Birzeit University
Mathematics Department
Math 234

97/100

Second Exam First ~~Semester 2016/2017~~

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Question 1 (32%). Mark each of the following by True of False

(1) (F) The set $W = \{p(x) \in P_5 : \text{degree of } p(x) \text{ is odd}\}$ is a subspace of P_5.

(2) (F) If V is a vector space with $\dim(V) = n$, then any set of $n+1$ vectors is a spanning set for V.

(3) (F) If V is a vector space of dimension n, then any subset of V containing more than n vectors is linearly independent.

(4) (F) Every linearly independent set of vectors in $\mathbb{R}^3$ has exactly 3 vectors.

(5) (F) If $S = \{v_1, \cdots, v_n\}$ and v is not in Span(S), then $\{v_1, \cdots, v_n, v\}$ are linearly independent.

(6) (T) A basis for a nonzero vector space cannot contain the zero vector.

(7) Let V be a vector space, $v_1, v_2, \ldots v_n \in V$ be linearly independent, and $v \in V$. Mark each of the following by true or false

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- (F) The vectors $v_1, v_2, \ldots v_n, v$ are linearly independent.
- (T) The vectors $v_1, v_2, \ldots v_{n-1}$ are linearly independent.

(8) Let V be a vector space, $\{v_1, v_2, \ldots v_n\}$ a spanning set for V, and $v \in V$. Mark each of the following by true or false

- (T) The vectors $v_1, v_2, \ldots v_n, v$ form a spanning set for V.
- (F) The vectors $v_1, v_2, \ldots v_{n-1}$ form a spanning set for V.

(9) (T) If V is a vector space of dimension n, then any subset of V containing less than n vectors cannot span V.

(10) (T) If $v_1, v_2, \cdots, v_n \in V$ are linearly independent, and $\dim(V) = n$, then $v_1, v_2, \cdots, v_n$ form a spanning set for V.

(11) (T) Let V be a vector space with $\dim(V) = 3$. If $v_1, v_2, v_3 \in V$ with span$(v_1, v_2, v_3) = V$, then v_1, v_2, v_3 are linearly independent.

(12) (F) If $f_1, f_2, \cdots, f_n \in C^{n-1}[a, b]$ and $W[f_1, f_2, \cdots, f_n](x) = \boxed{0}$ for all $x \in [a, b]$, then $f_1, f_2, \cdots, f_n$ are linearly dependent.

(13) (T) Every spanning set for $\mathbb{R}^3$ contains at least 3 vectors.

(14) (F) Every set of 4 polynomials in P_3 can be reduced to form a basis for P_3.

$-15+14 = -1$

Question 2 (30%). Circle the most correct answer

(1) The transition matrix from the ordered basis $[e_1, e_2]$ of $\mathbb{R}^2$ to the ordered basis $[\begin{pmatrix}-3\\7\end{pmatrix}, \begin{pmatrix}-2\\5\end{pmatrix}]$ is

(a) $\begin{pmatrix}-3 & -7\\2 & 5\end{pmatrix}$

(b) $\begin{pmatrix}-3 & -2\\7 & 5\end{pmatrix}$

(c) $\begin{pmatrix}3 & 7\\-2 & -5\end{pmatrix}$

(d) $\begin{pmatrix}5 & 2\\-7 & -3\end{pmatrix}$

$\begin{pmatrix}-3 & -2\\7 & 5\end{pmatrix}^{-1}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

(2) If V is a vector space with $\dim(V) = 4$, then

(a) any subset of V which has less than 4 elements is linearly independent

(b) any subset of V which has exactly 4 elements form a basis for V

(c) any subset of V which has more than 4 vectors is linearly dependent

(d) any subset of V which has more than 4 elements is a spanning set

(3) $\dim(\,\mathrm{span}(\begin{pmatrix}1\\-2\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}))$ is

$c_1\begin{pmatrix}1\\-2\\2\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\2\end{pmatrix} + c_3\begin{pmatrix}1\\1\\2\end{pmatrix} = 0$

(a) 3

(b) 2

(c) 1

(d) 0

$c_1 = +2c_3$

$c_2 = -3c_3$

(4) If V is a vector space and $\{v_1, v_2, \cdots, v_n\}$ is a spanning set for V and $v_{n+1} \in V$, then

(a) $v_1, v_2, \cdots, v_{n+1}$ are linearly independent

(b) $v_1, v_2, \cdots, v_{n+1}$ are linearly dependent

(c) the set $\{v_1, v_2, \cdots, v_{n+1}\}$ is not a spanning set.

(d) none

(5) $\dim(\mathrm{span}(x^2, 3+x^2, x^2+1))$ is

(a) 3

(b) 2

(c) 1

(d) 0

$3(x^2+1) - 2(x^2)$

$3x^2+3-2x^2$

x^2+3

(6) Let $S = \{ax^2 + bx + a + b : a, b \in \mathbb{R}\}$. Then a basis for S and the dimension of S are

~~(a)~~ basis: $x^2 + 1, x$; dimension = 2

(b) basis: $x^2 + 1, x + 1$; dimension = 2

(c) basis: $x^2, 1$; dimensin = 2

~~(d)~~ basis: $x^2, x, 1$; dimension = 3

(7) If A is an $n \times n$ nonsingular matrix, then

(a) the columns of A form a spanning set for $\mathbb{R}^n$ ✓

(b) one of the columns of A is the zero vector ✗

(c) the columns of A are linearly dependent ✗

(d) none of the above

(8) Let V be a vector space with $\dim(V) = 5$ and S a subspace of V. If $v_1, v_2, v_3, v_4 \in S$ with $\boxed{\operatorname{span}(v_1, v_2, v_3, v_4) = S}$ then v_1, v_2, v_3, v_4

(a) form a basis for S

(b) are linearly dependent

(c) are linearly independent

(d) is not a spanning set for V

(9) If $v_1, v_2, \cdots, v_n$ span a vector space V and $\dim(V) = m$, then

(a) $n = m$ $\qquad n \geq m$

(b) $m \leq n$

(c) $n \leq m$

(d) none of the above

(10) One of the following sets is a subspace of P_3

~~(a)~~ $\{p(x) \in P_3 : p(1) = 0\}$ $\qquad x - x^2 \quad 1 - x - 1 \quad 1 - x^2 - 1$

~~(b)~~ $\{p(x) \in P_3 : p(1) = 1\}$

(c) $\{p(x) \in P_3 : p(x) = x^2 + bx + c, b, c \in \mathbb{R}\}$ $\qquad 2(x-1)$

(d) $\{p(x) \in P_3 : p(0) = 1\}$ $\qquad 2x - 2$ ✓

$x - x + x^2 - 1$

$2 - x^2 \quad 2 - x$

$2(2 - x^2)$

$4 - 2x^2$

Question 3 (25%). (a) Find all values of a for which $\{\begin{pmatrix} a^2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis for $\mathbb{R}^3$.

$$\begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{vmatrix} \neq 0$$

$$\begin{vmatrix} a^2 & 0 & 1 \\ 0 & a & 0 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} a^2 & 0 \\ 0 & a \\ 1 & 2 \end{matrix} \Rightarrow \boxed{\begin{matrix} a \neq 0 \\ a \neq \pm 1 \end{matrix}}$$

$$(a^3 + 0 + 0) - (0 + 0 + a) \neq 0$$

$$a^3 - a \neq 0 \qquad a(a^2 - 1) \neq 0$$

$$\boxed{a \neq 0} \qquad a^2 - 1 \neq 0 \qquad a^2 \neq 1 \qquad \boxed{a \neq \pm 1}$$

(b) Show that the vectors $1, e^x + e^{-x}, e^x - e^{-x}$ are linearly independent.

must show that th wronskian is not equal to zero.

$$W = \begin{vmatrix} 1 & e^x + e^{-x} & e^x - e^{-x} \\ 0 & e^x - e^{-x} & e^x + e^{-x} \\ 0 & e^x + e^{-x} & e^x - e^{-x} \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} e^x - e^{-x} & e^x + e^{-x} \\ e^x + e^{-x} & e^x - e^{-x} \end{vmatrix}$$

$$= (e^x - e^{-x})(e^x - e^{-x}) - [(e^x + e^{-x})(e^x + e^{-x})]$$

$$\cancel{e^{2x}} - 1 - 1 + \cancel{e^{-2x}} - [\cancel{e^{2x}} + 1 + 1 + \cancel{e^{-2x}}]$$

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$$= -2 - 2 = \boxed{-4} \neq 0 \Rightarrow \text{L. Independent}$$

a
b
c

(c) Extend $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$ to a basis for $\mathbb{R}^3$.

we have to add another vector that linearly independent with the two vectors.

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{vmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{matrix} \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \text{ can be added to obtain a basis}$$

$(0+0+0) - (0+0+2) = \boxed{-2} \neq 0$ lin. independent

$\Rightarrow$ basis for $\mathbb{R}^3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$

(d) Find a basis and dimension of the of the subspace of $\underline{\underline{\mathbb{R}^4}}$ consisting of all vectors of the form $(a+c, -a+b, -b-c, a+b+2c)^T$.

$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \leftarrow a = 1$, all others zero

$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \leftarrow b = 1$, all others zero

$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \leftarrow c = 1$, all others zero

$\Rightarrow$ dimension $= 3$

basis $= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \right\}$

$1 - 2 \cdot \frac{1}{3}$

Question 4 (15%). (a) Find the transition matrix from the ordered basis $E = [1, x+1, x^2]$ to the ordered basis $F = [1+x, x+x^2, 2+x^2]$.

(b) Find the coordinate vector of $p(x) = 2x^2 - (x+1) - 2$ with respect to $F = [1+x, x+x^2, 2+x^2]$.

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

From E to $F \Rightarrow T = F^{-1} \cdot E$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] R_2 - R_1 \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array}\right] \cdot \frac{1}{3} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$T = F^{-1} \cdot E$

b) $[X]_F = T \cdot \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

$$= \begin{bmatrix} \frac{1}{3} & 1 & -\frac{2}{3} \\ -\frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$$

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