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Mathematics Department
Math 243: set theory

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Second Exam **First Semester 2012/2013**

Q1:) True or false 28

T 1) If R, S are equivalence relations on A then $R \cap S$ is an equivalence relation on A

T 2) If R, S are equivalence relations on A then $R \cup S$ is an equivalence relation on A

F 3) If R, S are equivalence relations on A then $R - S$ is an equivalence relation on A

T 4) If R is an equivalence relation on A then R^{-1} is an equivalence relation on A

T 5) If R, S are equivalence relations on A then RoS is an equivalence relation on A

T 6) If R is an equivalence relation on A and aRb then $[a] = [b]$

F 7) If R is a relation on A and aRb then $[a] = [b]$

T 8) If R is a relation on A then R^{-1} is a relation on A

F 9) If R is a relation on A and aRb then both ab, and ba are edges of the directed graph of R

F 10) If R is a one to one relation on A then R is a function on A

T 11) If R is a relation on $A \times B$ then R is a function on $A \times B$ iff R is bijective

T 12) If f is a function on A then f is relation on A

F 13) If f is a function from X into Y, and A, B subsets of X then $f(A \cap B) = f(A) \cap f(B)$

T 14) If f is a function from X into Y, and A, B subsets of Y then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

F 15) If f is a function from X into Y, and A subsets of Y then $f(f^{-1}(A)) = A$

20 **Q2:)** If A be the set of integers, define a relation R on A as follows, for any $a, b \in Z$ aRb iff $a - b = 7k$ for some $k \in Z$

1) Show R is an equivalence relation on A

2) Find $R[1]$

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Q4:) If f is a function from X into Y, and A, B subsets of X, and C, D subsets of Y. Prove or disprove each of the following statements

1) $f(A \cap B) = f(A) \cap f(B)$

2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

3) $f(f^{-1}(C)) = C$

4) $f(A \cup B) = f(A) \cup f(B)$

5) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

6) $A \subseteq f^{-1}(f(A))$

Q5:) If $A = \{1, 2, 3, 4\}$, and $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (3,1), (2,1), (3,2), (2,3), (1,3)\}$ is a relation on A

$R^{-1} = \{(1,1), (2,2), (3,3), (4,4), (2,1), (3,1), (1,3), (1,2), (2,3), (3,2)\}$

1) Is R an equivalence relation

2) Find $R[1]$

3) Find $R^{-1}\{1,2\}$

Question 2

$a - b = 7k$

1) 1) reflexive: Let ~~x~~ $m \in X$, show mRm

$m - m = 7k$

$0 = 7k$, $\exists$ $k = 0$ integer $\in Z$ so $(m,m) \in R$ so it is reflexive

2) Symmetric: ~~let mRn, nRm $\in R$~~ (m,n) (n,m)

let $(m,n) \in R$, mRn, show nRm

since mRn so $m - n = 7k$ $\exists$ $k \in Z$

$-(n-m) = 7k \Rightarrow n - m = 7\underbrace{(-k)}_{\in Z}$ so nRm

since mRn and nRm so it is symmetric

3) Transitive: let $(m,n), (n,s) \in R$, so mRn and nRs, show mRs

since mRn so $\Rightarrow m - n = 7k_1$ — 1

since nRs so $\Rightarrow n - s = 7k_2$ — 2

by adding 1 and 2 $\Rightarrow$ ~~m-n~~ $m - s = 7\underbrace{(k_1 + k_2)}_{\in Z}$ so mRs

so it is a transitive

Since the relation R is reflexive, Symmetric, transitive on A, so it is an equivalence relation

2) Find $R[1]$

$R[1] = \{a - 1 = 7k, k \in Z\} = \{a = 7k + 1, k \in Z\}$ k: is any integer

~~$= \{ \ldots, 8, 15, \ldots \}$~~ $= \{8, 15, 22, \ldots\}$

Question 3

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

1) Find $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

2) $\alpha = (1\ 2)(3\ 4)$

Question 4

1) False , The true is $f(A \cap B) \subseteq f(A) \cap f(B)$

$RHS \not\subseteq LHS$, but $LHS \subseteq RHS$

counter example : constan function.

Let $f(x) = 1$, $\forall x \in \mathbb{R}$, A, B is asubset of X

~~let A be afunctions on R~~

let A be $\Rightarrow f(A) = 1$, $x \in [1,2]$

let B be $\Rightarrow f(B) = 1$, $x \in [3,5]$

$A \cap B = \emptyset \Rightarrow f(A \cap B) = \emptyset$

$f(A) \cap f(B) = 1 \neq f(A \cap B)$ so it is afalse statment

2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ True

$LHS \subseteq RHS$

Let $x \in f^{-1}(C \cap D)$ so $\exists\, y \in C \cap D$, such that $y = f(x)$

since $y \in C \cap D$, so $y \in C$, so $x = f^{-1}(y) \in f^{-1}(C)$

and ~~so~~ $y \in D$, so $x = f^{-1}(y) \in f^{-1}(D)$

since $x \in f^{-1}(C)$ and $f^{-1}(D)$ so $x \in f^{-1}(C) \cap f^{-1}(D)$

so $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$

$RHS \subseteq LHS$

Let $x \in f^{-1}(C) \cap f^{-1}(D)$, so $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$,

~~so $\exists\, y \in C \ni f(x) = y$, so, $x = f^{-1}(y) \in f^{-1}(C)$~~

~~and $\exists\, z \in D \ni f(x) = z$, so $x = f^{-1}(z) \in f^{-1}(D)$~~

~~since $x \in f^{-1}(C)$ and $f^{-1}(D)$ so x~~

so $\exists y \in C$, such that $f(x) = y$

and $\exists y \in D$, such that $f(x) = y$

so $y \in C$ and D so $y \in (C \cap D) \Rightarrow x \in f^{-1}(C \cap D)$

so $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$

3) $f(f^{-1}(C)) = C$ False , counterexample, constant function

$f: R \longrightarrow$ constant

$f(x) = 2$, $\forall x \in R$

f^{-1} : constant $\longrightarrow R$

4) $f(A \cup B) = f(A) \cup f(B)$ True

LHS $\subseteq$ RHS

let $y \in f(A \cup B)$, so $\exists\, x \in A \cup B \ni f(x) = y$

~~so x~~ since $x \in A \cup B$ so $x \in A$ or $x \in B$

so $f(x) \in f(A)$ or $f(x) \in f(B)$ so $f(x) = y \in f(A) \cup f(B)$

so $f(A \cap B) \subseteq f(A) \cup f(B)$

RHS $\subseteq$ LHS

$A \subseteq A \cup B$ so $f(A) \subseteq f(A \cup B)$ ①

$B \subseteq B \cup A$ so $f(B) \subseteq f(A \cup B)$ ② by adding

$f(A) \cup f(B) \subseteq f(A \cup B)$ so RHS $\subseteq$ LHS

5) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ True

LHS $\subseteq$ RHS

Let $x \in f^{-1}(C \cup D)$ so $y \in C \cup D$ $\ni$ $y = f(x)$

since $y \in C \cup D$ so $y \in C \Rightarrow x \in f^{-1}(C)$

or $y \in D \Rightarrow x \in f^{-1}(D)$

so $x \in f^{-1}(C)$ or $f^{-1}(D)$ so $x \in f^{-1}(C) \cup f^{-1}(D)$

so $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

RHS $\subseteq$ LHS

$C \subseteq C \cup D$ so $f^{-1}(C) \subseteq f^{-1}(C \cup D)$

$D \subseteq C \cup D$ so $f^{-1}(D) \subseteq f^{-1}(C \cup D)$

so $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$.

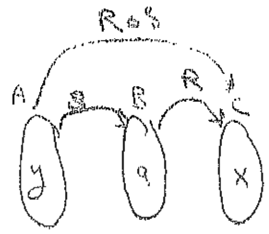
6) $A \subseteq \overleftarrow{f}(f(A))$ True

let $x \in A$ so $f(x) \in f(A)$ so $\cancel{f^{-1}(f(A)) \in f}$

$\overleftarrow{f}(f(x)) \in \overleftarrow{f}(f(A))$

so LHS $\subseteq$ RHS

Question #2(20%) a)Prove thaif A,B,C be sets, and let $R \subseteq B\times C, S \subseteq A\times B$ be relations then $(R\circ S)^{-1} = S^{-1}\circ R^{-1}$



proof

$(x,y) \in (R\circ S)^{-1} \Leftrightarrow (y,x) \in R\circ S$

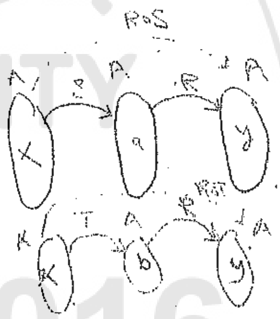
$\Leftrightarrow \exists a ; (y,a) \in S$ and $(a,x) \in R$

$\Leftrightarrow \exists a \in A ; (a,y) \in S^{-1}$ and $(x,a) \in R^{-1}$

$\Leftrightarrow (x,y) \in S^{-1}\circ R^{-1}$

b)Prove tha if R,S,T be relations from A to A then $(R\circ S)\cup(R\circ T) \subseteq R\circ(S\cup T)$

$R,S,T \subseteq A\times A$



proof

let $(x,y) \in (R\circ S)\cup(R\circ T) \Rightarrow (x,y)\in R\circ S$ or $(x,y)\in R\circ T$

$\Rightarrow \exists a ; (x,a)\in S$ and $(a,y)\in R$

or $\exists b ; (x,b)\in T$ and $(b,y)\in R$

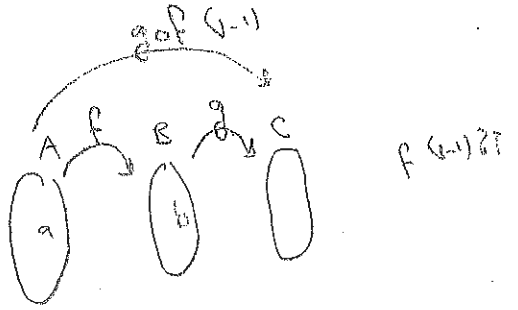
$\Rightarrow ((x,b)\in S$ or $(x,b)\in T)$ and $(b,y)\in R$

$\Rightarrow (x,b)\in (S\cup T)$ and $(b,y)\in R$

$\Rightarrow (x,y) \in R\circ(S\cup T)$

Question #3(20%) Let A, B, C be nonempty set and let $f : A \to B$, $g : B \to C$ be functions.

a) Show that if $g \circ f$ is one to one, then $f : A \to B$ is one to one.



proof: suppose $f(a) = f(b)$

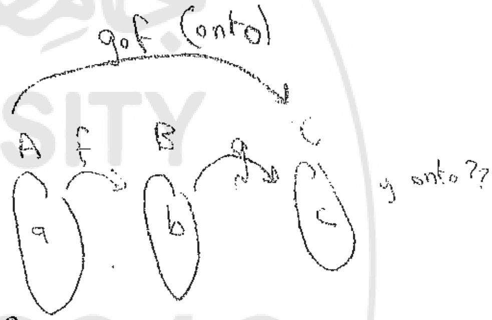
$\Rightarrow g(f(a)) = g(f(b))$

$\Rightarrow g \circ f(a) = g \circ f(b)$

$\Rightarrow$ ~~[illegible]~~ $a = b$ since $g \circ f$ is (1-1)

$\Rightarrow f$ is one to one

b) Show that if $g \circ f$ is onto, then $g : B \to C$ is onto.



proof let $c \in C \Rightarrow \exists a \in A$; $g \circ f(a) = c$ (onto)

$\Rightarrow (a, c) \in g \circ f$

$\Rightarrow \exists b \in B$; $(a, b) \in f$ and $(b, c) \in g$

$\Rightarrow \exists b \in B$; $(b, c) \in g$

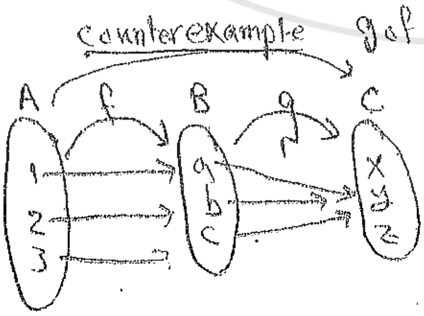
$\Rightarrow g(b) = c$

g is onto

c) Show that the converse of (a) is not true

converse : If $f : A \to B$ is one to one, then $g \circ f$ is one to one.

False counterexample



f is one to one but $g \circ f$ is not one to one