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Birzeit University- Mathematics Department
Math234, Linear Algebra

First Hour Exam
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Section 1

Question 1(24%) Answer by True or False:

1. It is always true that $\det(AB) = \det(BA)$ for any $n \times n$ matrices A, B. (..✓..)
2. If A and B are nonsingular $n \times n$ matrices that satisfy $AB = BA$ then $A^{-1}B^{-1} = B^{-1}A^{-1}$. (..✓..)
3. If $AB = O$ then either $A = O$ or $B = O$. (..X..)
4. There is a nonzero 2×2 matrix such that $\det(2A) = 2\det(A)$. (..X..) 4 det(A)
5. The matrix AA^T is symmetric. $(AA^T)^T = A^TA$. (..X..)
6. A system with more unknowns than equations is always consistent . (..X..) Usually
7. If $\vec{x}$ is a solution to $A\vec{z} = \vec{b}$ and $\vec{y}$ is a solution to $A\vec{w} = \vec{0}$ then $\vec{x} + \vec{y}$ is a solution to the system $A\vec{z} = \vec{b}$. $A(x+y) = Ax + Ay = b$. (..✓..)
8. If A and B are nonsingular $n \times n$ matrices then $A + B$ is nonsingular. (..X..)
9. The product of two elementary matrices is an elementary matrix. (..X..)
10. $(A + B)(A - B) = A^2 - B^2$. (..X..) $AB \neq BA$
11. If A is $n \times n$ matrix in which $r_i = cr_j$ then $\det(A) = 0$. (..✓..)
12. The transpose of an elementary matrix is an elementary matrix of the same type. (..✓..)
13. All singular matrices are row equivalent. (..X..)
14. If A is nonsingular then adj(A) is nonsingular. (..✓..) non Singular
15. $\det(A^k) = (\det(A))^k$. (..✓..)
16. We can use Cramer's rule to solve any linear system. (..X..)
Just for non-Singular Matrices.

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Question 2(30%) Circle the correct answer (**Show your work**):

1. Consider the homogeneous system $A\vec{x} = \vec{0}$ then
   (a) If $\vec{x}$ is a solution then $c\vec{x}$ is a solution for any constant c.
   (b) If $\vec{x}$ and $\vec{y}$ are two solutions then $\vec{x}+\vec{y}$ is also a solution.
   (c) The system is always consistent.
   (d) All of the above.

2. If $\vec{b} = \vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ then the 3×4 system $A\vec{x} = \vec{b}$ is
   (a) Inconsistent.
   (b) Has only one solution.
   (c) Has infinitely many solutions.
   (d) None of the above.

$X = 1$

3. If A is a nonsingular matrix then
   (a) The reduced row echelon form of A is the identity matrix.
   (b) A is the product of elementary matrices.
   (c) $\det(A) \neq 0$.
   (d) All of the above.

Consider the matrix

$$A = \begin{pmatrix} 1 & \alpha & \beta & 0 \\ 0 & \alpha & \beta & 1 \\ 0 & 0 & \alpha & \beta \end{pmatrix} = 0$$

Use this matrix to answer questions (4,5,6)

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 0$$

4. The matrix is in row echelon form if
   (a) $\alpha = 0$.
   (b) $\beta = 0$.
   (c) $\alpha = 1$.
   (d) $\beta = 1$.

5. The matrix is in reduced row echelon form if
   (a) $\alpha = \beta = 0$.
   (b) $\alpha = \beta = 1$.
   (c) $\alpha = 1, \beta = 0$.
   (d) $\alpha = 0, \beta = 1$.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & \cancel{1} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$x_1 + \cancel{x_2} + x_3 = 6$

$x_1 = -x_3$

$-1 \cancel{x} = 0$

$\boxed{x_2 = 0}$

$x_3 + x_4 = 0$ $\boxed{x_3 = -x_4}$

6. If $\alpha = \beta = 1$, the set of all solutions of the system $A\vec{x} = \vec{0}$ is

(a) $(a, -a, a, a)^T$, $a \in \mathbb{R}$.

(b) $(a, 0, a, a)^T$, $a \in \mathbb{R}$.

(c) $(a, 0, -a, a)^T$, $a \in \mathbb{R}$.

(d) $(a, 0, a, -a)^T$, $a \in \mathbb{R}$.

7. If $\vec{x}_1$ and $\vec{x}_2$ are two solutions of the system $A\vec{x} = \vec{b}$ then $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution if

(a) $c_1 = c_2 = 0$.

(b) $c_1 + c_2 = 1$.

(c) $c_1 + c_2 = 0$.

(d) $c_1 = c_2 = 1$.

~~$x_1, x_2 = 0$ if $c_1 + c_2 = 0$~~

$A(c_1x_1 + c_2x_2) = c_1(Ax_1) + c_2(Ax_2)$

$c_1 b + c_2 b$

8. If A is a 3×3 matrix with $\det(A) = 2$ then $\det(2\text{adj}(A)) =$

(a) 8.

(b) 16.

(c) 32.

(d) 64.

$\det(A) = 2$

$\frac{8}{\det(A)} = \frac{1}{(\det(A))}$

$4 * 8 = 32.$

9. One of the following statements is false:

(a) If $\det(A) = 1$ then $A^{-1} = \text{adj(A)}$.

(b) $A(\text{adj(A)}) = I$.

(c) If A is singular then $\det(A) = 0$.

(d) If A is singular then the system $A\vec{x} = \vec{0}$ has nontrivial solution.

10. The triangular factorization of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is

(a) $L = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

(b) $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

(c) $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$

(d) $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

$L_{21} = 3$

$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

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Question 3(15%) Find the inverse of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{pmatrix}$$

$13 - 16 + 6 - 3$
$26 - 40 + 14 = 40 - 40 = 0.$
$39 - 86 + 18 =$
$39 + 18 = 57$

Then find the solution of the system $A\vec{x} = (3, 0, 1)^T$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right)$$

must be zero.

$row_2 = row_2 - 2r_1$
$r_3 = r_3 - 3r_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right)$$

$r_3 = r_3 - r_2$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & +1 & +1 & +1 & -1 \end{array}\right)$$

$r_2 = r_2 - r_3$
$r_1 = r_1 - 3r_3$

$\Downarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & -3 & 3 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$r_1 = r_1 - 2r_2$

$-2+6 = 4$
-3
1

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$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 3 & 7 & 9 & 0 & 0 & 1 \end{array}\right)$$

$row_2 = row_2 - 2r_1$

$r_3 = r_3 - 3r_1$

must be zero.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right)$$

$r_3 = r_3 - r_2$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & +1 & +1 & +1 & -1 \end{array}\right)$$

$r_2 = r_2 - r_3$

$r_1 = r_1 - 3r_3$

$\Downarrow$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & -3 & 3 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$r_1 = r_1 - 2r_2$

$-2+6 = 4$

-3

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & 1 \\ 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\vec{X} = A^{-1} b$$

$$= \begin{pmatrix} 4 & -3 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 12+1 \\ -9+1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 13 \\ -8 \\ 2 \end{pmatrix}$$

Solution $(13, -8, 2)$

(x_1, x_2, x_3)

Question 4(15%) A matrix A is called orthogonal if $AA^T = A^TA = I$.

(a) Show that type I elementary matrices are orthogonal. Hint: use the fact that type I matrices are symmetric.

type I is symetric $\Rightarrow E_1^T = E_1$

$E_1 E_1^T = E_1 E_1 = E_1^2$

type I $E^T = E^{-1}$

AND we Now that $E_1^{-1} = E_1$ (Non singular)

$\therefore \quad E_1^2 = E_1 E_1 = E_1 E_1^{-1} = I$

AND :- $E_1^T E_1 = E_1 E_1 = E_1^{-1} E_1 = I.$

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(b) Show that the product of two orthogonal matrices is orthogonal.

A is orth. then $AA^T = A^TA = I$, B is orth. then $BB^T = B^TB = I$.

$A^T = A^{-1} = (A^T)^{-1}$ $\qquad B^T = B^{-1} = (B^T)^{-1}$

$(AB)(AB)^T = AB(B^TA^T) = AB(B^{-1}A^{-1}) = AB(AB)^{-1} = I$

$(AB)^T(AB) = (B^TA^T)(AB) = (B^{-1}A^{-1})(AB) = (AB)^{-1}(AB) = I.$

$\therefore$ AB is orthogonal.

(c) Show that if A is orthogonal then $\det(A) = \pm 1$.

$A^TA = I$ if we take the determinant

$\det(A^TA) = \det(I)$ but $\det(A^T) = \det(A)$

$[\det(A)]^2 = 1$ $\qquad$ " $\det(A^2) = (\det(A))^2$

$\therefore \det(A) = \pm 1.$

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Question 5(16%) Use Gauss-Jordan reduction to solve the following linear system

$$
\begin{aligned}
\alpha x_1 + x_2 + x_3 &= 1\\
x_1 + \alpha x_2 + x_3 &= 1\\
x_1 + x_2 + \alpha x_3 &= 1
\end{aligned}
$$

Consider the following cases:

(a) $\alpha = 1$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 \end{array}\right) \qquad r_2 = r_2 - r_1 \quad r_3 = r_3 - r_1$$

$$\boxed{\text{We have infinitely many solution}}$$

free varible

lead Varible

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right)$$

"Gauss-Jordan reduction"

$X_1 + X_2 + X_3 = 1$, let $X_1 = B$, $X_2 = \alpha$ $\} \in \mathbb{R}$

but because we have two free Varible and reduce row echelon form.

$\vec{X} = (B, \alpha, 1-\alpha-B)$ $\alpha, B \in \mathbb{R}$

(b) $\alpha = 0$.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 \end{array}\right) \qquad r_3 = r_3 - r_2.$$

must be zero

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 0 & 1 & -1 & 0 \end{array}\right) \qquad r_3 = r_3 - r_1.$$

must be zero.

$$\Downarrow$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 0 & 0 & -2 & -1 \end{array}\right) \qquad r_1 = r_1 + \tfrac{1}{2} r_3$$

$$\Downarrow$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & \tfrac{1}{2}\\ 1 & 0 & 1 & 1\\ 0 & 0 & -2 & -1 \end{array}\right) \qquad r_2 = r_2 + \tfrac{1}{2} r_3$$

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$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$ replace r_1 & r_2

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$$

Using back substitution

$X_3 = \frac{1}{2}$

$X_2 = \frac{1}{2}$

$X_1 = \frac{1}{2}$

Solution $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$