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Birzeit University
Mathematics Department
Math 234

First Exam — First Semester 2006/2007

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Question 1 (1.5 points each). **Mark each of the following by True of False**

(1) (..F...) If A, B are square $n \times n$ matrices such that $AB = 0$, then A and B are singular.

(2) (..T...) If $A = LU$ is the LU-factorization and A is nonsingular then U is nonsingular.

(3) (..F...) If A, B are $n \times n$ symmetric matrices then AB is symmetric.

$(AB)^T = B^T A^T = BA$

(4) (..T...) If A is symmetric and skew symmetric then A must be a zero matrix.(A is skew symmetric if $A^T = -A$).

(5) (..T...) If the system $AX = b$ is consistent then b is a linear combinations of the columns of A.

$b = \alpha a_1 + \alpha a_2 + \cdots$

$b = x_1 a_1 + x_2 a_2 + \cdots$

(6) (..T...) If A, B are square $n \times n$ matrices and AB is singular then A or B is singular.

(7) (..F...) If A is row equivalent to B then $\det(A) = \det(B)$.

A row equi. to I

det A ≠ det I

(8) (..T...) If the coefficient matrix of the system $Ax = 0$ is singular then the system has infinitely many solutions.

homogeneous ⇒ trivial sol

singular ⇒ there are another solution

(9) (..T...) In the square linear system $AX = b$, if A is singular and b is a linear combination of the columns of A then the system has a infinitely many solution.

$X = A^{-1} b$ ⇒ one sol

singular ⇒ there are another sol

(10) (..T..) A square matrix A is nonsingular iff its RREF (reduced row echelon form) is the identity matrix.

(11) (..T...) If $AB = AC$, and $|A| \neq 0$, then $B = C$.

A is nonsingular

$A^{-1}AB = A^{-1}AC$

$IB = IC$

(12) (..F...) If $det(A) = det(B)$, then $A = B$

(13) (..T...) In the linear system $Ax = b$, if $b = a_1 = a_2 + 3a_4$ then the system has infinite solutions.

$a_1 - a_2 - 3a_4 = 0$ ⇒ many solutions

(14) (..F...) If one row in an echelon form of an augmented matrix of the linear system $Ax = b$ is $[0005|0]$ then the system is inconsistent.

$x_4 = 0$ is a solution

(15) (..T...) The vector $(0, 0, 0)^T$ is a linear combination of the vectors $(1, 2, 3)^T, (1, 4, 1)^T, (2, 3, 1)^T$

(16) (..T...) In the linear system $Ax = 0$, if $a_1 = a_2 + 3a_4$ then the system has infinite solutions.

(17) (..T...) If A is a singular matrix, then A^T is also singular.

$\det(A) = \det(A^T) = 0$

(18) (..F...) If A and B are singular matrices, then $A + B$ is also singular.

$\det A+B \neq \det A + \det B$

(19) (..T..) If A is an $n \times n$ matrix with $A = A^{-1}$, then $\det(A) = \pm 1$.

$A(A^{-1}) = I$

$AA = I$

(20) (..F..) If A is a 3×4-matrix, $b \in \mathbb{R}^3$, and the system $Ax = b$ is consistent, then $Ax = b$ has a unique solution.

underdetermined ⇒ free variables

$A = A^{-1}$

$|A| = |A^{-1}|$

$|A| = \frac{1}{|A|} \Rightarrow |A|^2 = 1$

$|A| = \mp 1$

(21) (.F...) If A, B, C are $n \times n$ nonsingular matrices, then $A^2 - B^2 = (A+B)(A-B)$.

(22) (.T...) If A is a singular matrix and U is the row echelon form of A, then $\det(U) = 0$.

REF

احد الصفوف أو الأعمدة الصفار ⇐ بسبب Elementry

(23) (.F...) If $A = LU$ is an LU-factorization of a singular matrix A, then U is singular.

(24) (..T..) Any two $n \times n$-nonsingular matrices are row equivalent.

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(25) (.F...) If A is a 3×4-matrix, $b \in \mathbb{R}^3$, and the system $Ax = b$ is consistent, then $Ax = b$ has a unique solution.

$Ax=0$ has one sol

$4\times 3 \Rightarrow$ احد الصفوف صفر من نحصل zero sol

Question 2 (2 points each). Circle the most correct answer

المعادلات اكثر المجاهيل

(1) If A is a $\boxed{4 \times 3}$ matrix such that $Ax = 0$ has only the zero solution, and $b = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 0 \end{pmatrix}$, then the system $Ax = b$

A is nonsingular and row-eq to I

(a) is either inconsistent or has one solution
(b) has exactly one solution
(c) is either inconsistent or has an infinite number of solutions
(d) is inconsistent

(2) If A an 3×3 matrix such that $Ax = 0$ has a nonzero solution, then

(a) A is singular
(b) A is nonsingular
(c) A is row equivalent to the identity
(d) $|A| \neq 0$

Homogeneous + nonzero solution $\Rightarrow$ singular

one sol $\rightarrow$ nonsingular

inconsistent, many sol $\rightarrow$ singular

(3) Let $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$. Then the values of a that make A singular are (is)

(a) 1
(b) 2
(c) 1, 2
(d) None

أو مباشرة upper $2 * 1 * a = 0$ singular

$2\begin{vmatrix} 1 & 0 \\ 0 & a \end{vmatrix} = 2(a-0) = 2a =$

$2a = |A|$, $2a = 0$ singular, $a = 0$

$2a = 0$, $a = 0$

(4) Let $(1,2,0)^T$ and $(2,1,1)^T$ be the first two columns of a 3×3 matrix A and $(1,1,1)^T$ be a solution of the system $Ax = (4,2,5)^T$. Then the third column of the matrix A is

(a) $(1,1,4)^T$.
(b) $(1,-1,-4)^T$.
(c) $(4,-1,1)^T$.
(d) $(1,-1,4)^T$.

$\begin{bmatrix} 1 & 2 & x_1 \\ 2 & 1 & x_2 \\ 0 & 1 & x_3 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \\ \\ \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

$3 + x_1 = 4$, $x_1 = 1$

$3 + x_2 = 2$, $x_2 = -1$

$1 + x_3 = 5$, $x_3 = 4$

$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$

$1 + 2 + a = 4$, $a = 1$

$2 + 1 + b = 2$, $b = -1$

$0 + 1 + c = 5$

(5) If E is an elementary matrix then one of the following statements is not true

(a) E is nonsingular.

(b) E^{-1} is an elementary matrix.

(c) E^T is an elementary matrix.

(d) $E + E^T$ is an elementary matrix.

(6) If $AB = 0$, where A and B are $n \times n$ matrices. Then

(a) either $A = 0$ or $B = 0$

(b) both A, B are singular.

(c) both A, B are nonsingular.

(d) either A or B is singular

sing/nonsing

$|AB| = |0|$

$\det A \det B = 0$

sing one of them is singular

nonsing

(7) If B is a 3×3 matrix such that $B^2 = B$. One of the following is always true

(a) B is nonsingular.

(b) $\det(B) = 0$.

(c) $B^5 = B$.

(d) $B = I$.

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$\det(B)\det(B) = \det B$

$(\det(B))(\det B - 1) = 0$

(8) If A and B are $n \times n$ matrices such that $Ax = Bx$ for some non zero $x \in \mathbb{R}^n$. Then

(a) A and B are nonsingular.

(b) A and B are singular.

(c) $A - B$ is singular.

(d) none.

$Ax - Bx = 0$

$(A-B)x = 0$

matrix

$Zx = 0$

$x \neq 0 \Rightarrow |Z||x| = 0$

singular

$(A-B)\bar{x} = 0$

(9) If A is a 3×3 matrix with $\det(A) = -2$. Then $\det(adj(A)) =$

(a) 4.

(b) -4.

(c) 8.

(d) -8.

$\det(adj(A)) = |adj A| = |A|^{n-1}$

$n > 1$

$\Rightarrow (-2)^2 = 4$

OR

$A\, adj(A) = \det(A) I$

$A\, adj(A) = -2I$

$\frac{A}{-2}\, adj(A) = I$

$\det(\frac{-1}{2}A) \det adj(A) = 1$

$(-\frac{1}{2})^3 \det(A)$

$-\frac{1}{8}(-2) \det(adj(A)) = 1$

(10) The adjoint of the matrix $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ is

(a) $\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^T$

$\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$

$A\, adj(a) = \det A\, I$

(11) Assume that the last row in the row echelon form of a 4×4 linear system is $[0 \quad 0 \quad 0 \quad a-3 | b-4]$. The system has one solution if

(a) $b \neq 4$.

(b) $a \neq 3$.

(c) $a \neq 3$ and $b \neq 4$.

(d) $a = 3$, $b = 4$.

a ≠ 3
b any value either 4

(12) Let A be a 4×4 matrix. If the homogeneous system $Ax = 0$ has only the trivial solution then

(a) A is nonsingular.

(b) A is row equivalent to I.

(c) RREF of A is I.

(d) all of the above

nonsingular

(13) An $n \times n$ matrix A is invertible if

(a) there exists a matrix B such that $AB = I$

(b) $|A| = 0$ ⇒ singular ×

(c) $Ax = 0$ ha a nonzero solution × singular

(d) All of the above ×

$x_1 + 2x_2 + 3x_3 = 0$

(14) Let A be a 3×3-matrix such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then

one sol
one sol
2 sol
homogeneous & b has a solution ⇒ A is singular

(a) The system $Ax = 0$ has only one solution

(b) A is the zero matrix

(c) There are elementary matrices $E_1, E_2, \cdots, E_k$ such that $A = E_1 E_2 \cdots E_k$

(d) A is singular.

(15) If A is a nonsingular $n \times n$-matrix, then

(a) There is a singular matrix C such that $A = CI$.

(b) The system $Ax = 0$ has a nontrivial (nonzero) solution. ⇒ one sol

(c) $\det(A) = 1$

(d) There are elementary matrices $E_1, E_2, \cdots, E_k$ such that $A = E_1 E_2 \cdots E_k$.

$\boxed{12}$

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(16) If A and B are two nonsingular $n \times n$-matrices, then

(a) There is a singular matrix C such that $A = CB$.

(b) The system $(AB)x = 0$ has a nontrivial (nonzero) solution.

(c) $\det(A) = \det(B)$

(d) There is a nonsingular matrix C such that $A = CB$

nonsing nonsing nonsing

$A = CB$
$C = AB^{-1}$

(17) If A, B, C are 3×3-matrices, $\det(a) = 9$, $\det(B) = 2$, $\det(C) = 3$, then $\det(3C^T BA^{-1}) =$

(a) 16

(b) 6

(c) 2

(d) 18

$\frac{27 \times 3 \times 2 \times 1}{9}$

$\frac{3 \times 2 \times 9}{27}$

Question 3 (20 points). (1) Let $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 4 \\ 1 & 2 & \alpha & \beta \end{array}\right)$ be the Augmented matrix of a linear system.

(a) For what values of α, β is the system consistent

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$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & (\alpha-1) & (\beta-2) \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & (\alpha-1) & (\beta-2) \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & (\alpha-2) & (\beta-5) \end{pmatrix}$$

the system is ~~in~~ inconsistent if $\alpha = 2$ and $\beta \neq 5$

" " " Consistent " $\alpha \neq 2$ or $\beta = 5$

(b) For what values of α, β is the system inconsistent

$-7 \times 4 - 2 = -14$

(2) (5 points) If $(2A)^{-1} = \begin{pmatrix} -3 & 1 \\ 2 & 4 \end{pmatrix}$, then $A =$

$(2A)(2A)^{-1} = I$

$2A = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \to A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(2A) = \frac{1}{\det(2A)^{-1}}$

$\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 1 \\ 2 & 4 \end{pmatrix}$ $= \frac{1}{-14}$

5

~~$2a = -3$ $2b = 1$~~

~~$2c = 2$ $2d = 4$~~   ~~in~~ $A =$

$$\frac{\begin{pmatrix} 2d & -2b \\ -2c & 2a \end{pmatrix}}{4ad - 4cb} = \begin{pmatrix} -3 & 1 \\ 2 & 4 \end{pmatrix}$$

$\frac{-1}{14} =$

$\cancel{\mathcal{U}}$ $\mathcal{U}$

/10

$\frac{10}{4}$ =

$\frac{-3}{24}$ $\frac{1}{4}$ $\frac{10}{3}$

$\frac{30}{65}$

second First

(3) Let A be an $n \times n$-matrix. Prove that if $AA^T = A$, then A is symmetric and $A^2 = A$.

$A^T = A$

$*AA^T = A$ A is symmetric if $A^T = A$

but $A = AA^T$

why $A^T = AA^T = A$

since $A^T = A$ and $AA^T = A \rightarrow AA = A$

$A^2 = A$

4

First
$AA^T = A$
$A^{T^T}A^T = A$
$A^TA^T = A$
$AA = A$
$A^2 = A$
$A = A$

second

(4) Let $A, B, A+B$ be $n \times n$ nonsingular matrices. Show that $A^{-1} + B^{-1}$ is invertible. Hint consider $A^{-1}(A+B)B^{-1}$

$A^{-1}(A+B)B^{-1} = A^{-1} + B^{-1}$

$(A^{-1}(A+B)B^{-1})C = I$

$(A+B)C = AB$

$AC + BC = AB$

by multiplying the left side by A^{-1} and the right side by $B^{-1}A^{-1}$

$B^{-1}A^{-1}(A+B)CA^{-1}$

$B^{-1}CA^{-1}+$

$(A+B)C = AB$

$C = (A+B)^{-1}AB$

$A^{-1}(A+B)B^{-1}$ is nonsingular because $\det(A^{-1}(A+B)B^{-1})$

$\Rightarrow |A^{-1}||A+B||B^{-1}|$

$\Rightarrow \frac{|A+B|}{|A||B|} \neq 0$ since A, B, $A+B$ nonsingular

$A^{-1}(A+B)B^{-1}$

$\Rightarrow (A^{-1}A + A^{-1}B)B^{-1}$

$\Rightarrow (I + A^{-1}B)B^{-1}$

$\Rightarrow IB^{-1} + A^{-1}BB^{-1}$

$\Rightarrow IB^{-1} + A^{-1}I$

$\Rightarrow B^{-1} + A^{-1}$

4

$\frac{6}{4} =$ $\frac{17}{2}$ $\frac{3}{4}$ $\frac{1}{4}$ 10 34 8 $\frac{30}{35}$

Question 4 Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \\ 1 & 3 & 2 \end{pmatrix}$, then

(a) the LU-factorization of A is given by $L = \begin{pmatrix} \quad \end{pmatrix}$ and $U = \begin{pmatrix} \quad \end{pmatrix}$

5

$$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \\ 0 & \frac{10}{3} & (2-\frac{2}{3}) \\ & & (\frac{4}{3}) \end{pmatrix} \to \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & \frac{-17}{6} \end{pmatrix}$$

$5 \times \frac{-10}{12}$ =

$\frac{-25}{6} + \frac{4}{3}$

$\frac{-17}{6}$ $\frac{8}{6}$

(b) $A^{-1} =$

3

$$L = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & \frac{-17}{6} \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{10}{12} & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 3 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 3 & -1 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 & 1 & 0 \\ 0 & -10 & -4 & 1 & 0 & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & -10 & -4 & 1 & 0 & -3 \end{array}\right)$$

$\frac{50}{4} - \frac{16}{4} =$

$\frac{34}{4} = \frac{17}{2}$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{17}{2} & 1 & \frac{10}{4} & -3 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{2}{17} & \frac{10}{34} & \frac{-6}{17} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & \frac{-3}{4} & 1 \\ 0 & 1 & \frac{5}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{2}{17} & \frac{10}{34} & \frac{-6}{17} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & \frac{-3}{4} & 1 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}\right)$$

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$\frac{-6}{17} \times \frac{-5}{4} = \frac{30}{}$