Birzeit University
Mathematics Department
Math 234

Second Exam — First Semester 2007/2008

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Question 1 (54%) *True or False*

(1) Let A be a 4×6 matrix. Mark each of the following by true or false

- (..F..) *The columns of A are linearly independent.*
- (..F..) *The rows of A are linearly dependent.*
- (..F..) $4 \leq \operatorname{rank}(A) \leq 6$.
- (..F..) *The columns of A form a spanning set for* $\mathbb{R}^4$.
- (..T..) *The columns of A are linearly dependent.*
- (..F..) *The system* $Ax = 0$ *has only the zero solution.*

(2) Let A be a 3×3 matrix and $\operatorname{rank}(A) = 3$. *Mark each of the following by true or false*

- (..T..) *The columns of A are linearly independent.*
- (..T..) *The columns of A form a spanning set for* $\mathbb{R}^3$.
- (..T..) *The system* $Ax = 0$ *has only the zero solution.*
- (..T..) *The reduced row echelon form of A is I.*
- (..T..) *The rows of A are linearly independent.*
- (..F..) *The nullity of A is 3.*
- (..T..) *The system* $Ax = b$ *is consistent for every* $b \in \mathbb{R}^3$.

(3) (..F...) *Every linearly independent set of vectors in* $\mathbb{R}^4$ *has exactly 4 vectors.*

(4) (..T..) *Every spanning set for* $\mathbb{R}^3$ *contains at least 3 vectors.*

(5) (..F...) *Every set of 4 polynomials in* P_3 *can be reduced to form a basis for* P_3.

~~*(6)* (......) *If* $S = \{v_1, \ldots, v_n\}$ *and v is not in Span(S), then* $\{v_1, \ldots, v_n, v\}$ *are linearly independent.*~~

(7) (..F...) *If V is a vector space of dimension n, then any subset from V of more than n vectors spans V.*

(8) (..F...) *If V is a vector space of dimension n, then any subset from V of less than n vectors is linearly independent.*

(9) (..T..) *A basis for a nonzero vector space can't contain the zero vector.*

(10) (..F..) *If A and B are row equivalent, then the nonzero rows of anyone of them forms a basis for the row space of the other.*

(11) (...) If U is the row echelon form or the reduced row echelon form of A, then the rank of A is equal to the number of lead 1's in U.

(12) (..T..) Let U be the row echelon form of A, then the nullity of A is equal to the number of free variables in U.

(13) (..F..) Two row equivalent matrices have the same column space.

(14) (..T..) Two row equivalent matrices have the same null space.

(15) (..T..) If A is a square matrix, and the nullity of A is not zero, then the columns of A are linearly dependent.

(16) (..T..) If A is a 5×7 matrix, then nullity $A \geq 2$.

(17) (..T..) If A is an $m \times n$ matrix, $m > n$, then either the rows or the columns of A are linearly dependent.

(18) (..F..) If A is a singular matrix, then the null space of A is $\{0\}$.

(19) (..F..) If A is a 3×3 matrix and $Ax = 0$ has a nontrivial solution, then the nullity of A is either 1 or 2.

(20) Let V be a vector space, $v_1, v_2, \ldots v_n \in V$ be linearly independent, and $v \in V$. Mark each of the following by true or false

(..F..) The vectors $v_1, v_2, \ldots v_n, v$ are linearly independent.
(..F..) The vectors $v_1, v_2, \ldots v_{n-1}$ are linearly dependent.

(21) Let V be a vector space, $\{v_1, v_2, \ldots v_n\}$ a spanning set for V, and $v \in V$. Mark each of the following by true or false

(..T..) The vectors $v_1, v_2, \ldots v_n, v$ form a spanning set for V.
(..F..) The vectors $v_1, v_2, \ldots v_{n-1}$ form a spanning set for V.

(22) (..T..) If A is an $n \times n$ matrix and the row space of A is $\mathbb{R}^{1\times n}$, then the column space of A is $\mathbb{R}^n$.

(23) (..F..) The transition matrix T from the ordered basis $[e_1, e_2]$ to the ordered basis $[(-1,1)^T, (-1,2)^T]$ is $T = \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}$.

Question 2 (1) [4%] For what values of λ is the set of vectors $\{t^2+3, 2t^2+\lambda^2+2\}$ linearly dependent in P_3

$t^2+3, 2t^2+\lambda^2+2$ are L.D

iff $\exists$ c_1, c_2 not all zero such that $c_1(t^2+3)+c_2(2t^2+\lambda^2+2)=0$

iff $\begin{vmatrix} 1 & 2 \\ 3 & \lambda^2+2 \end{vmatrix} = 0$ iff $\lambda^2=4$

iff $\lambda = \pm 2$.

or L.D iff

$\alpha(t^2+3) = 2t^2+\lambda^2+2$

iff $\alpha = 2$ and $6 = \lambda^2+2$

iff $\lambda^2 = 4$

iff $\lambda = \pm 2$.

Wronskian not true.

(2) [..%] *Find a basis and dimension of the of the subspace of* $\mathbf{R}^4$ *consisting of all vectors of the form* $(a+c, -a+b, -b-c, a+b+2c)^T$.

$$S=\left\{\begin{pmatrix}a+c\\-a+b\\-b-c\\a+b+2c\end{pmatrix}\right\}=\left\{a\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}+c\begin{pmatrix}1\\0\\-1\\2\end{pmatrix} : a,b,c\in\mathbb{R}\right\}$$

since $\begin{pmatrix}1\\0\\-1\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$, so the three vectors are L.D.

$\Rightarrow S=\left\{a\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}+b\begin{pmatrix}0\\1\\-1\\1\end{pmatrix} : a,b\in\mathbb{R}\right\}$ and since $\begin{pmatrix}1\\-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}$ are L.I.

so a basis for S is $\left\{\begin{pmatrix}1\\-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}\right\}$.

(3) [4%] *Find a basis for* $\mathbf{R}^3$ *that includes* $\{(1,0,2)^T,(0,1,3)^T\}$.

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix}\right\}=\left\{\alpha\begin{pmatrix}1\\0\\2\end{pmatrix}+\beta\begin{pmatrix}0\\1\\3\end{pmatrix} : \alpha,\beta\in\mathbb{R}\right\}.$$

choose $v\notin\operatorname{span}\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix}\right)$, for example $v=\begin{pmatrix}1\\1\\1\end{pmatrix}\notin\operatorname{span}\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix}\right)$

(take $\alpha=\beta=1$) $\Rightarrow x=\begin{pmatrix}1\\1\\5\end{pmatrix}\in\operatorname{span}\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix}\right)$

so $\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}$ are L.I. and since $\dim(\mathbb{R}^3)=3$

$\Rightarrow\left\{\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\3\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

(4) [5%] *Find all values of* k *for which* $\{(k^2,0,1)^T,(0,k,2)^T,(1,0,1)^T\}$ *is a basis for* $\mathbf{R}^3$.

The given set is a basis for $\mathbb{R}^3$ iff $\begin{pmatrix}k^2&0&1\\0&k&0\\1&2&1\end{pmatrix}$ is nonsingular

iff $\begin{vmatrix}k^2&0&1\\0&k&0\\1&2&1\end{vmatrix}\neq 0$ iff $k^2(k)+(-k)\neq 0$

iff $k(k^2-1)\neq 0$

iff $k\neq 0, k\neq 1, k\neq -1$.

is a basis $\boxed{\text{iff } k\neq 0,\ k\neq\pm 1}$

(5) [4%] Do the vectors: $A_1 = \begin{pmatrix} -1 & 6 \\ 0 & -6 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & -8 \\ -1 & -4 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ form a basis for $\mathbb{R}^{2\times 2}$

• L.I? Solve $c_1\begin{pmatrix} -1 & 6 \\ 0 & -6 \end{pmatrix} + c_2\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & -8 \\ -1 & -4 \end{pmatrix} + c_4\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ —(*)

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 0 & -1 & -1 & 0 \\ -6 & 0 & -4 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -8 & 6 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -8 & 6 \\ 0 & 0 & 7 & -6 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

so (*) has only the zero solution $c_1 = c_2 = c_3 = c_4 = 0$.

so A_1, A_2, A_3, A_4 are L.I.

and since $\dim(\mathbb{R}^{2\times 2}) = 4$, and A_1, A_2, A_3, A_4 are L.I, they form a basis.

(6) [4%] *Show that the vectors* $1, e^x + e^{-x}, e^x - e^{-x}$ *are linearly independent.*

$$W[1, e^x + e^{-x}, e^x - e^{-x}](x) = \begin{vmatrix} 1 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \end{vmatrix} = (e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})$$

$$= (e^{2x} + e^{-2x} - 2) - (e^{2x} + e^{-2x} + 2) = -4 \neq 0$$

so $1, e^x + e^{-x}, e^x - e^{-x}$ are L.I.

(7) [6%] *Let* $E = [2, 3x, x^2]$, $F = [1 + x, 1 + x^2, 1 + x + x^2]$ *be two bases for* P_3

(a) *Find the transition matrix from* E *to* F.

(b) *Find the coordinate vector of* $p(x) = 2x^2 - x - 1$ *with respect to* F.

(a)
$$\left.\begin{aligned} 1+x &= \tfrac{1}{2}(2) + \tfrac{1}{3}(3x) + 0x^2 \\ 1+x^2 &= \tfrac{1}{2}(2) + 0(3x) + 1\cdot x^2 \\ 1+x+x^2 &= \tfrac{1}{2}(2) + \tfrac{1}{3}(3x) + 1\cdot x^2 \end{aligned}\right\} \Rightarrow T_{F\to E} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 1 \end{pmatrix}.$$

$$\Rightarrow T_{E\to F} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 2 & -3 & 0 \\ -2 & 3 & 1 \end{pmatrix}.$$

$\boxed{p(x) = -\frac{1}{2}(2) - \frac{1}{3}(3x) + 2(x^2)}$

b) $$[p(x)]_F = T_{E\to F}[p(x)]_E = \begin{pmatrix} 2 & 0 & -1 \\ 2 & -3 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{3} \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$