MATH 234
MIDTERM EXAM

Student Name: Key I Student Number:

Instructor: Section:

Question 1. (22 points) Answer by true or false:

1. F If A is row equivalent to B, then $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}$ have the same solution set.
2. F If a system of linear equations is overdetermined and consistent, then it must have infinitely many solutions.
3. F If A, B, C are $n \times n$ matrices such that $AB = AC$, then $B = C$.
4. T Span$\{1 + x, 1 - x\}$ is a subspace of P_2
5. T Any nonsingular matrix can be written as a product of elementary matrices.
6. F If A is row equivalent to B then both A and B are nonsingular.
7. T If span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = R^3$ then span$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}\} = R^3$ for any $\mathbf{x} \in R^3$.
8. T If A is singular then adj(A) is singular.
9. T If A is singular and U is the row echelon form of A, then $\det(U) = 0$.
10. F Suppose that $\{f_1, f_2, \ldots, f_n\} \subseteq C^{n-1}[a, b]$. If $W[f_1, \ldots, f_n] = 0$, where W denotes the Wronskien, then $f_1, f_2, \ldots, f_n$ are linearly dependent.
11. F Every linear system can be solved using Cramer's Rule.
12. F If E_1 and E_2 are elementary $n \times n$ matrices, then E_1E_2 is elementary.
13. T $(AB)C = A(BC)$ for all matrices A, B, and C when multiplication is allowed.
14. T If A and B are $n \times n$ matrices and A is singular, then AB is singular.
15. T If A, B are $n \times n$ matrices, $|A| = 2$ and $|B| = -2$, then $|A^{-1}B^T| = -1$.
16. F If A and B are symmetric $n \times n$ matrices then AB is symmetric.
17. T A linear homogeneous system which has a nonzero solution must have infinitely many solutions.
18. T If S is a set of vectors that are linearly independent in a vector space V then any nonempty subset of S is linearly independent
19. F If S is a set of vectors that are linearly independent in a vector space V then any subset of V containing S is linearly independent
20. F If S is a subspace of V then any set of vectors in S that spans S also spans V.
21. F If A is a singular $n \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every vector $\mathbf{b} \in R^n$.
22. T The set $\{1, \sin^2 x, \cos^2 x\}$ is linearly dependent in $C[0, \pi]$.

Question 2 (36 points) Circle the most correct answer:

1. One of the following is a subspace of $R^{n\times n}$

   (a) All singular $n \times n$ matrices
   (b) All upper triangular $n \times n$ matrices
   (c) All nonsingular $n \times n$ matrices
   (d) All triangular $n \times n$ matrices

2. The set of vectors $\{(1, a)^T, (b, 1)^T\}$ is a spanning set for R^2 if

   (a) $a \neq 1$ and $b \neq 1$
   (b) $ab \neq 1$
   (c) $ab = 1$
   (d) $a \neq b$

3. One of the following matrices is in reduced row echelon form

   (a) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

4. One of the following is a spanning set of R^3

   (a) $\{(1, -1, 1)^T, (-4, 4, -4)^T, (3, 0, 5)^T\}$
   (b) $\{(2, 0, 0)^T, (0, 3, -4)^T\}$
   (c) $\{(2, 0, 0)^T, (0, 2, 0)^T, (1, 1, 0)^T, (0, -3, 0)^T\}$
   (d) $\{(-1, -1, -1)^T, (-1, -2, -1)^T, (-1, 0, 0)^T\}$

5. Suppose that $\mathbf{y}$ and $\mathbf{z}$ are both solutions to $A\mathbf{x} = \mathbf{0}$ then

(a) $A\mathbf{x} = \mathbf{0}$ has exactly two solutions
(b) $\mathbf{y} = \mathbf{z}$
(c) $\mathbf{y} + \mathbf{z}$ is a solution to $A\mathbf{x} = \mathbf{0}$
(d) None of the above

6. The set $\{(1,1,1)^T, (1,1,c)^T, (1,c,1)^T\}$ is linearly independent in R^3 if

(a) $c \neq 1$
(b) $c = -1$
(c) $c = 1$
(d) c is any real number

7. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ then the $(2,3)$ entry of A^2 is

(a) 0
(b) 1
(c) 2
(d) 3

8. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} a & 4d & a+g \\ b & 4e & b+h \\ c & 4f & c+i \end{bmatrix}$. If $|A| = 3$ then $|B| =$

(a) 12
(b) 6
(c) 3
(d) 24

9. If A is a 3×3 matrix with $\det(A) = 2$ then $\det(\text{adj}(A))$ is

(a) 2
(b) 4
(c) 0
(d) $\frac{1}{2}$

10. Let $A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$. Then the adjoint of A is

(a) $\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

11. One of the following is not a subspace of P_3

(a) $\{p \in P_3 \,|p(1) = p(-1)\}$
(b) $\{p \in P_3 \,|p(2) = 0\}$
(c) $\{p \in P_3 \,|p(2) = p(5)\}$
(d) $\{p \in P_3 \,|p(0) = 2\}$

12. One of the following sets is linearly independent in P_3

(a) $\{2x, 2 - x, x^2\}$
(b) $\{1 + x, 1 - x, 1\}$
(c) $\{x, x^2, 2x + 3x^2\}$
(d) $\{2, 2 - x, x\}$

13. One of the following sets is a subspace of $C[-1, 1]$

(a) $\{f(x) \in C[-1, 1]\ ;\ f(1) = 0\}$
(b) $\{f(x) \in C[-1, 1]\ ;\ f(1) = 1\}$
(c) $\{f(x) \in C[-1, 1]\ ;\ f(1) = -1\}$
(d) $\{f(x) \in C[-1, 1]\ ;\ f(1) = 0 \text{ or } f(-1) = 0\}$

14. Suppose A and B are $n \times n$ nonsingular matrices. Then

(a) AB is nonsingular
(b) $B^T A^{-1}$ is nonsingular
(c) ABA^{-1} is nonsingular
(d) all of the above

15. The vectors $1, x, x^2, x^2 + x - 1$

   (a) are linearly independent in P_3
   (b) are linearly independent in P_4
   (c) span P_3
   (d) span P_4

16. consider the linear system $\begin{array}{lll} 4x_1 & +kx_2 & = 4 \\ kx_1 & +x_2 & = -2 \end{array}$.

   (a) The system has a unique solution if
      i. $k = 2$
      ii. $k \neq -2,\ 2$
      iii. $k \neq -2$
      iv. none of the above
   (b) The system has infinitely many solutions if
      i. $k = 2$
      ii. $k \neq 2$
      iii. $k = -2$
      iv. none of the above
   (c) The system is inconsistent if
      i. $k = 2$
      ii. $k \neq -2$
      iii. $k = -2$
      iv. none of the above

Question 3. (14 points) Recall that the null space of an $m \times n$ matrix A is the set

$$N(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\}.$$

1. Prove that $N(A)$ is a subspace of R^n.

$N(A)$ is nonempty since $A\vec{x}=\vec{0}$ always has the trivial sol $\Rightarrow \vec{0} \in N(A)$

1) let $\vec{x} \in N(A)$ & let α be a scalar

$A(\alpha\vec{x}) = \alpha(A\vec{x}) = \alpha.\vec{0} = \vec{0} \Rightarrow \alpha\vec{x} \in N(A)$

2) let $\vec{x}, \vec{y} \in N(A)$

$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0}+\vec{0} = \vec{0} \Rightarrow \vec{x}+\vec{y} \in N(A)$

2. Find the null space of the matrix $A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

lead variables : x_1, x_3

free " : x_2, x_4

let $x_2 = t$, $x_4 = s \longrightarrow x_3 = -s$

$x_1 = 2t+s$

$$\vec{x} = \begin{bmatrix} 2t+s \\ t \\ -s \\ s \end{bmatrix}, \quad t, s \in R$$

Question 4 (8 points) Let A be an $n \times n$ matrix satisfying $A^T A = A$. Prove the following:

1. A is symmetric.
2. $A = A^2$.
3. Find all possible values of $\det(A)$.

1. $A = A^T A$

$A^T = (A^T A)^T = A^T (A^T)^T = A^T A = A$

2. $A = A^T A = A \cdot A$ since $A^T = A$

$= A^2$

3. $A = A^T A \Rightarrow |A| = |A^T A|$

$= |A^T| |A|$

$= |A| \cdot |A| = |A|^2$

$\Rightarrow |A|^2 - |A| = 0$

$|A| (|A| - 1) = 0$

$\Rightarrow |A| = 0, 1$