Birzeit University
Mathematics Department
Math234
Quiz#4

Instructor: Dr. Ala Talahmeh — First Semester 2021/2022
Name:... — Number:...............
Time: 15 minutes — Date: 23/12/2021

Exercise#1 [**5 marks**]. Let $W = \{ax^3 + bx^2 + cx + d : c = a - b,\ d = a + b + c\}$ be a subspace of the vector space P_4. Find a **basis** and **dimension** for W.

Solution. $f(x) \in W \Rightarrow f(x) = ax^3 + bx^2 + (a-b)x + a + b + a - b$

$= a(x^3 + x + 2) + b(x^2 - x)$

$\therefore \text{Span}(x^3+x+2,\ x^2-x) = W$

- Linear Independence:

Let $c_1(x^3+x+2) + c_2(x^2-x) = 0$

$x^3: c_1 = 0$

$x^2: c_2 = 0$

$x: c_1 - c_2 = 0$

$x^0: 2c_1 = 0$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & 0 \end{array}\right] \xrightarrow[-2R_1+R_4]{-R_1+R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow c_1 = c_2 = 0$$

lin. indep.

$\therefore \{x^3+x+2,\ x^2-x\}$ form a basis for W and $\dim W = 2$.

Exercise#2 [**5 marks**]. Let $E = [1,\ x,\ x^2]$ and $F = [1,\ 1+x,\ 1+x+x^2]$ be two ordered bases for P_3. Find the **transition matrix** S from E to F and use it to find $[p(x)]_F$ where $p(x) = 2 + 5x - x^2$.

Solution.

$$S_{E\to F} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore\ S_{E\to F} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[p(x)]_F = [2 + 5x - x^2]_F = (S_{E\to F})\,[p(x)]_E$$

Notice $p(x) = 2(1) + 5(x) + (-1)(x^2)$

$$\therefore\ [p(x)]_E = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$\therefore\ [p(x)]_F = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix}$$