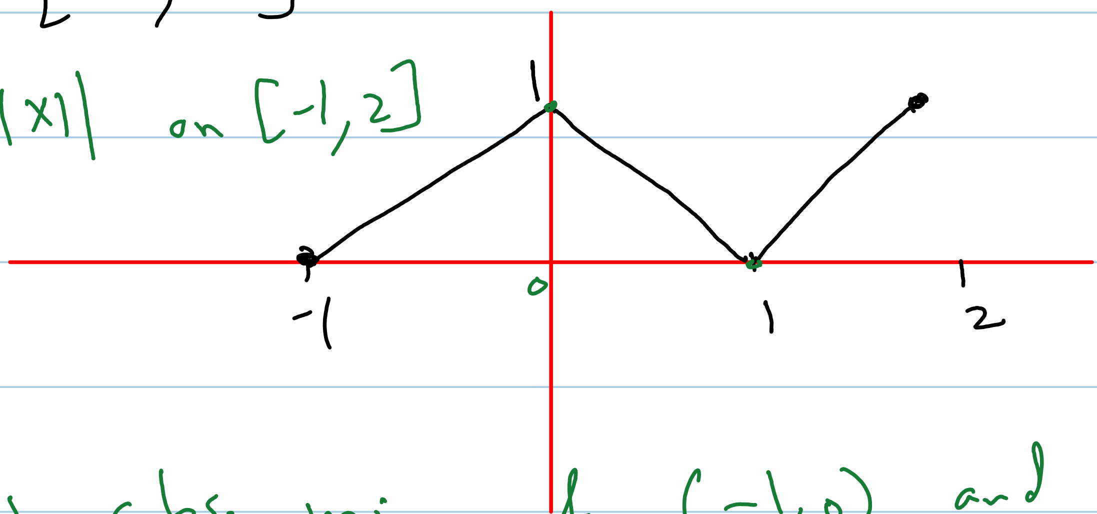






on $[-1, 2]$

$y = \big|1-|x|\big|$ on $[-1,2]$



$f(2) = \big|1-|2|\big|$
$= |-1| = 1$

f has abs. min. at $(-1, 0)$ and $(1, 0)$

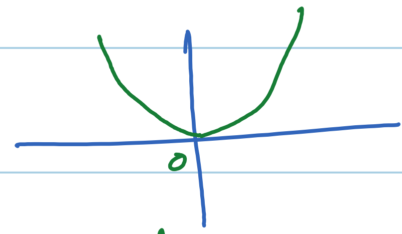
" " " max. at $(0, 1)$ and $(2, 1)$.

In Problems 19–26, find c such that $f'(c) = 0$ and determine whether $f(x)$ has a local extremum at $x = c$.

19. $f(x) = x^2$ $\quad f'(x) = 2x \quad f'(c) = 2c = 0 \Rightarrow \boxed{c = 0}$

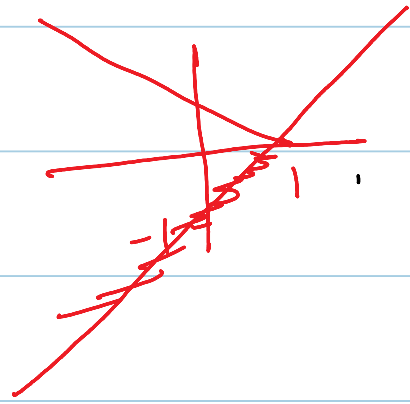
$f'(0) = 0$

From the graph



f has abs. min. at $x=0$.

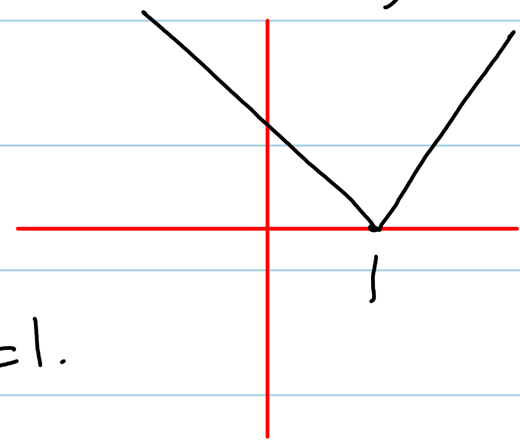
(28) $f(x) = |x-1| = \begin{cases} x-1, & x \geq 1 \\ -x+1, & x < 1 \end{cases}$



$$f'(x) = \begin{cases} 1, & x > 1 \\ -1, & x < 1 \\ \text{DNE}, & x = 1 \end{cases}$$

$\therefore \quad f'(1)$ DNE

$y = |x-1|$



f has abs. min. at $x=1$.

5.2 Monotonicity and Concavity

التزايد والتناقص التقعر

5.2.1 Monotonicity

Definition A function f defined on an interval I is called **(strictly) increasing** on I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

and is called **(strictly) decreasing** on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

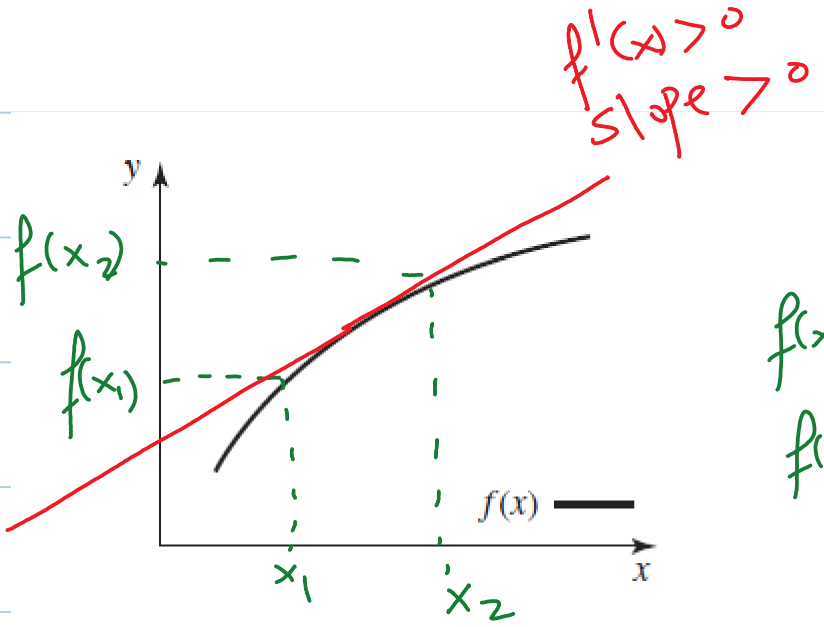


Figure 5.22 An increasing function.

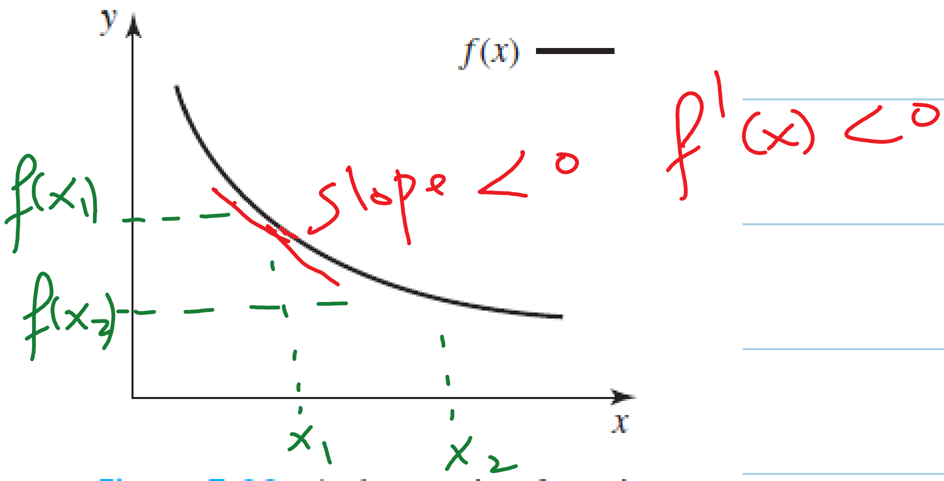


Figure 5.23 A decreasing function.

First-Derivative Test for Monotonicity Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.

EXAMPLE 1 Determine where the function

$$\boxed{f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3, \quad x \in \mathbf{R}}$$

is increasing and where it is decreasing.

Sol. f is Cont. on $\mathbb{R}$.

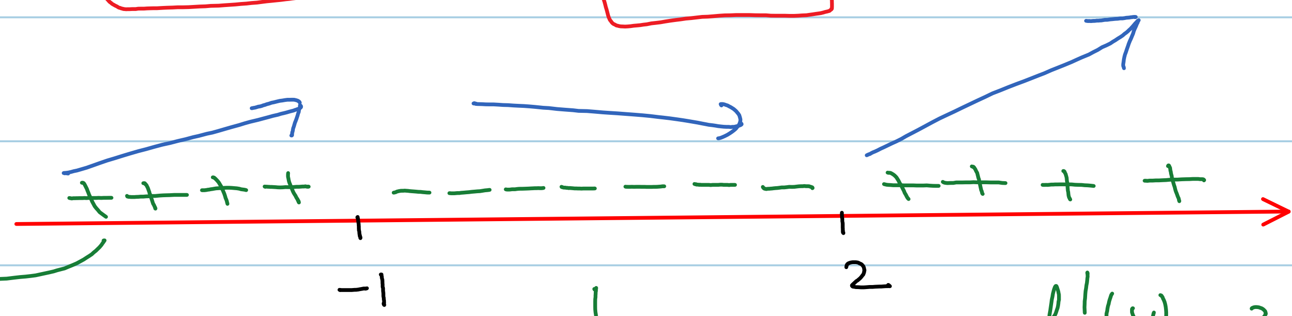
$$f'(x) = 3x^2 - \frac{3}{2}(2x) - 6$$

$$= 3x^2 - 3x - 6 = 3(x^2 - x - 2)$$

$$= 3(x-2)(x+1) = 0$$

$\boxed{x=2}$ or $\boxed{x=-1}$

f'



$f'(x) = 3(x-2)(x+1)$

$f'(-2) = 3(-4)(-1) > 0$

$f'(0) = 3(-2)(1) < 0$

$f'(4) = 3(2)(5) > 0$

f is increasing on $(-\infty, -1] \cup [2, \infty)$.

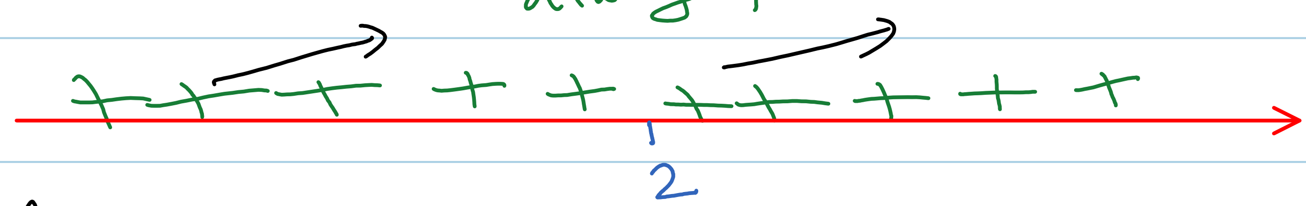
f is decreasing on $[-1, 2]$.

(6) $y = (x-2)^3 + 3$, $x \in \mathbb{R}$.

y is cont. on $\mathbb{R}$

$y' = 3(x-2)^2 = 0 \Rightarrow \boxed{x=2}$

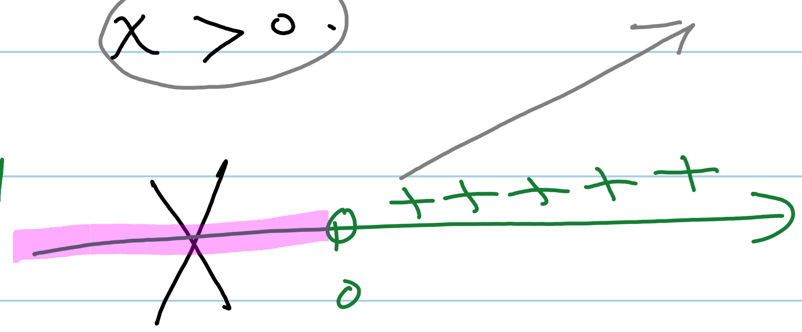
always positive



$\therefore f$ is increasing on $(-\infty, \infty)$.

(18) $y = \ln x$, $x > 0$.

$y' = \frac{1}{x} > 0$ y'



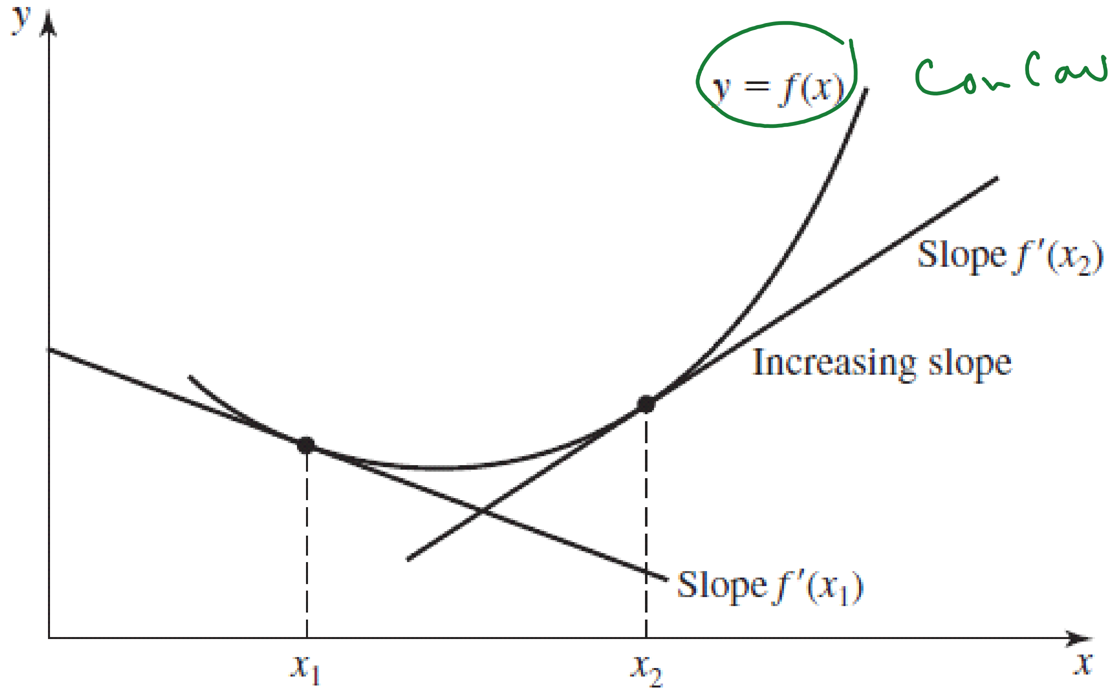
$\therefore f$ is increasing on $(0, \infty)$.

5.2.2 Concavity

مقعر للأعلى

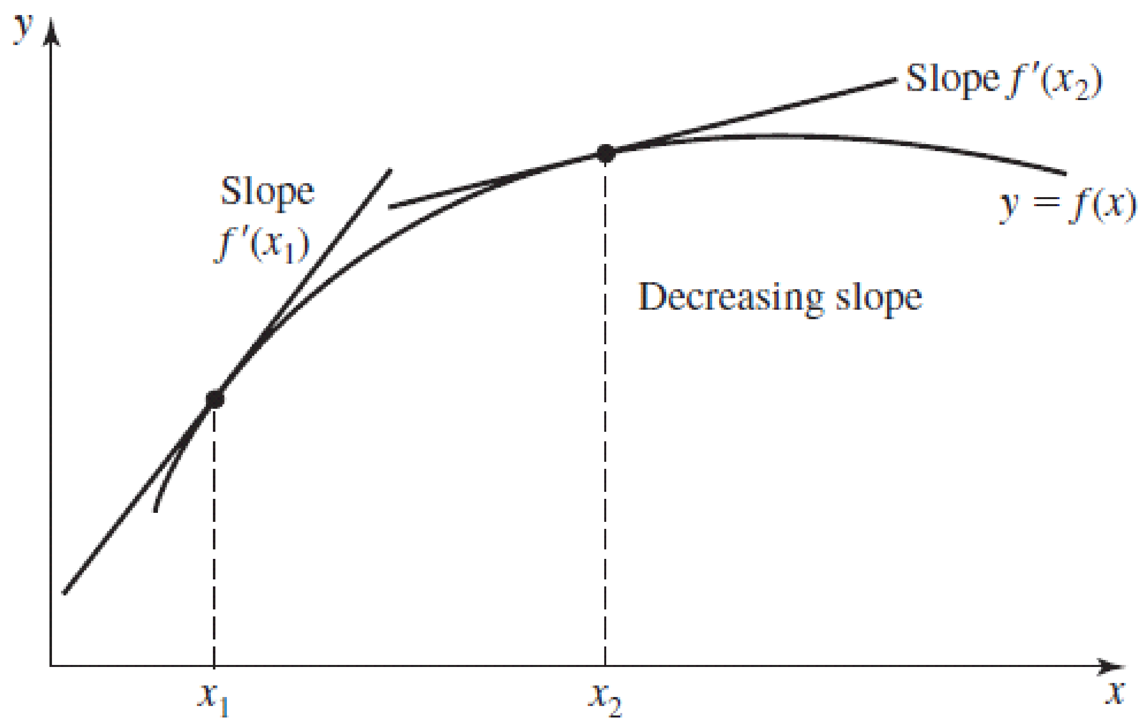
Definition A differentiable function $f(x)$ is **concave up** on an interval I if the first derivative $f'(x)$ is an increasing function on I. $f(x)$ is **concave down** on an interval I if the first derivative $f'(x)$ is a decreasing function on I.

مقعر للأسفل



Concave up.

Figure 5.29 A function is concave up if its derivative is increasing.



Concave down.

Figure 5.30 A function is concave down if its derivative is decreasing.

Second-Derivative Test for Concavity Suppose that f is twice differentiable on an open interval I.

(a) If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

(b) If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

EXAMPLE 3 Determine where the function

$$f(x) = x^3 - \frac{3}{2}x^2 - 6x + 3, \quad x \in \mathbf{R}$$

is concave up and where it is concave down.

Sol. f is cont. and diff'ble on $\mathbb{R}$.

$$f'(x) = 3x^2 - 3x - 6$$

$$\boxed{f''(x) = 6x - 3} = 0 \Rightarrow x = \frac{3}{6} = \frac{1}{2}$$

$f''(x)$: $- - - - - - - -$ (concave down) $\frac{1}{2}$ $+ + + + + +$ (concave up)

$f''(0) = -3 < 0$ $\qquad$ $f''(1) = 6 - 3 > 0$

f is concave up on $(\frac{1}{2}, \infty)$

" down on $(-\infty, \frac{1}{2})$.

13. $y = \dfrac{1}{(1+x)^2}, x \neq -1$

Where f is increasing, decreasing, concave up, concave down.

Sol. $y = (1+x)^{-2}$

$$y' = -2(1+x)^{-3}(1) = \frac{-2}{(1+x)^3}$$

$$y'' = -2(-3)(1+x)^{-4}(1) = \frac{6}{(1+x)^4}$$

Now, $\boxed{y' = \dfrac{-2}{(1+x)^3}}$

$y' = 0 \Rightarrow -2 = 0$ (impossible)

y' DNE $\Rightarrow (1+x)^3 = 0 \Rightarrow \boxed{x = -1}$

$f'(x)$: $+ + + + +$ | -1 | $- - - - - - - - - - - -$

$f'(-2) = \dfrac{-2}{(-1)^3} > 0$ $f'(0) = \dfrac{-2}{(1)^3} < 0$

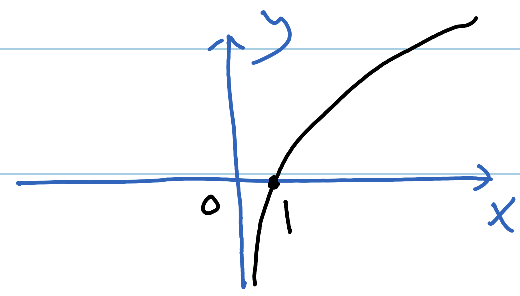
f is increasing on $(-\infty, -1)$
decreasing on $(-1, \infty)$.

Concavity $y'' = \dfrac{6}{(1+x)^4}$

$+ + + + + \quad -1 \quad + + + +$

f is concave up on $(-\infty, -1) \cup (-1, \infty)$.

(18) $y = \ln x, \; x > 0.$



$y' = \dfrac{1}{x} > 0$; $y'' = \dfrac{-1}{x^2} < 0$

y': $\times$ | 0 | $+ + + + +$

y'': $\times$ | 0 | $- - - - - - - -$

$y = \ln x$ is increasing on $(0, \infty)$ and concave down on $(0, \infty)$

5.3.1 — نقاط الانعطاف — الرسم

5.3 Extrema, Inflection Points, and Graphing

5.3.2 — 5.3.3

5.3.1 Extrema

A continuous function has a local minimum at c if the function is decreasing to the left of c and increasing to the right of c. A continuous function has a local maximum at c if the function is increasing to the left of c and decreasing to the right of c.

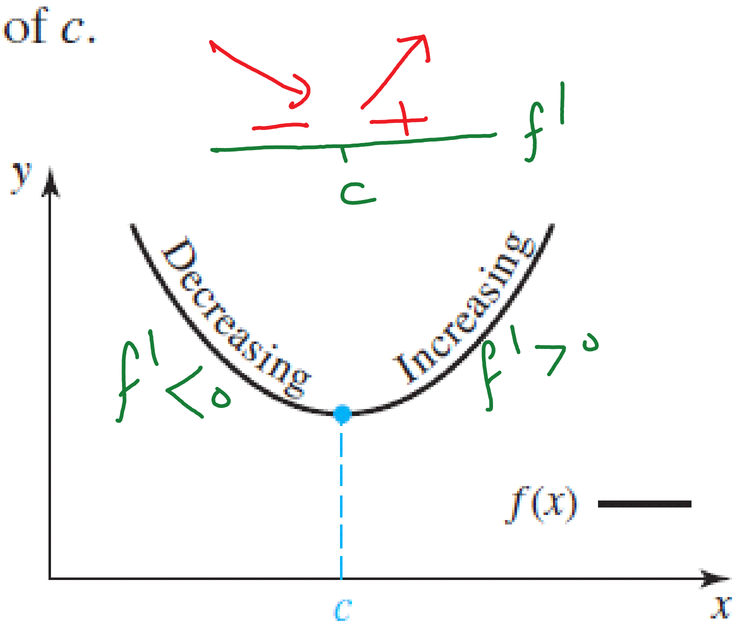


Figure 5.36 The function $y = f(x)$ has a local minimum at $x = c$.

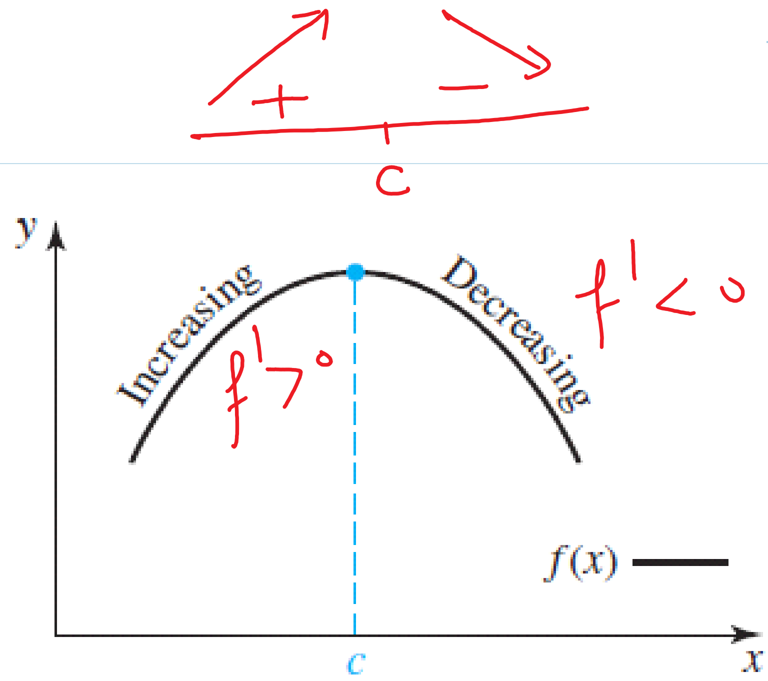


Figure 5.37 The function $y = f(x)$ has a local maximum at $x = c$.

1. Find all numbers c where $f'(c) = 0$.
2. Find all numbers c where $f'(c)$ does not exist.
3. Find the endpoints of the domain of f.

نقاط طرفية

Critical points
النقاط الحرجة
$c \in Dom(f)$.

EXAMPLE 1 Find all local and global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2, \qquad x \in \mathbf{R}$$

Sol. f is cont. and diff ble on $(-\infty, \infty)$ since it is a poly.

$$f'(x) = \frac{3}{2}\cdot 4x^3 - 2(3x^2) - 6(2x) = 0$$

$$= 6x^3 - 6x^2 - 12x = 0$$

$$\Rightarrow 6x(x^2 - x - 2) = 0$$

$$\boxed{f'(x) = 6x(x-2)(x+1) = 0}$$

$\boxed{x=0}$, $\boxed{x=2}$, $\boxed{x=-1}$ $\in D_f$

$f'(x)$ $- - - - -$ -1 $+ + + +$ 0 $- - - -$ 2 $+ + + + +$

$$f(-1) = \frac{3}{2}(-1)^4 - 2(-1)^3 - 6(-1)^2 + 2$$
$$= \frac{3}{2} + 2 - 6 + 2 = -\frac{1}{2}$$

$(-1, -\frac{1}{2})$ local min.

$$f(2) = \frac{3}{2}(2)^4 - 2(2)^3 - 6(2)^2 + 2$$
$$= 24 - 16 - 24 + 2 = -14$$

$(2, -14)$ local min. (global).

$f(0) = 2 \quad (0, 2)$ local max.

f is increasing on $[-1, 0] \cup [2, \infty)$.
decreasing on $(-\infty, -1] \cup [0, 2]$.

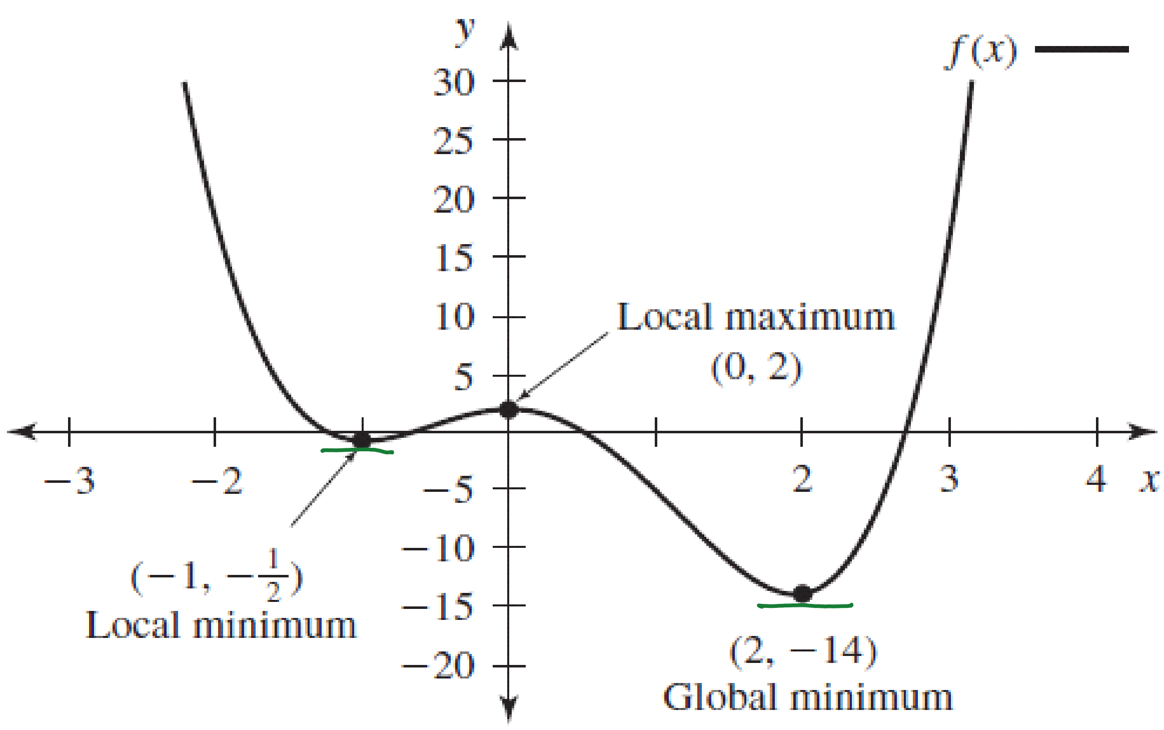


Figure 5.40 The graph of $f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2$ in Example 1.

EXAMPLE 2 Find all local and global extrema of $D_f = (-\infty, \infty)$.

$$f(x) = x(1-x)^{2/3}, \quad x \in \mathbf{R}$$

$$f'(x) = (x)\,\frac{2}{3}(1-x)^{-\frac{1}{3}}(-1) + (1-x)^{\frac{2}{3}}(1).$$

$$= \frac{-2x}{3(1-x)^{\frac{1}{3}}} + \frac{(1-x)^{\frac{2}{3}}}{1}$$

$$= \frac{(-2x)(1) + (1-x)^{\frac{2}{3}}\left(3(1-x)^{\frac{1}{3}}\right)}{3(1-x)^{\frac{1}{3}}(1)}.$$

$$= \frac{-2x + 3(1-x)}{3(1-x)^{\frac{1}{3}}} = \frac{-5x+3}{3(1-x)^{\frac{1}{3}}}$$

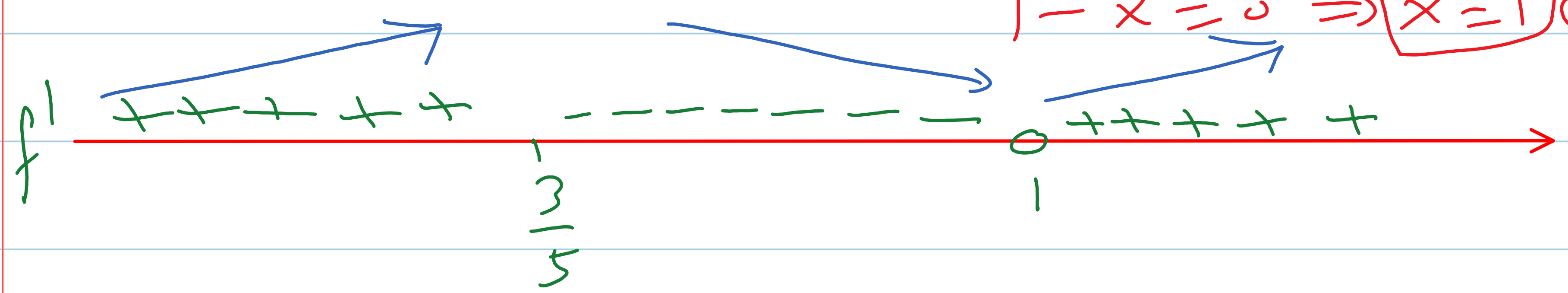
$$\boxed{f'(x) = \frac{-5x+3}{3(1-x)^{\frac{1}{3}}}}$$

$$f'(x) = 0 \Rightarrow -5x+3 = 0 \Rightarrow \boxed{x = \frac{3}{5}} \in D_f$$

$$f'(x) \text{ DNE} \Rightarrow \frac{3(1-x)^{\frac{1}{3}}}{3} = \frac{0}{3}$$

$$\left[(1-x)^{\frac{1}{3}}\right]^3 = (0)^3$$

$$1 - x = 0 \Rightarrow \boxed{x = 1} \in D_f$$



$\left(\frac{3}{5}, f\left(\frac{3}{5}\right)\right)$ local max.

$(1, f(1)) = (1, 0)$ local min.

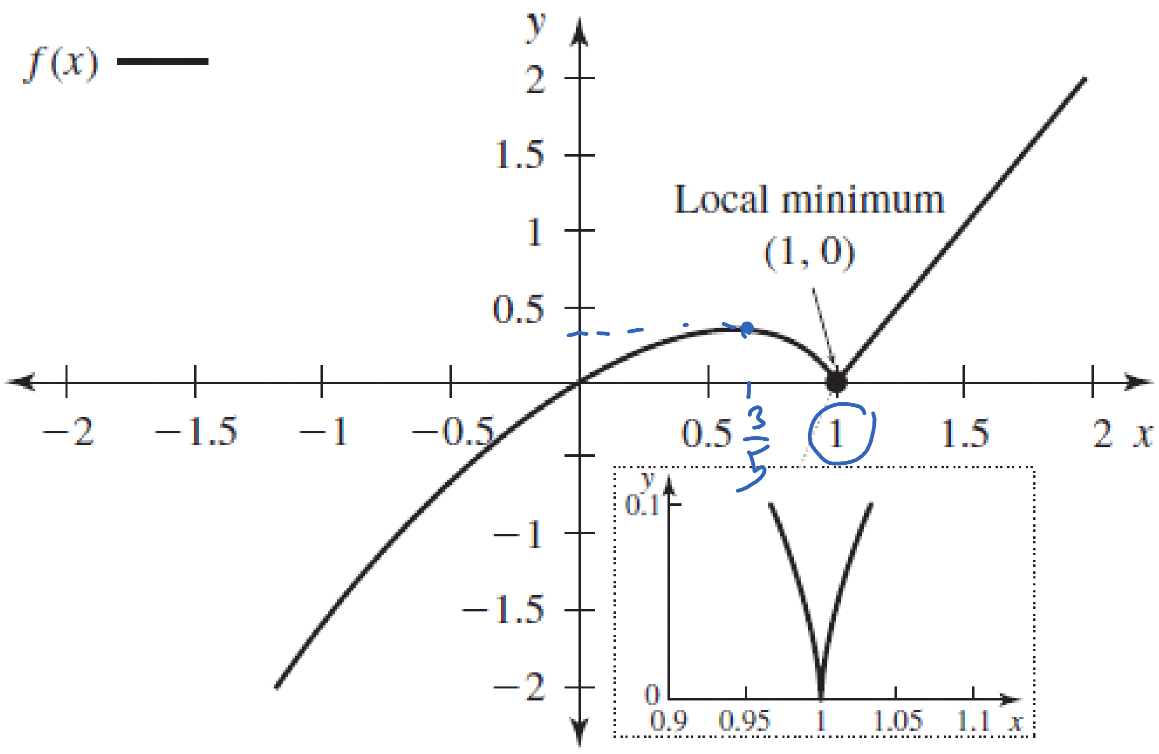


Figure 5.41 The graph of $f(x) = x(1-x)^{2/3}$.

The Second-Derivative Test for Local Extrema Suppose that f is twice differentiable on an open interval containing c.

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

If $f'(c)=0$ and $f''(c)=0$, test fails. In this case use 1st derivative test.

EXAMPLE 1 Find all local and global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2, \quad x \in \mathbf{R}$$

using the 2nd derivative test.

Sol. $f'(x) = 6x^3 - 6x^2 - 12x = 6x(x-2)(x+1)$

$$\boxed{f''(x) = 18x^2 - 12x - 12}$$

$f'(x) = 0 \Rightarrow \boxed{x=0}, \boxed{x=2}, \boxed{x=-1}$

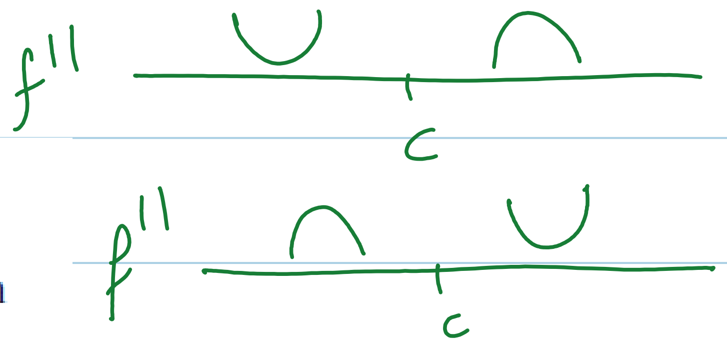
$f''(0) = -12 < 0 \Rightarrow (0, f(0))$ local max.

$f''(2) = 18(4) - 12(2) - 12 > 0 \quad (2, f(2))$ local min.

$f''(-1) = 18 + 12 - 12 = 18 > 0 \quad (-1, f(-1))$ local min.

5.3.2 Inflection Points

Inflection points are points where the concavity of a function changes—that is, where the function changes from concave up to concave down or from concave down to concave up.



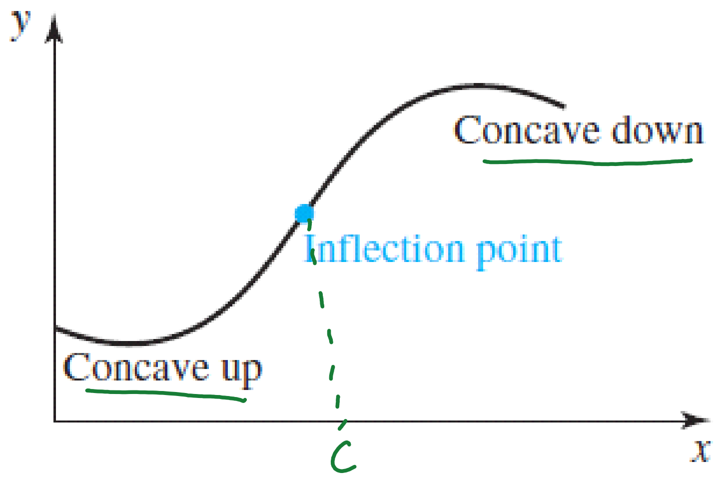


Figure 5.42 Inflection point.

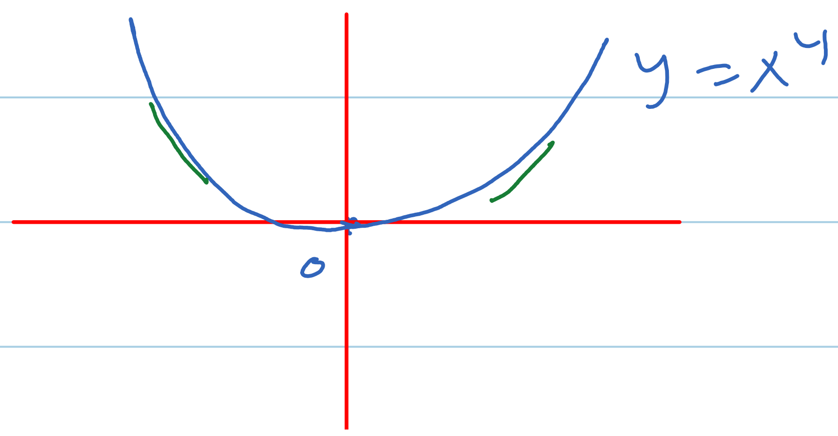
If $f(x)$ is twice differentiable and has an inflection point at $x = c$, then $f''(c) = 0$. The converse is not true.

$$\left(f''(c) = 0 \not\Longrightarrow f \text{ has inflection point at } x = c \right)$$

ex. $y = x^4 \Rightarrow y' = 4x^3$, $\boxed{y'' = 12x^2}$

+ + (0) concave up on $(-\infty, \infty)$

$y''(0) = 12(0)^2 = 0$ but f does not have inflection points at $x = 0$



EXAMPLE 3 Show that the function

$$f(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 2x + 1, \quad x \in \mathbf{R}$$

has an inflection point at $x = 1$.

Sol: f is cont. + diff'ble on $\mathbb{R}$.

$$f'(x) = \frac{3}{2}x^2 - \frac{6}{2}x + 2$$

$$= \frac{3}{2}x^2 - 3x + 2$$

$$f''(x) = 3x - 3 = 0 \Rightarrow \boxed{x = 1}$$

f'' $\quad - \; - \; - \; - \; - \; - \quad | \quad + \; + \; + \; +$

$(1, f(1)) = (1, 2)$ is the inflection point.

f is concave up on $(1, \infty)$ and concave down on $(-\infty, 1)$.

ex. $f(x) = x^2$

$f'(x) = 2x$, $\boxed{f''(x) = 2} = 0$ impossible $\quad > 0$

$\therefore \quad + \; + \; + \; + \; + \; + \; + \; +$

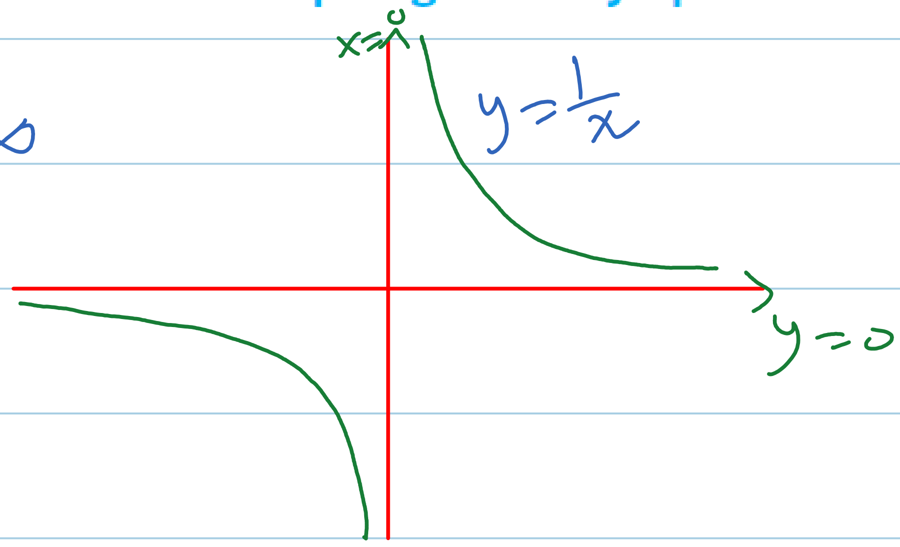
$\therefore$ f is concave up on $(-\infty, \infty)$

There is no inflection points.

5.3.3 Graphing and Asymptotes

$\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$

$\lim\limits_{x\to 0^-} \frac{1}{x} = \infty$

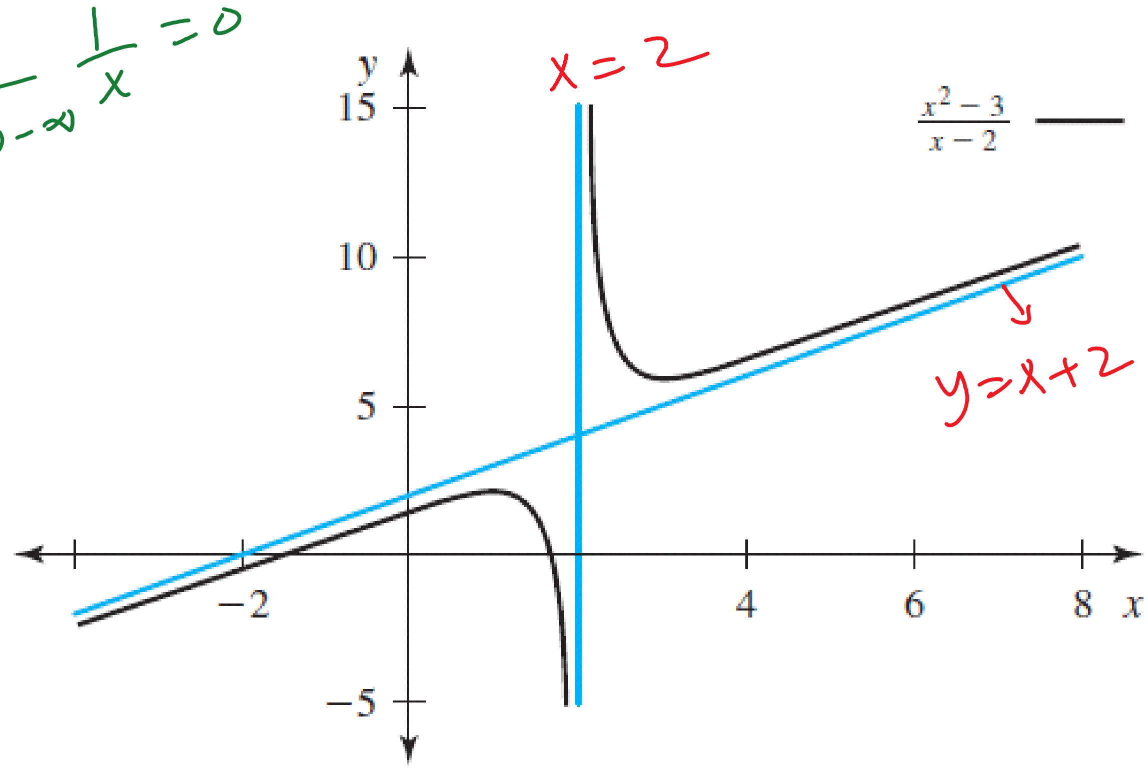


$x=0$ is vertical asymptote.

$y=0$ is Horizontal asymptote.

$\lim\limits_{x\to \infty} \frac{1}{x} = 0$

$\lim\limits_{x\to -\infty} \frac{1}{x} = 0$



$x=2$ is vertical asymptote

$\boxed{y=x+2}$ is oblique asymptote.

Figure 5.47 The graph of $f(x) = \frac{x^2-3}{x-2}$ with oblique asymptote $g(x) = x+2$ (and a vertical asymptote at $x = 2$).

H.A

Definition A line $y = b$ is a **horizontal** asymptote if either

$$\lim_{x\to -\infty} f(x) = b \quad \text{or} \quad \lim_{x\to \infty} f(x) = b$$

V.A

A line $x = c$ is a **vertical** asymptote if

$$\lim_{x\to c^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to c^+} f(x) = -\infty$$

or

$$\lim_{x\to c^-} f(x) = +\infty \quad \text{or} \quad \lim_{x\to c^-} f(x) = -\infty$$

ex. $f(x) = \frac{x-2}{2x-6}$. Find H.A, V.A

Sol. $\lim\limits_{x\to\infty} \frac{x-2}{2x-6} = \frac{1}{2}$, $\lim\limits_{x\to -\infty} \frac{x-2}{2x-6} = \frac{1}{2} \Rightarrow y = \frac{1}{2}$ is H.A.

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$$f(x) = \frac{x-2}{2x-6}$$

$$\lim_{x \to 3^+} \frac{x-2}{2x-6} = \frac{1}{0^+}$$

$$= +\infty \Rightarrow \boxed{x = 3} \text{ is V.A.}$$

ex. $\lim_{x \to 2} \frac{x-2}{x^2-4}$. Find H.A, V.A.

Sol. H.A $\lim_{x \to \pm\infty} \frac{x-2}{x^2-4} = 0 \Rightarrow \boxed{y = 0}$ is H.A.

V.A $f(x) = \frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \frac{1}{x+2}, \; x \neq 2$

$\therefore \lim_{x \to -2^+} \frac{1}{x+2} = +\infty \Rightarrow \boxed{x = -2}$ is V.A

$$\lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4} \neq \infty \text{ or } -\infty.$$

$x = 2$ is Not V.A. (hole)

ex. Find H.A and V.A of $f(x) = \frac{x^2-3}{x-2}$

sol. H.A

$$\lim_{x\to\infty} \frac{x^2-3}{x-2} = \lim_{x\to\infty} \frac{\frac{x^2}{x}-\frac{3}{x}}{\frac{x}{x}-\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{x-\frac{3}{x}}{1-\frac{2}{x}}$$

$$= \frac{\infty-0}{1-0} = \boxed{\infty}$$

$$\lim_{x\to-\infty} \frac{x^2-3}{x-2} = \cdots = \lim_{x\to-\infty} \frac{x-\frac{3}{x}}{1-\frac{2}{x}}$$

$$= \frac{-\infty-0}{1-0} = \boxed{-\infty}$$

There is No H.A.

V.A $\boxed{x=2}$ $\quad \lim_{x\to 2^+} \frac{x^2-3}{x-2} \quad \frac{4-3}{0^+} = \frac{1}{0^+}$

$= +\infty$.

Oblique Asymptote (O.A). for $y=f(x)$

$y = mx+b$

$$\lim_{x\to\infty} \left[f(x)-(mx+b)\right] = 0 \quad \text{or}$$

$$\lim_{x\to-\infty} \left[f(x)-(mx+b)\right] = 0$$

How to find an O.A ?

We ~~use~~ long division.

The simplest case of an oblique asymptote occurs with a rational function in which the degree of the numerator is one higher than the degree of the denominator.

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ex. Find O.A of $f(x) = \dfrac{x^2-3}{x-2}$

ضرب

$$\begin{array}{r|l} & x+2 \\ \hline x-2 & x^2-3 \\ & \mp x^2 \pm 2x \\ \hline & 2x-3 \\ & -2x \pm 4 \\ \hline & 1 \end{array}$$

$\dfrac{x^2}{x} = x$

$\dfrac{2x}{x} = 2$

$$\therefore f(x) = (x+2) + \frac{1}{x-2}$$

$$f(x) - (x+2) = \frac{1}{x-2}$$

$$\lim_{x \to \pm\infty} [f(x) - (x+2)] = \lim_{x \to \pm\infty} \frac{1}{x-2} = 0$$

$\therefore y = x+2$ is O.A .

Ex. $f(x) = \dfrac{2x^2-5}{x+2}$ find all asymptotes. (H.A, V.A, O.A).

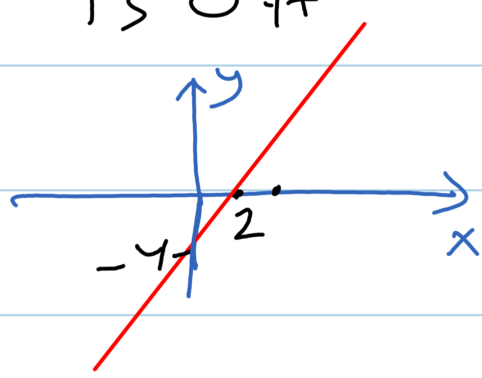
Sol. There is No H.A .

V.A $\boxed{x = -2}$ (check!).

O.A

$$\begin{array}{r|l} & 2x-4 \\ \hline x+2 & 2x^2-5 \\ & -2x^2 \mp 4x \\ \hline & -4x-5 \\ & \pm 4x \pm 8 \\ \hline & 3 \end{array}$$

$y = 2x-4$ is O.A .



EXAMPLE 5 Sketch the graph of the function

$$f(x) = \frac{2}{3}x^3 - 2x + 1, \quad x \in \mathbf{R}$$

step1 Domain $= (-\infty, \infty)$.

step2 Asymptotes : None

step3 $f'(x) = 2x^2 - 2 = 0 \Rightarrow x = \pm 1$

f': $+ + + +$ ($\nearrow$) on $(-\infty, -1)$; $- - - - - -$ ($\searrow$) on $(-1, 1)$; $+ + +$ ($\nearrow$) on $(1, \infty)$

f is $\nearrow$ on $(-\infty, -1] \cup [1, \infty)$

$\searrow$ on $[-1, 1]$.

$(-1, f(-1)) = (-1, \frac{7}{3})$ local max.

$(1, f(1)) = (1, -\frac{1}{3})$ local min.

step4. $f''(x) = 4x = 0 \Rightarrow \boxed{x = 0}$

$f''(x)$: $- - - - -$ ($\frown$) on $(-\infty, 0)$; $+ + + +$ ($\smile$) on $(0, \infty)$

f is concave up on $(0, \infty)$ and down on $(-\infty, 0)$

$(0, f(0)) = (0, 1)$ is the inflection point.

step5. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{2}{3}x^3 - 2x + 1 \quad (\infty - \infty)$

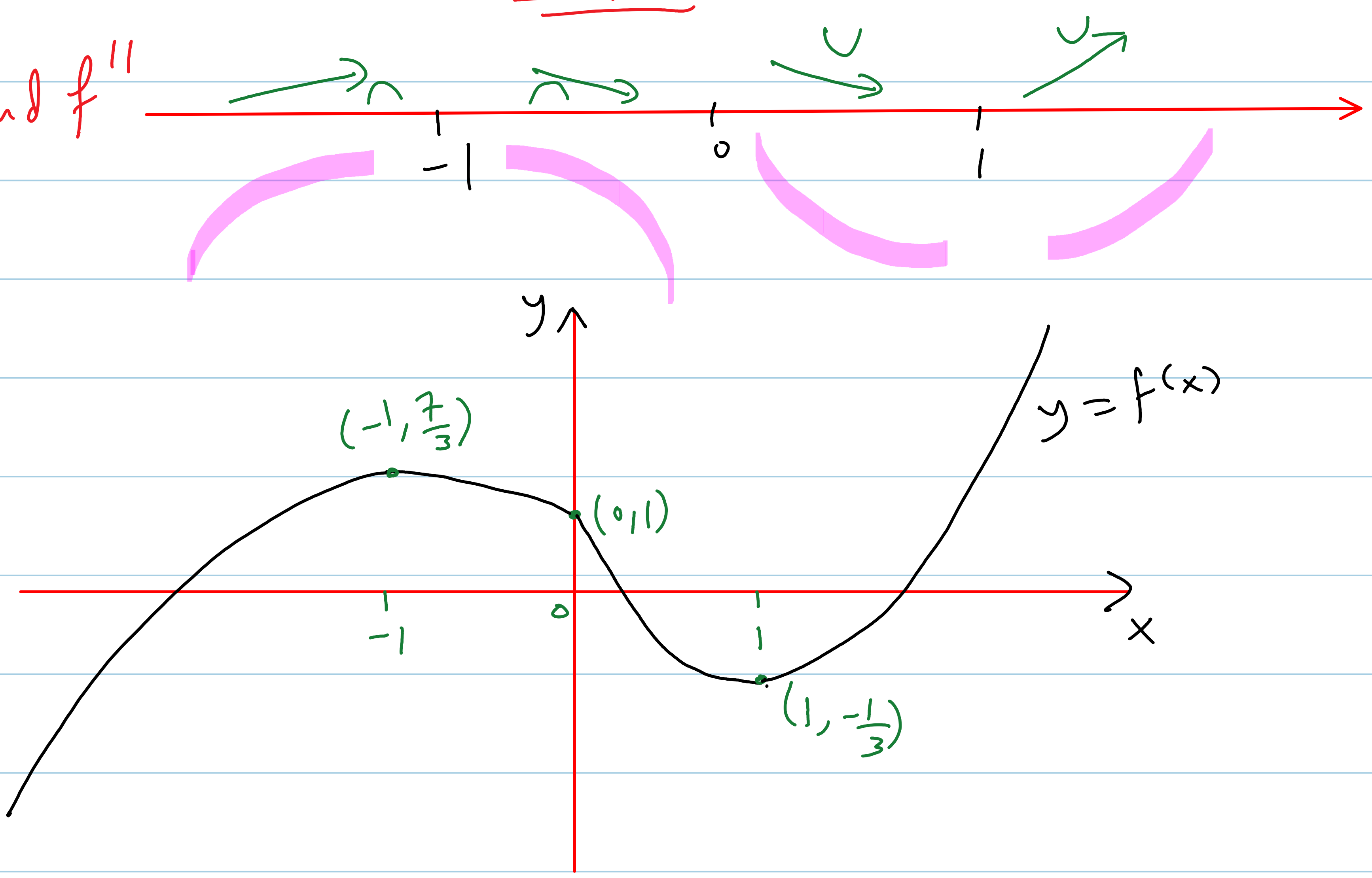
$$= \lim_{x\to\infty} x^3\left(\frac{2}{3} - \frac{2}{x^2} + \frac{1}{x^3}\right)$$

$$= \infty\left(\frac{2}{3} - 0 + 0\right) = \infty$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3\left(\frac{2}{3} - \frac{2}{x^2} + \frac{1}{x^3}\right) = -\infty$$

Step6 graph

f' and f''



ex. Sketch the graph of $f(x)$ together with its asymptotes. , where

$$f(x) = \frac{x^2-3}{x-2}.$$

Solution. (1) Domain $= (-\infty, 2) \cup (2, \infty)$.

(2) Asymptotes $\boxed{x = 2}$ V.A

There is No H.A

O.A $\boxed{y = x+2}$ ← لاحظ/ي المعادلة السابقة

(3) $$f'(x) = \frac{(x-2)(2x)-(x^2-3)(1)}{(x-2)^2}$$

$$= \frac{2x^2-4x-x^2+3}{(x-2)^2} = \boxed{\frac{x^2-4x+3}{(x-2)^2}}$$

Critical values $f'(x) = 0 \Rightarrow x^2-4x+3=0$

$(x-1)(x-3) = 0$

f' DNE if $(x-2)^2 = 0 \Rightarrow \boxed{x=2} \notin D_f$ $\quad \boxed{x=1} \in, \quad \boxed{x=3} \in$

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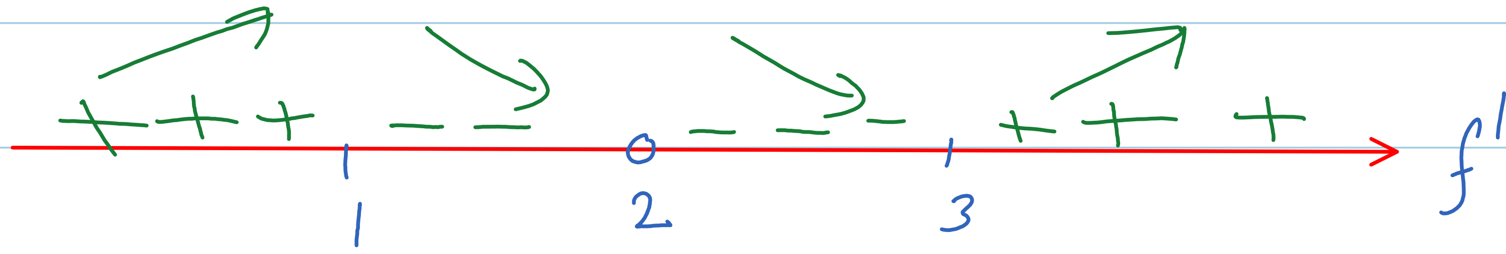


f is $\nearrow$ on $(-\infty, 1] \cup [3, \infty)$.

$\searrow$ on $[1, 2) \cup (2, 3]$.

$(1, f(1)) = (1, 2)$ local max.

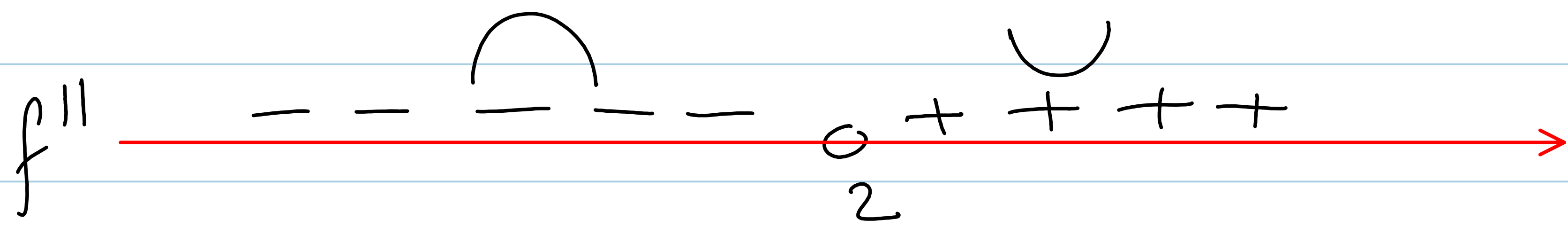
$(3, f(3)) = (3, 6)$ local min.

(4) $f'(x) = \dfrac{x^2 - 4x + 3}{(x-2)^2}$

$f''(x) = \dfrac{2}{(x-2)^3}$ (تحقق / s).

$f'' = 0 \Rightarrow 2 = 0$ impossible

f'' DNE $(x-2)^3 = 0 \Rightarrow \boxed{x = 2}$

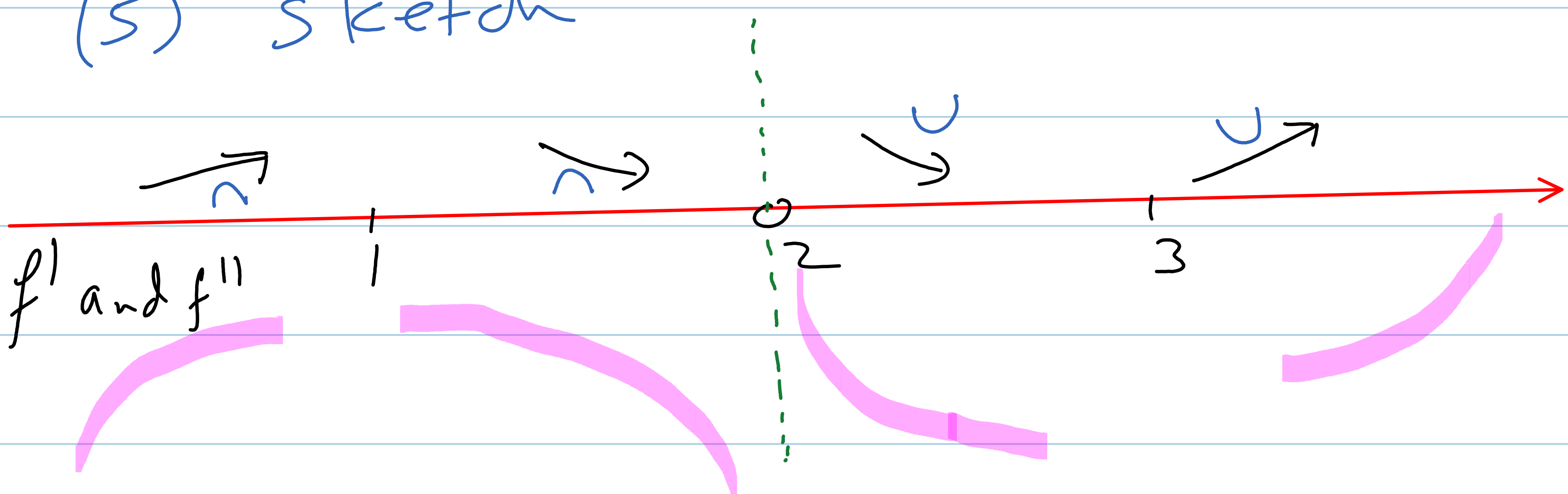


f is concave up on $(2, \infty)$.

concave down on $(-\infty, 2)$

There is no inflection pts.

(5) sketch



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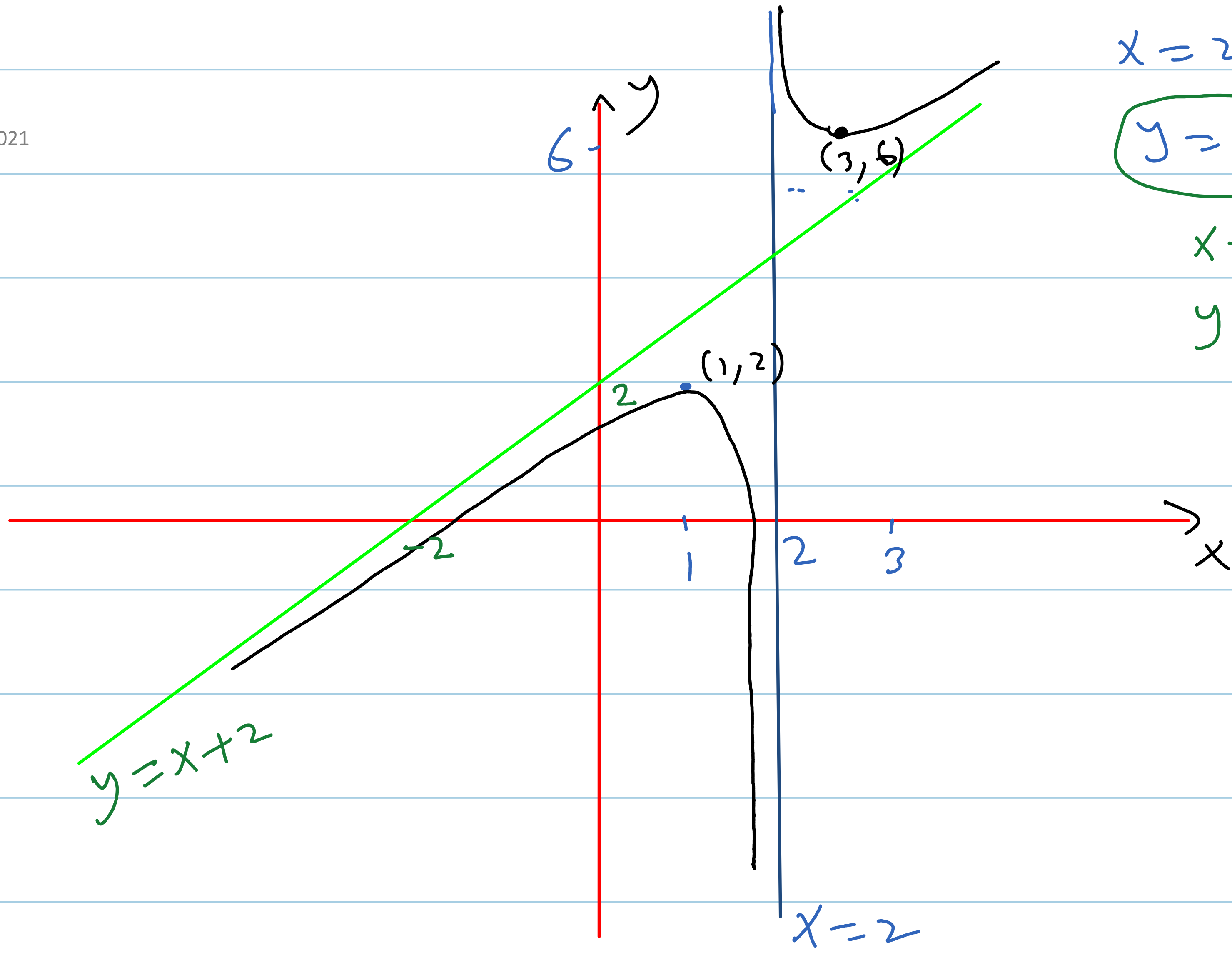


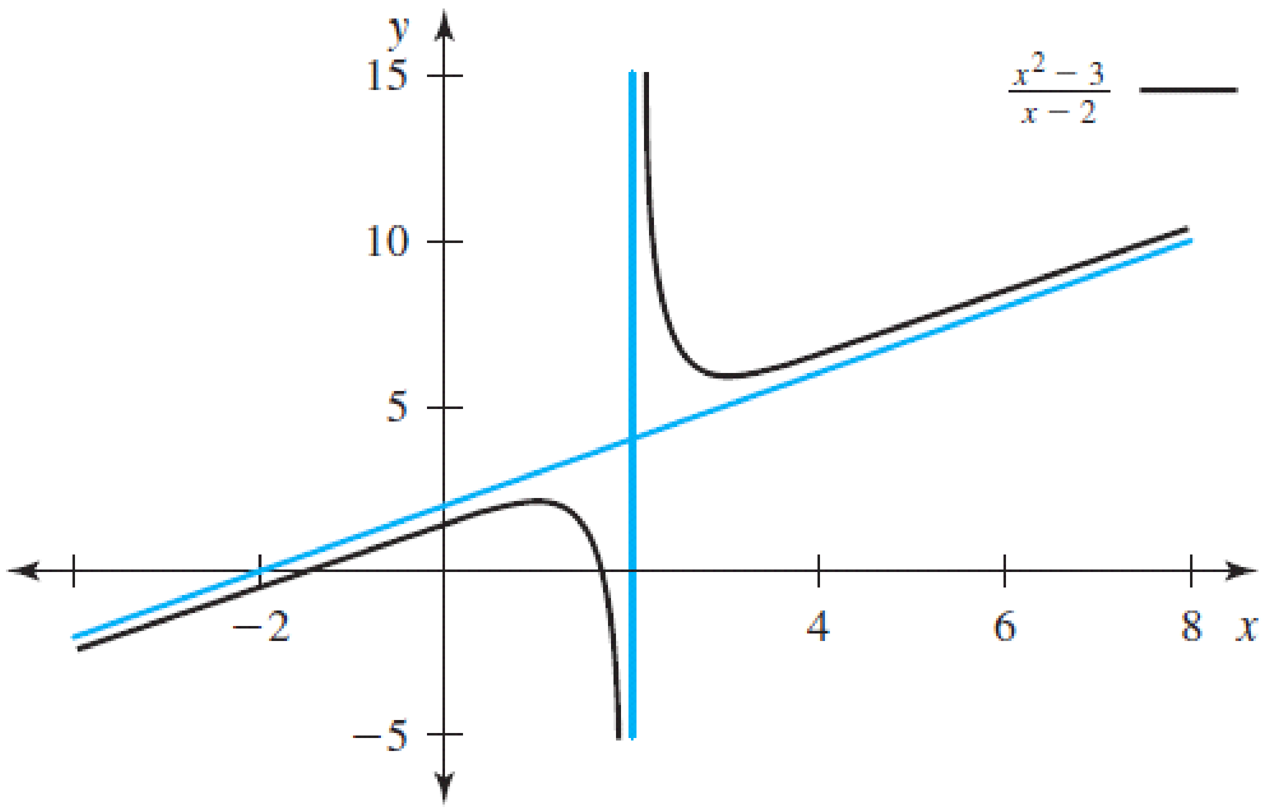
$x = 2$ V.A

$\boxed{y = x + 2}$ O.A

$x = 0 \Rightarrow \boxed{y = 2}$

$y = 0 \Rightarrow x + 2 = 0$

$\boxed{x = -2}$



5.4 Optimization

EXAMPLE 1 **Chemical Reaction** Consider the chemical reaction

$$\boxed{A + B \to AB}$$

a A
b con B
x AB

The reaction rate is given by the function

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$$\boxed{R(x) = k(a - x)(b - x), \quad 0 \le x \le \min(a, b)}$$

where x is the concentration of the product AB and $\min(a, b)$ denotes the minimum of the two values of a and b. The constants a and b are the concentrations of the reactants A and B at the beginning of the reaction. To be concrete, we choose $k = 2$, $a = 2$, and $b = 5$. Then

$$\boxed{R(x) = 2(2 - x)(5 - x) \quad \text{for } 0 \le x \le 2}$$

We are interested in finding the concentration x that maximizes the reaction rate:

Solution.

$R(x) = 20 - 14x + 2x^2, \quad 0 \le x \le 2$

$R'(x) = -14 + 4x = 0 \Rightarrow x = \frac{14}{4} = \frac{7}{2} \notin (0,2)$

$R'(x)$: 0 – – – – – – – – 2

Endpoints $x = 0$, $x = 2$

$R(0) = 20$, $R(2) = 20 - 28 + 8 = 0$

$\Rightarrow$ R is decreasing on $[0,2]$

the absolute max. of R is $R(0) = 20$.

Thus, the reaction rate is maximal when the concentration of the product AB is equal to 0.

EXAMPLE 2 **Crop Yield** Let $Y(N)$ be the yield of an agricultural crop as a function of nitrogen level N in the soil. A model that is used for this relationship is

$$\boxed{Y(N) = \frac{N}{1+N^2} \quad \text{for } N \geq 0} \qquad \text{Domain } N \in [0, \infty)$$

(where N is measured in appropriate units). Find the nitrogen level that maximizes yield.

Solution:
$$Y'(N) = \frac{(1+N^2)(1) - (N)(2N)}{(1+N^2)^2}$$

$$= \frac{1-N^2}{(1+N^2)^2}$$

$Y'(N) = 0$

$1 - N^2 = 0$

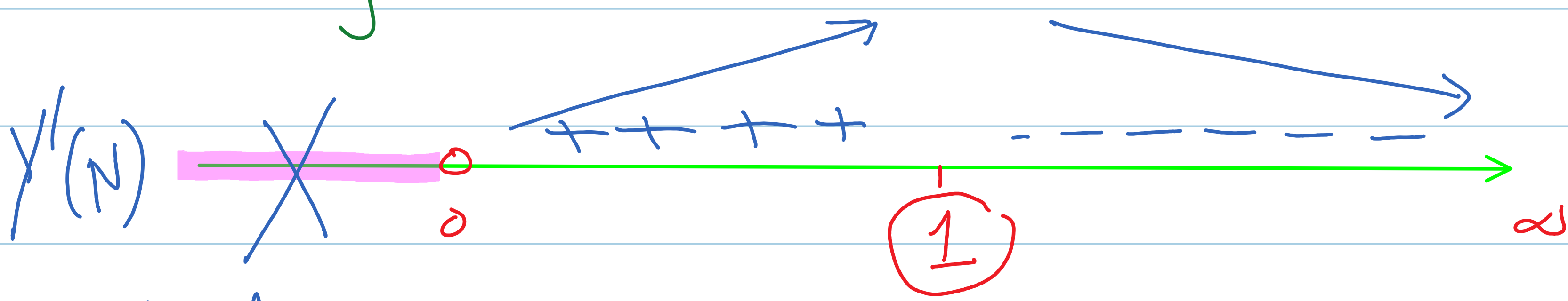
$N = \pm 1$

$\boxed{N=-1}$ reject , $\boxed{N=1}$

$Y'(N)$ undefind

$(1+N^2)^2 = 0$

ϕ



Endpoints

$Y(0) = \boxed{0}$

$Y(1) = \frac{1}{1+1^2} = \frac{1}{2}$

$$\lim_{N\to\infty} Y(N) = \lim_{N\to\infty} \frac{N}{1+N^2} = \boxed{0}$$

We conclude the global max. of $Y(N)$ is $\left(\frac{1}{N}, \frac{1}{2}\right)$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad Y(N)$

Thus, $N = 1$ is the nitrogen level that maximizes yield.

EXAMPLE 3 **Maximizing Area** A field biologist wants to enclose a rectangular study plot. She has 1600 ft of fencing. Using this fencing, determine the dimensions of the study plot that will have the largest area.

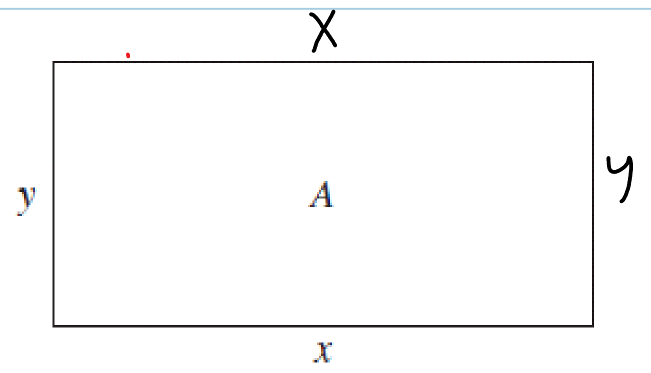


Figure 5.52 The rectangular study plot in Example 3.

Given

$$\boxed{2x + 2y = 1600}$$

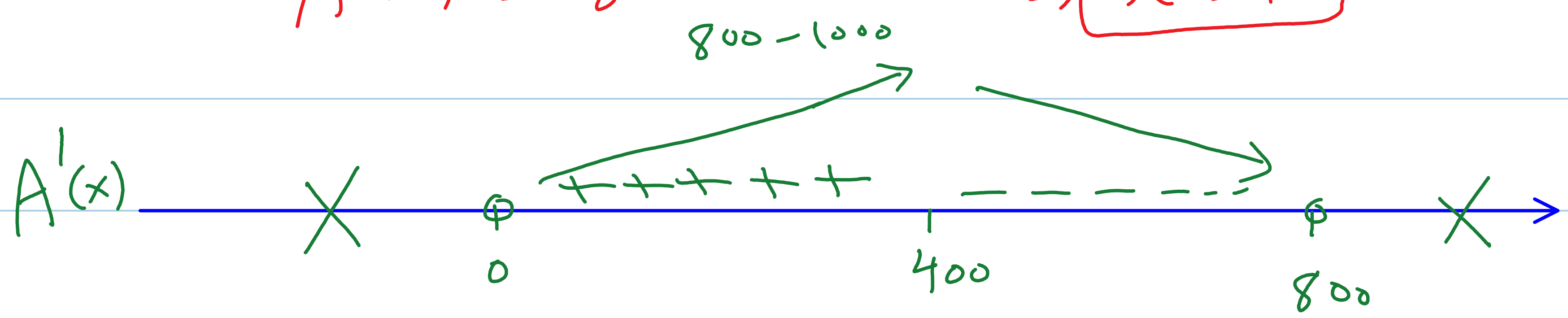
$$\text{Area} = xy$$

but $\boxed{x + y = 800} \Rightarrow \boxed{y = 800 - x}$

$$\therefore \text{Area} = x(800 - x)$$

$$\boxed{A(x) = 800x - x^2, \quad 0 \le x \le 800}$$

$$A'(x) = 800 - 2x = 0 \Rightarrow \boxed{x = 400}$$



$$A(0) = 0$$

$$A(400) = (800 - 400)(400) = \boxed{160000}$$

$$A(800) = (800 - 800)(800) = 0$$

The area is maximized when $x = 400$, $y = 800 - 400 = 400$ and the maximum area $= (400)(400) = 160\,000 \text{ ft}^2$.

Rmk: For a rectangle with fixed perimeter the max. area occurs when the rectangle is a square.

EXAMPLE 4 **Minimizing Material** Aluminum soda cans are shaped like a right circular cylinder and hold about 12 ounces of liquid. The production of aluminum requires a lot of energy, so it is desirable to design soda cans that use the least amount of material. What dimensions would such an optimal soda can have?

r, h

Notice. 12 ounces $= 0.355$ liter.

1 liter $= 1000 \text{ cm}^3$

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Volume $= \boxed{\pi r^2 h = 355 \text{ cm}^3}$

$$\boxed{h = \frac{355}{\pi r^2}}$$

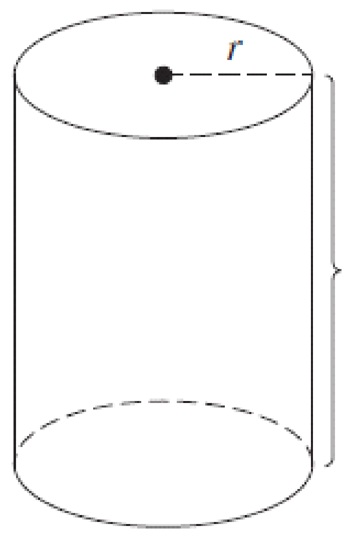
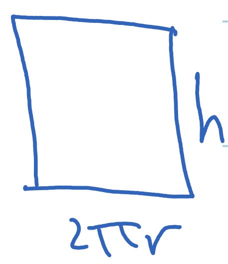


Figure 5.53 A right circular cylinder with height h and radius r.



Surface area $= 2\pi r^2 + 2\pi r h$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right), \quad r > 0$$

$$\boxed{A(r) = 2\pi r^2 + \frac{710}{r}, \quad r > 0}$$

$$A'(r) = 4\pi r - \frac{710}{r^2} = 0$$

$$4\pi r = \frac{710}{r^2}$$

$$r^3 = \frac{710}{4\pi} = \frac{355}{2\pi}$$

$$\boxed{r = \sqrt[3]{\frac{355}{2\pi}}}$$

$$A''(r) = 4\pi + 2(710)r^{-3}$$

$$= 4\pi + \frac{1420}{r^3}$$

$$A''\left(\sqrt[3]{\frac{355}{2\pi}}\right) = 4\pi + \frac{1420}{\left(\frac{355}{2\pi}\right)} > 0$$

$\therefore\ r = \left(\frac{355}{2\pi}\right)^{\frac{1}{3}}$ is where a local min. occurs.

$r > 0 \quad r \in (0, \infty)$

$$\lim_{r \to 0^+} A(r) = \lim_{r \to 0^+} \left(2\pi r^2 + \frac{710}{r}\right) = 0 + \infty = \infty$$

$$\lim_{r \to \infty} A(r) = \lim_{r \to \infty} \left(2\pi r^2 + \frac{710}{r}\right) = \infty + 0 = \infty$$

$$h = \frac{355}{\pi r^2} = \frac{355}{\pi\left(\frac{355}{2\pi}\right)^{\frac{2}{3}}} = \cdots$$

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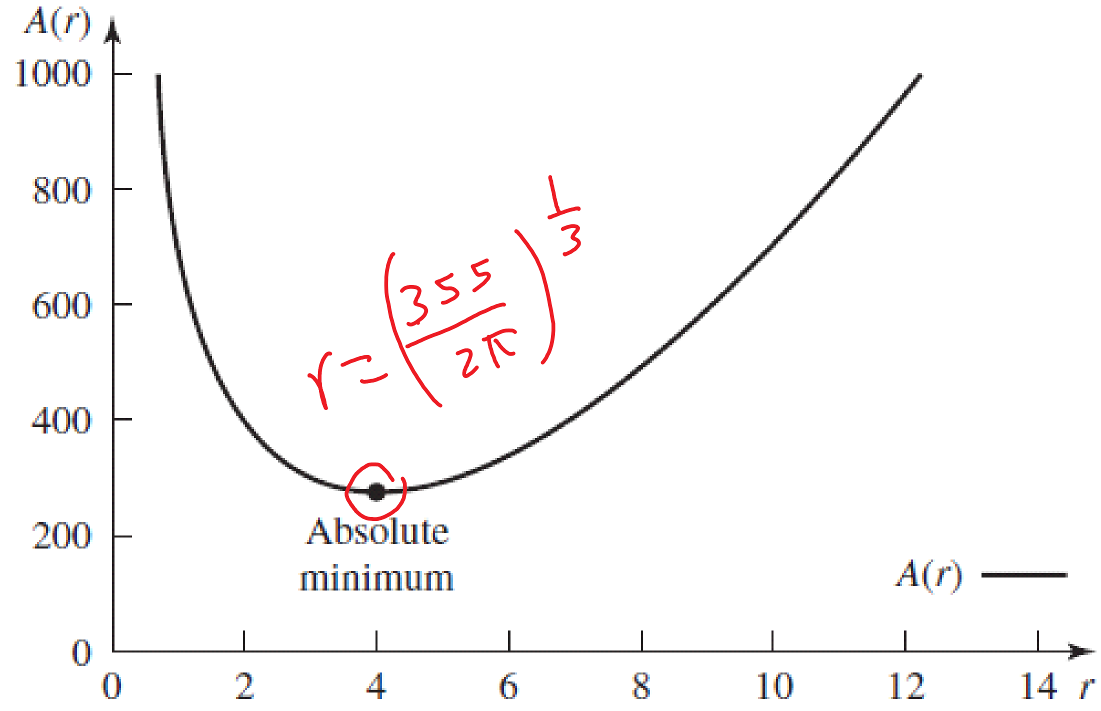


Figure 5.54 The surface area A of a right circular cylinder

5.5 L'Hospital's Rule

L'Hospital's Rule Suppose that f and g are differentiable functions and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \qquad \left(\frac{0}{0}\right)$$

or

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty \qquad \left(\frac{\infty}{\infty}\right)$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L$$

ex. (1) $\displaystyle \lim_{x \to 2} \frac{x^6 - 64}{x^2 - 4} \qquad \left(\frac{0}{0}\right)$

$$= \lim_{x \to 2} \frac{6x^5}{2x} = \lim_{x \to 2} (3x^4) = 3(2)^4 = 48$$

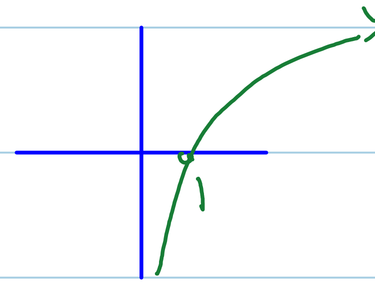
(2) $\displaystyle \lim_{x \to 0} \frac{1 - \cos^2 x}{\sin x} \qquad \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{-2\cancel{\cos x}(-\sin x)}{\cancel{\cos x}} = \lim_{x \to 0} (2\sin x) = 2\sin 0 = 0$$

(3) $\displaystyle \lim_{x \to \infty} \frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right)$

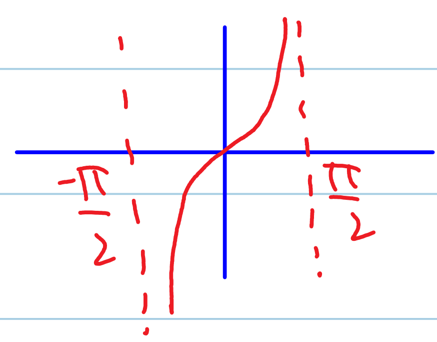
$$= \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

$y = \ln x$



$x \to \infty$, $\ln x \to \infty$

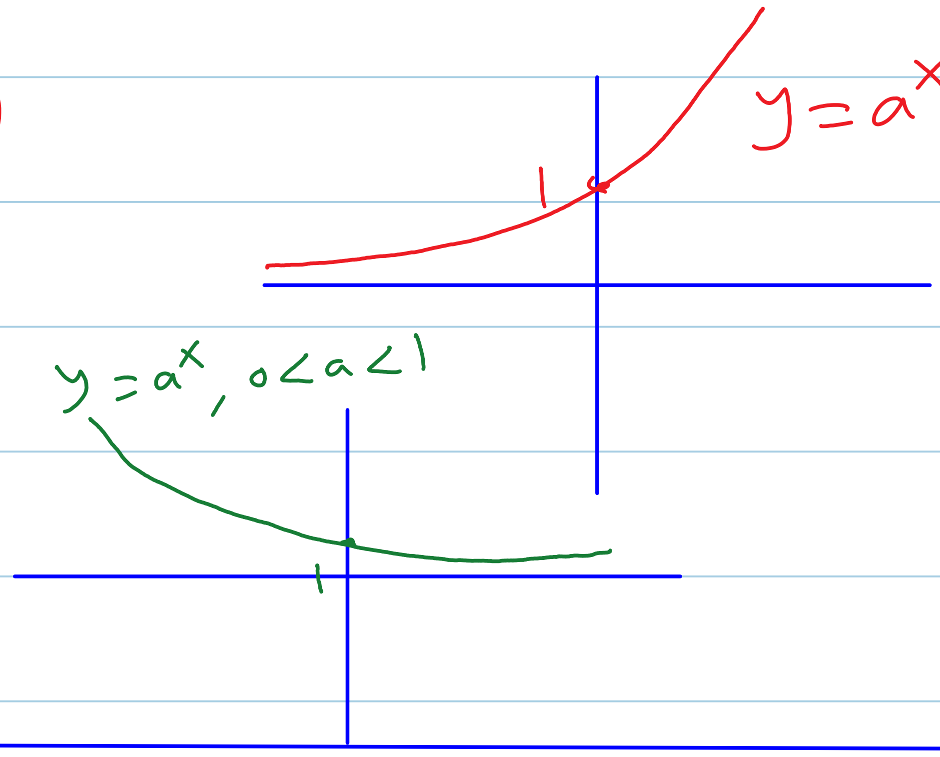
(4) $\displaystyle \lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{1 + \tan x} \qquad \left(\frac{\infty}{\infty}\right)$



$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\sec^2 x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-} 1 = 1.$$

⑤ $\lim_{x\to\infty} \frac{e^x}{x}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty} \frac{e^x}{1} = \infty$$



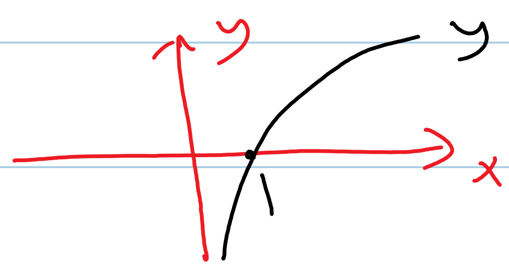
$$\underbrace{0\cdot\infty}, \quad \underbrace{\infty-\infty}, \quad \underbrace{0^0, \ \infty^0, \ 1^\infty}_{\lim_{x\to a}(f(x))^{g(x)}}$$

$0\cdot\infty$ $\quad \lim_{x\to a}[\underset{0}{f(x)}\,\underset{\infty}{g(x)}] = \lim_{x\to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)} \overset{\text{or}}{=} \lim_{x\to a} \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$

which is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Now we can use L'Hôpital

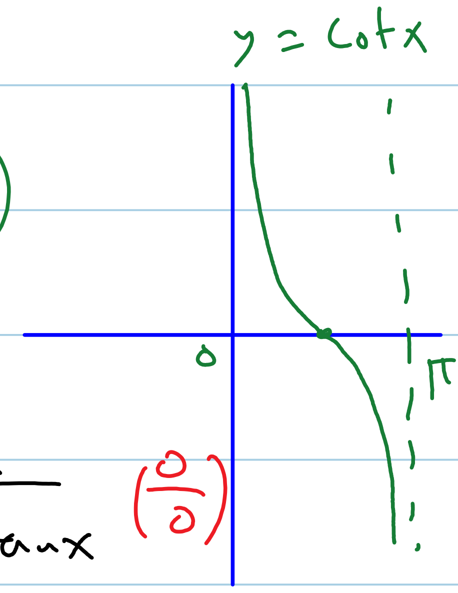
⑥ $\lim_{x\to 0^+} x\ln x$ $\quad 0\cdot(-\infty)$



$x\to 0^+,\ \ln x\to -\infty$

$$= \lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+}(-x) = 0.$$



7) $\lim_{x\to 0^+}(x\cot x)$ $(0\cdot\infty)$

$$=\lim_{x\to 0^+}\frac{x}{\left(\frac{1}{\cot x}\right)}=\lim_{x\to 0^+}\frac{x}{\tan x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 0^+}\frac{1}{\sec^2 x}=\frac{1}{1}=1.$$

8) $\lim_{x\to \frac{\pi}{2}^-}(\tan x-\sec x)$ $(\infty-\infty)$

$$=\lim_{x\to\frac{\pi}{2}^-}\left(\frac{\sin x}{\cos x}-\frac{1}{\cos x}\right)$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{\sin x-1}{\cos x}\quad\left(\frac{0}{0}\right)$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{-\sin x}=\frac{0}{-1}=0.$$

$\lim_{x\to a}(f(x))^{g(x)}$ of the type $0^0, \infty^0, 1^\infty$

$$\lim_{x\to a}(f(x))^{g(x)}=\lim_{x\to a}e^{\ln(f(x))^{g(x)}}\qquad e^{\ln A}=A$$

$$=\lim_{x\to a}e^{g(x)\ln f(x)}=e^{\lim_{x\to a}[g(x)\ln f(x)]}$$

(9) $$\lim_{x\to 0^+} x^x \qquad (0^0)$$

$$= \lim_{x\to 0^+} e^{\ln(x^x)}$$

$$= e^{\boxed{\lim_{x\to 0^+} x\ln x}} \qquad 0\cdot\infty$$

$$= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= e^{\lim_{x\to 0^+} \left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right)} \qquad \boxed{e^A = \exp(A)}$$

$$= e^{\lim_{x\to 0^+}(-x)} = e^0 = 1.$$

(10) $$\lim_{x\to \frac{\pi}{4}^-} (\tan x)^{\tan(2x)} \qquad (1^\infty)$$

$$= \lim_{x\to\frac{\pi}{4}^-} \exp\left[\ln(\tan x)^{\tan(2x)}\right]$$

$$= \exp\left[\lim_{x\to\frac{\pi}{4}^-}\left(\tan(2x)\ln(\tan x)\right)\right] \qquad (\infty\cdot 0)$$

$$= \exp\left[\lim_{x\to\frac{\pi}{4}^-} \frac{\ln(\tan x)}{\cot(2x)}\right] \qquad \left(\frac{0}{0}\right)$$

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$$= \exp\left[\lim_{x\to\frac{\pi}{4}^-} \frac{\left(\frac{\sec^2 x}{\tan x}\right)}{-\csc^2(2x)(2)}\right]$$

$$= \exp\left[\lim_{x\to\frac{\pi}{4}^-} \left(\frac{1}{\cos^{\cancel{2}} x}\cdot\frac{\cancel{\cos x}}{\sin x}\cdot\left(-\frac{1}{2}\right)\sin^2(2x)\right)\right]$$

$$= \exp\left[-\frac{1}{2}\lim_{x\to\frac{\pi}{4}^-} \frac{1}{\cos x \sin x}(2\sin x\cos x)^2\right]$$

$$= \exp\left[-2\lim_{x\to\frac{\pi}{4}^-} (\sin x \cos x)\right]$$

$$= \exp\left[-2\ \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\right]$$

$$= e^{-1} = \frac{1}{e}.$$

The End of ch5

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6 Integration

التكامل

تكامل
Integral $\int$

6.1 The Definite Integral

التكامل المحدود

6.1.1, 6.1.2, 6.1.3

$$\int_a^b f(x)\,dx$$

upper limit.
variable of integration
integrand
lower limit of integration

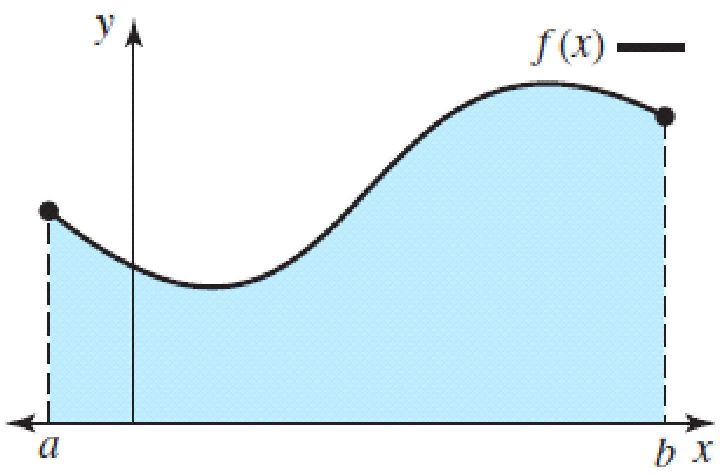
1. If f is integrable on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, then

مساحة

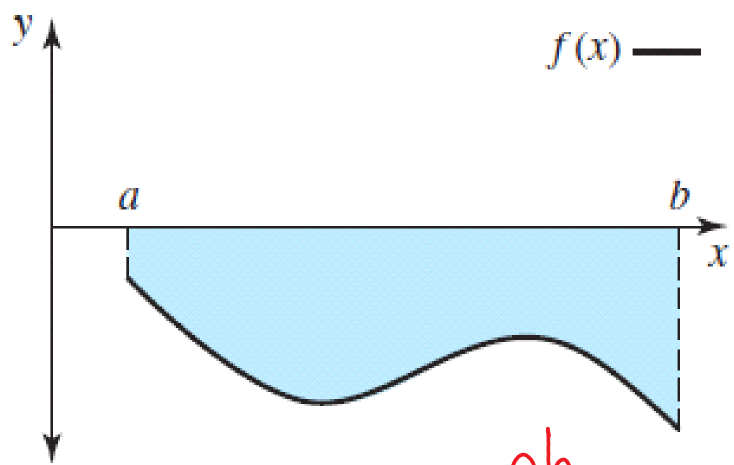
$$\int_a^b f(x)\,dx = \left[\begin{array}{l}\text{the area of the region between the} \\ \text{graph of } f \text{ and the } x\text{-axis from } a \text{ to } b\end{array}\right]$$

2. If f is integrable on $[a, b]$, then

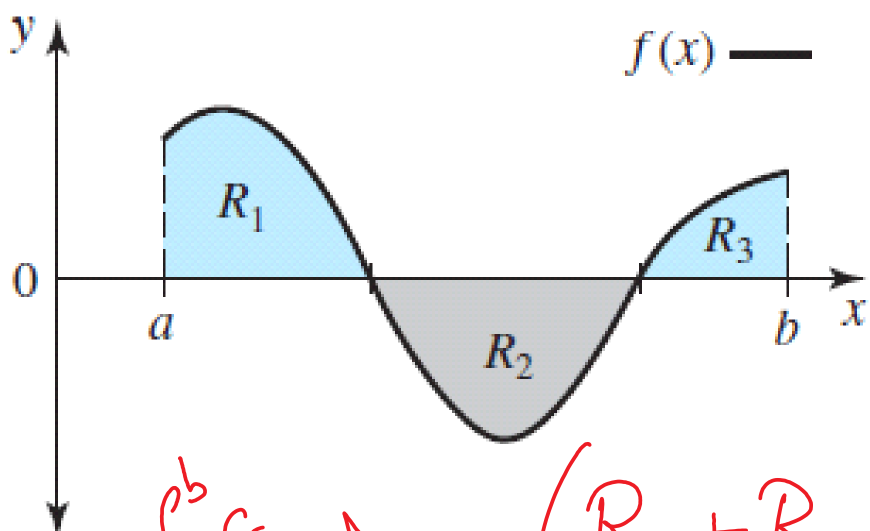
$$\int_a^b f(x)\,dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$



Area $= \int_a^b f(x)\,dx$



Area $= -\int_a^b f(x)\,dx$



$$\int_a^b f(x)\,dx = (R_1 + R_3) - R_2$$

$=$ (area above x-axis) $-$ (area below x-axis).

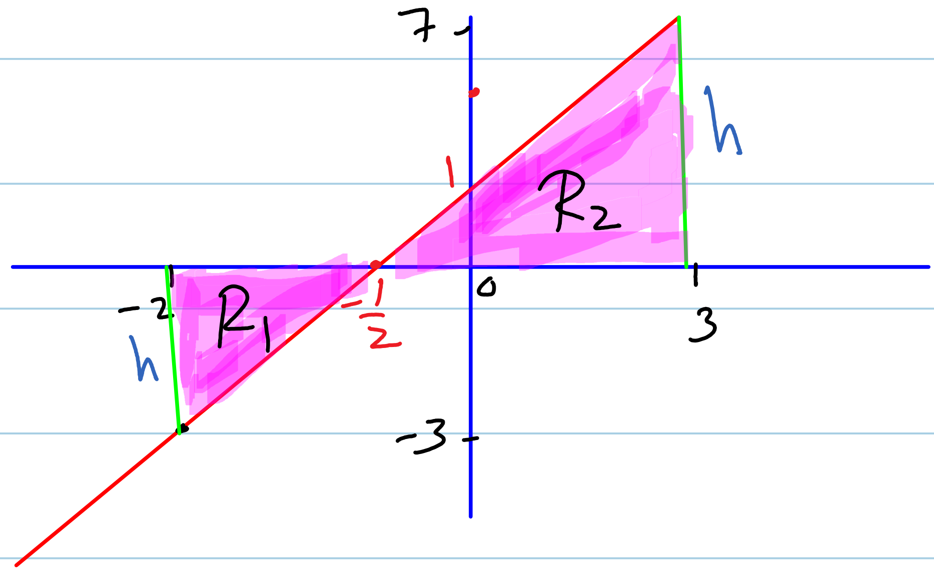
EXAMPLE 9 Find the value of $\int_{-2}^{3}(2x+1)\,dx$

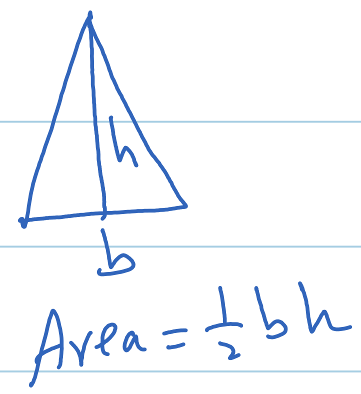
$y = 2x+1$, $[-2, 3]$ $x=-2, y=-3$

$x=3, y=7$

$x=0 \Rightarrow y=1$

$y=0 \Rightarrow 2x+1=0 \Rightarrow x=-\frac{1}{2}$





$$\int_{-2}^{3}(2x+1)\,dx = R_2 - R_1$$

$$= \frac{1}{2}\left(\frac{7}{2}\right)(7) - \frac{1}{2}\left(\frac{3}{2}\right)(3)$$

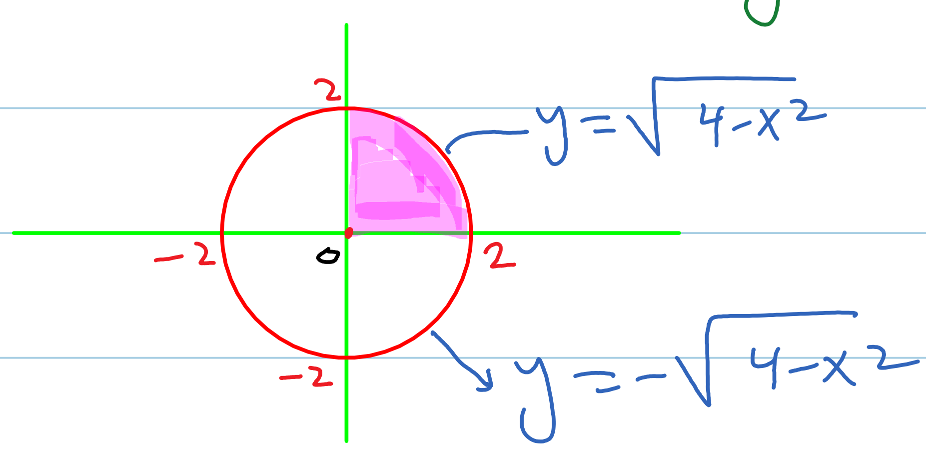
$$= \frac{49}{4} - \frac{9}{4} = \frac{40}{4} = 10 .$$

EXAMPLE 11 Find the value of

$$\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$y = \sqrt{4-x^2}$ $y^2 = 4-x^2$

$x^2+y^2=4$ Circle.



$$\therefore \int_{0}^{2}\sqrt{4-x^2}\,dx = \frac{1}{4} \text{ area of the circle } = \frac{1}{4}\pi(2)^2 = \pi.$$

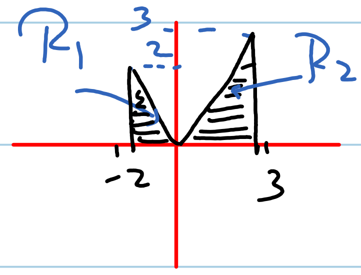
ex. $\int_{-2}^{2} \sqrt{4-x^2}\, dx = \frac{1}{2}\pi(2)^2 = 2\pi.$

ex. $\int_{-2}^{3} |x|\, dx$

$= R_1 + R_2$

$= \frac{1}{2}(2)(2) + \frac{1}{2}(3)(3)$

$= \frac{4}{2} + \frac{9}{2} = \frac{13}{2}.$



6.1.3 Properties of the Riemann Integral

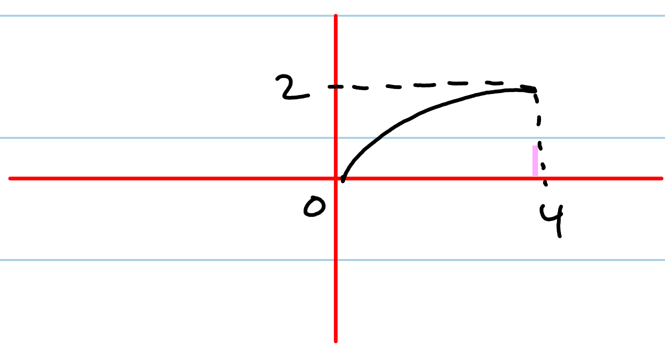
Properties Assume that f and g are integrable over $[a, b]$.

1. $\int_a^a f(x)\, dx = 0$ and
2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
3. If k is a constant, then
$$\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$$
4. $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
5. If f is integrable over an interval containing the three numbers a, b, and c, then
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$
a c b
6. If $\boxed{f(x) \geq 0}$ on $[a, b]$, then $\boxed{\int_a^b f(x)\, dx \geq 0.}$
7. If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$
8. If $m \leq f(x) \leq M$ on $[a, b]$, then
min. max
$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

(9) $\int_a^b k\, dx = k(b-a).$

ex. Show that $0 \leq \int_0^4 \sqrt{x}\, dx \leq 8$

$f(x) = \sqrt{x}$, $[0, 4]$



$0 \leq x \leq 4$

$\boxed{0 \leq \sqrt{x} \leq 2}$

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$$0(4-0) \le \int_0^4 \sqrt{x}\,dx \le 2(4-0)$$

$$0 \le \int_0^4 \sqrt{x}\,dx \le 8$$

ex: If $\int_1^3 f(x)\,dx = 5$, find $\int_3^1 (2f(x)-2)\,dx$

Sol. $\int_3^1 (2f(x)-2)\,dx = \int_3^1 2f(x)\,dx - \int_3^1 2\,dx$

$$= 2\int_3^1 f(x)\,dx - \int_3^1 2\,dx$$

$$= -2\int_1^3 f(x)\,dx - 2(1-3)$$

$$= -2(5) - 2(-2)$$

$$= -10 + 4 = -6.$$

(Q65) $\int_{-2}^2 \left(\sqrt{4-x^2} - 2\right) dx = \int_{-2}^2 \sqrt{4-x^2}\,dx - \int_{-2}^2 2\,dx$

$$= \frac{1}{2}\pi(2)^2 - 2(2--2)$$

$$= 2\pi - 8.$$

6.2 The Fundamental Theorem of Calculus

6.2.1 The Fundamental Theorem of Calculus (Part I)

The Fundamental Theorem of Calculus (FTC) (Part I) If f is continuous on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(u)\,du \qquad a \le x \le b$$

a x b

is continuous on $[a, b]$ and differentiable on (a, b), with

$$\frac{d}{dx} F(x) = f(x)$$

EXAMPLE 1 Compute

$$\frac{d}{dx} \int_0^x (\sin u - e^{-u})\,du = \sin x - e^{-x}$$

for $x > 0$.

EXAMPLE 2 Compute

$$\frac{d}{dx} \int_3^x \frac{1}{1+u^2}\,du = \frac{1}{1+x^2}$$

for $x > 3$.

EXAMPLE 3 Compute

$$\frac{d}{dx} \int_0^{x^2} (u^3 - 2)\,du, \qquad x > 0$$

$$= \left[(x^2)^3 - 2\right] \frac{d}{dx}(x^2)$$

$$= (x^6 - 2)(2x)$$

$$= 2x^7 - 4x.$$

EXAMPLE 4 Compute

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du$$

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du = -\frac{d}{dx}\int_{1}^{\sin x} u^2\,du$$

$$= -(\sin x)^2 \cdot \frac{d}{dx}(\sin x)$$

$$= -\sin^2 x \cos x .$$

EXAMPLE 5 For $x \in \mathbf{R}$, compute

مهم

$$\frac{d}{dx}\int_{x^2}^{x^3} e^u\,du$$

$$\frac{d}{dx}\int_{x^2}^{x^3} e^u\,du = e^{x^3}\frac{d}{dx}(x^3) - e^{x^2}\frac{d}{dx}(x^2).$$

$$= e^{x^3}\cdot 3x^2 - e^{x^2}(2x)$$

Leibniz's Rule If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for u between $g(x)$ and $h(x)$, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

بشكل عام

ex. 37 $\left[\int_{2-x^2}^{x+x^3} \sin t\,dt\right]' = \sin(x+x^3)\cdot(1+3x^2) - \sin(2-x^2)(-2x)$

36 $\frac{d}{dx}\int_{x^3}^{x^4} \ln(1+t^2)\,dt = \ln(1+x^8)\cdot 4x^3 - \ln(1+x^6)\cdot 3x^2.$

التكاملات غير المحدودة

6.2.2 Antiderivatives and Indefinite Integrals

$$\int f(x)\,dx$$

$$\int f(x)\,dx = F(x) + C$$ antiderivatives

$$\Rightarrow F'(x) = f(x).$$

ex. $\int x^4\,dx = \frac{x^5}{5} + C$

In general, $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$

ex. $\int (e^x + \sin x)\,dx = e^x - \cos x + C.$

ex. $\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0.$

ex. $\int \sec^2 x\,dx = \tan x + C.$

ex. $\int 5^x\,dx = \frac{5^x}{\ln 5} + C$

In general, $(a^x)' = a^x \ln a \Rightarrow a^x = \frac{(a^x)'}{\ln a}$

$$\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C}$$

ex. $\int 2^x\,dx = \frac{2^x}{\ln 2} + C.$

ex. $\int \frac{1}{1-\sin^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx$

$= \tan x + C.$

ex. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C.$

ex. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C.$

Summary.

TABLE 6-1 A Collection of Indefinite Integrals

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\int e^x\,dx = e^x + C$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$
$\int \cos x\,dx = \sin x + C$	$\int \sin x\,dx = -\cos x + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x \tan x\,dx = \sec x + C$	$\int \csc x \cot x\,dx = -\csc x + C$
$\int \tan x\,dx = \ln\lvert\sec x\rvert + C$	$\int \cot x\,dx = -\ln\lvert\csc x\rvert + C$
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

The Fundamental Theorem of Calculus (Part II) Assume that f is continuous on $[a, b]$; then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$; that is, $F'(x) = f(x)$.

ex. $\int_{-1}^{2} (x^2 - 3x)\,dx = \left(\frac{x^3}{3} - \frac{3x^2}{2}\right)\Big|_{-1}^{2}$

$= \left[\frac{2^3}{3} - \frac{3(2)^2}{2}\right] - \left[\frac{(-1)^3}{3} - \frac{3(-1)^2}{2}\right]$

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$$= \left(\frac{8}{3} - 6\right) - \left(\frac{-1}{3} - \frac{3}{2}\right)$$

$$= \frac{9}{3} - 6 + \frac{3}{2} = -3 - \frac{3}{2} = -\frac{3}{2}$$

ex. $\displaystyle\int_{-5}^{-1} \frac{1}{x}\,dx = \ln|x| \Big|_{-5}^{-1}$

$$= \ln|-1| - \ln|-5|$$

$$= \cancel{\ln 1}^{\,0} - \ln 5 = -\ln 5$$

ex. $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec^2 x\,dx = \tan x \Big|_{0}^{\pi/4}$

$$= \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1.$$