Exam April 2018, questions and answers

Linear Algebra (جامعة بيرزيت)

Birzeit University
Mathematics Department
Math 234

Second Exam-answers

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Q1 (36 points) Answer the following statements by true or false:

(1) $(\ldots\ldots F)$ The coordinate vector of $2+2x$ with respect to the basis $[2x, 4]$ is $(1,2)^t$

(2) $(\ldots\ldots T)$ If two matrices are row equivalent, they must have the same null space.

(3) $(\ldots\ldots T)$ If A is an $n \times n$ invertible matrix, then the linear system $AX = b$ is consistent for every $b \in R^n$.

(4) $(\ldots\ldots T)$ Any subset of a vector space that does not contain the zero vector is not a subspace.

(5) $(\ldots\ldots T)$ The set $S = \{f \in C[-1,1] : f(0) = 0\}$ is a subspace of $V = C[-1,1]$

(6) $(\ldots\ldots T)$ $S = \{A \in R^{2\times 2} : a_{11} = 0\}$ is a subspace of $V = R^{2\times 2}$

(7) $(\ldots\ldots T)$ $S = \{v = (x,y) \in R^2 : x + y = 1\}$ is not a subspace of $V = R^2$

(8) $(\ldots\ldots F)$ Any subset of a vector space that contains the zero vector is a subspace.

(9) $(\ldots\ldots T)$ If $v_1, v_2, ..., v_n$ span a vector space V and v_n is a linear combination of $v_1, ..., v_{n-1}$, then $V = Span\{v_1, ..., v_{n-1}\}$.

(10) $(\ldots\ldots T)$ If two none zero vectors in a vector space V are linearly dependent, then one of them is a scalar multiple of the other.

(11) $(\ldots\ldots T)$ The vectors $(0,0,0)^T, (2,3,1)^T, (2,-5,3)^T$ are linearly dependent.

(12) $(\ldots\ldots T)$ If n vectors span a vector space V, then a collection of $m > n$ vectors in V is linearly dependent.

(13) $(\ldots\ldots T)$ If V is a vector space with dimension $n > 0$, then any set of $m < n$ vectors in V does not span V.

(14) $(\ldots\ldots F)$ The set $S = \{v_1, ..., v_n\}$ is a basis of a vector space V if every vector in V is a linear combination of the set S.

(15) $(\ldots\ldots F)$ If $v_1, v_2, ..., v_n$ are linearly dependent, then $v_1 \in Span\{v_2, ..., v_n\}$.

(16) $(\ldots\ldots F)$ A basis for the subspace $S = \{(a+b+2c, a+2b+4c, b+2c)^T, a, b, c \in R\}$ is $\{(1,1,0)^T, (1,2,1)^T, (1,2,1)^T\}$

(17) $(\ldots\ldots F)$ A basis for the subspace $S = \{f \in P_3 : f(0) = 0\}$ is $\{x^2 + x\}$

(18) $(\ldots\ldots T)$ The set of vectors $x, x-1, x^2-x-1, sinx, e^x$ are linearly independent

Q2 :(39 points) Choose the correct answer.

(1) Let u and v be distinct (not equal)vectors in R^n, and let B be a basis for R^n. Then

(a) the coordinate vector of u with respect to B never equals u

(b) the coordinate vector of v with respect to B equals v

(c) the coordinate vector of $u + v$ with respect to B equals the sum of the coordinate vector of u and the coordinate vector of v with respect to B. T

(d) None

(2) Let V and W be subspaces of R^n such that V is contained in W. Then

(a) V and W may have the same dimension even though they need not be equal

(b) every subset of W that spans W contains a set that spans V. T

(c) every basis for V can be extended to a basis for W. T

(d) None

(3) For any finite $n-$dimensional vector space V with a basis B

(a) The coordinate vector of any vector v in V is in R^n. T

(b) A subspace of V is a subset of V that contains a zero vector and is closed under the operation of addition

(c) The set of nonzero vectors in V is a subspace of V

(d) None

(4) For any vector space V,

(a) If V is finite-dimensional, then V is a subspace of R^n for some positive integer n

(b) If V is infinite-dimensional, then every infinite subset of V is linearly independent

(c) If V is finite-dimensional, then no infinite subset of V is linearly independent. T

(d) None

(5) An $n \times n$ matrix A is invertible if

(a) The columns of A are li

(b) The rows of A are li

(c) $N(A) = \{\mathbf{0}\}$

(d) all of the above. T

(6) Let S be a finite subset of a subspace W of R^n. Then S is a basis for W if

(a) S is linearly independent

(b) S spans W

(c) every vector in W is a linear combination of vectors in S

(d) None. T

(7) Suppose that W is a subspace of R^n. Then

(a) the dimension of W is greater than n

(b) every basis of R^n contains a basis of W

(c) every linearly independent subset of W has at most n vectors. T

(d) None

(8) One of the following is not a subspace in the corresponding space

(a) $S = \{f \in C(R) : f(1) = 0\}, V = C(R)$

(b) $S = \{A \in R^{2\times 2} : a_{11} = 0\}, V = R^{2\times 2}$

(c) $S = \{v = (x, y) \in R^2 : x + y = 1\}, V = R^2$. T

(d) $S = \{v = (x, y) \in R^2 : x + y = 0\}, V = R^2$

(9) For an finite dimensional vector space V,

(a) every infinite subset of V spans V

(b) every infinite subset of V is linearly independent.

(c) every finite subset of V can not span V.

(d) None. T

(10) The dimension of the null space of $\begin{pmatrix} 1 & 1 & 2 & 1 & 4 \\ 2 & -1 & 2 & -1 & 6 \\ 3 & 0 & 4 & 0 & 10 \end{pmatrix}$ is

(a) 0

(b) 1

(c) 2

(d) 3. T

(11) One of the following set of vectors are linearly independent

(a) $(1, 1, 2, 1, 4), (2, 2, 4, 2, 8)$

(b) $(1, 1, 2, 1, 4), (2, -1, 2, -1, 6), (0, 0, 0, 0, 0)$

(c) $x, 1, x^2 + 1$. T

(d) $(1, 2, 3), (0, 1, 0), (0, 0, 1), (1, 1, 1)$

(12) The dimension of the subspace $S = \{(a + b + 2c, a + 2b + 4c, b + 2c)^T, a, b, c \in R\}$ is

(a) 4

(b) 1

(c) 2. T

(d) 3

(13) A basis for the vector space spanned by $1 - x - x^2, 1 + x + x^2, 2 - x, 1 - x$ from this set of vectors is

(a) $1 - x - x^2, 1 + x + x^2, 2 - x$. T

(b) $1 - x - x^2, 1 + x + x^2$

(c) $1 - x - x^2, 1 + x + x^2, 2 - x, 1 - x$

(d) $1 - x - x^2, 1 - x$

Q3 (12 points):(a) If U, W are subspaces of a vector space V. Show that $U \cap W$ is a subspace of V

1.$0 \in U \cap W$, since $0 \in U$, and $0 \in W$. So $U \cap W \neq \phi$. **(2 points)**

2. Let $x, y \in U \cap W$. Then $x, y \in U$, and $x, y \in W$. since U, W are subspaces of V, so $x + y \in U$, and $x + y \in W$. So, $x + y \in U \cap W$. **(2 points)**

3. Let $x \in U \cap W, \alpha \in R$. So $x \in U$, and $x \in W, \alpha \in R$. Since U, W are subspaces of V, so $\alpha x \in U$, and $\alpha x \in W$. So, $\alpha x \in U \cap W$. **(2 points)**

so, $U \cap W$ is a subspace of V

(b) Let $S = \{(0, a)^t : a \in R\}$. Show that S is a subspace of R^2.

1.$0 \in S$, by taking $a = 0$. So $S \neq \phi$. **(2 points)**

2. Let $x, y \in S$, say, $x = (0, a)^t, y = (0, b)^t : a, b \in R$. Then $x + y = (0, a + b)^t : a + b \in R$, and so $x + y \in S$. . **(2 points)**

3. Let $x = (0, a)^t : a \in R, \alpha \in R$. So $\alpha x = (0, \alpha a)^t \in S$. So, S is a subspace of R^2.. **(2 points)**

Q4: (15 points)

1. Let $V = P_3$, and let $U = \{f \in V : f(0) = f(1) = 0\}$. Find a basis for U
   Let $U = \{f \in V; f(x) = ax^2 + bx + c, a, b, c \in R, f(0) = f(1) = 0\}$. So, $c = 0$, and $a + b + c = 0$, so $b = -a$.**(3 points)**. Thus, $U = \{ax^2 - ax, a \in R\}$. So a basis for U is $x^2 - x$. **(2 points)**
2. Let $V = R^{2\times 2}$, and let $S = \{A \in V : A^t = A\}$. Find a basis for S.
   $S = \{A \in V : A^t = A\}$, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **(2 points)** A basis for S is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.**(3 points)**
3. Let $V = P_2$, $B = [1 - x, 2 + x], F = [1 + 2x, 2 - 3x]$. Find the transition matrix S from B into F
   U_1 the transition matrix from $B = [1-x, 2+x] into E = [1, x]$ is $\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$. **(2 points)**
   U_2 the transition matrix from $F = [1 + 2x, 2 - 3x] into E = [1, x]$ is $\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}$. **(2 points)** So the transition matrix from B into F is $U = U_2^{-1}U_1 = \frac{1}{7}\begin{pmatrix} 1 & 8 \\ 3 & 3 \end{pmatrix}$. **(1 points)**
   Or $U = ([1 - x]_F, [2 + x]_F)$ **(1 points)**, **(2 points)** $[1 - x]_F$, $[2 + x]_F$ **(2 points)**

Q5: (10 points)

1. Let A be an $m \times n$ matrix with $N(A) \neq \{\mathbf{0}\}$. If the system $Ax = b$ is consistent, prove that $Ax = b$ has infinitely many solutions.

   Since $N(A) \neq \{\mathbf{0}\}$, so $Ax = 0$ has a free variable and so $Ax = b$ has a free variable. **(3 points)** And since it is consistent, so it has infinitely many solutions. **(2 points)**

   Or, the solutions of $Ax = b$ are of the form $x_0 + tz, t \in R$, where x_0 is a solution of $Ax = b$, and $z \in N(A)$

2. Let $V = R$ be the set of real numbers with usual addition and multiplication. Show that the only subspaces of V are $\{\mathbf{0}\}$, and R.

Let $S \neq \{\mathbf{0}\}$. So there exists $x \in S, x \neq 0$ **(2 points)**. So $\frac{1}{x} \in R$, **(1 points)**and so $\frac{1}{x}x = 1 \in S$ **(1 points)**. So if $a \in R$, then $a.1 = a \in S$. **(1 points)**. So $S = R$