Key

Form 1

BIRZEIT UNIVERSITY

MATHEMATICS DEPARTMENT
MATH234 -Second Exam-
First Semester 2017/2018

- **Name**.................................. • **Number**........................

Section	Teacher	Time
1	Reema Sbeih	S,M,W 11:00 - 11:50
2	Hasan Yousef	S,M,W 09:00 - 09:50
3	Hani Kabajah	T,R 09:30 - 10:50
4	Khaled Altakhman	T,R 09:30 - 10:50
5	Hasan Yousef	S,M,W 11:00 - 11:50
6	Mahmoud Ghannam	S,M,W 12:00 - 12:50

(Q1) [20 points] Fill the blanks with True (T) or False (F)

2 pts each

[T] (1) Any basis for $\mathbb{R}^{3\times 3}$ must contain exactly nine vectors.

[F] (2) If $S = \{(x+1, x)^T \; ; \; x \in \mathbb{R}\}$, then S is a subspace of $\mathbb{R}^2$

[T] (3) Any set of vectors containing the zero vector is linearly dependent.

[T] (4) $span(u, v) = span(u)$ iff v is a scalar multiple of u.

[F] (5) Any subset of linearly dependent vectors is linearly dependent.

[T] (6) If the vectors v_1, v_2, v_3 span P_3, then they are linearly independent.

[F] (7) If S is a subset of a vector space V and $0 \in S$, then S is a subspace of V.

[T] (8) Every set of vectors spanning $\mathbb{R}^4$ has at least four vectors.

[T] (9) If the vectors $v_1, v_2, \ldots, v_n$ span a vector space V and v_1 is a linear combinations of $v_2, \ldots, v_n$, then $V = span(v_2, \ldots, v_n)$.

[T] (10) If A is an $m \times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

(Q2) [45 points] Circle the most correct answer.

3 pts each

(1) If v_1, v_2, v_3 are vectors in $\mathbb{R}^3$ and $span(v_1, v_2, v_3) = span(v_2, v_3)$, then

(a) The vectors v_1, v_2, v_3 span $\mathbb{R}^3$.

(b) The vectors v_1, v_2, v_3 are linearly independent.

$\boxed{\text{(c)}}$ The vectors v_1, v_2, v_3 cannot form a basis of $\mathbb{R}^3$.

(d) $span(v_1, v_2, v_3) = span(v_1, v_2)$.

---

(2) One of the following sets is a basis of P_3.

$\boxed{\text{(a)}}$ $\{x+2, x-1, x^2\}$

(b) $\{x^2, x, 5, x+1\}$

(c) $\{x^2+x+1, x^2-x\}$

(d) $\{x^2+3, x^2+2, 1\}$

(3) One of the following statements is false.

(a) $dim(\mathbb{R}^{2\times 3}) = 6$.

(b) $dim(C^6[-1,1]) = 6$.

(c) $dim(P_6) = 6$.

(d) $dim(\{0\}) = 0$.

---

(4) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$, then the null space of A is

(a) $\{(\alpha, \alpha, \beta)^T \ ; \ \alpha, \beta \in \mathbb{R}\}$.

(b) $\{(-\alpha, \alpha, \beta)^T \ ; \ \alpha, \beta \in \mathbb{R}\}$.

(c) $\{(-\alpha, \alpha, 0)^T \ ; \ \alpha \in \mathbb{R}\}$.

(d) $\{(-\alpha, \alpha, \alpha)^T \ ; \ \alpha \in \mathbb{R}\}$.

---

(5) $dim(span(1, \sin^2 x, \cos^2 x)) =$

(a) 1

(b) 2

(c) 3

(d) ∞

---

(6) One of the following sets is a subspace of $\mathbb{R}^2$.

(a) $\{(a,b)^T \ ; \ ab = 0\}$

(b) $\{(a,b)^T \ ; \ b = 0\}$

(c) $\{(a,b)^T \ ; \ a^2 = b^2\}$

(d) $\{(a,b)^T \ ; \ a = 1\}$

---

(7) If the set $\{u_1, u_2, \ldots, u_n\}$ is a spanning set of a vector space V, then

(a) The set $\{u_1, u_2, \ldots, u_{n-1}\}$ is also a spanning set of V.

(b) The set $\{u_1, u_2, \ldots, u_n, u\}$ is also a spanning set of V for any $u \in V$.

(c) $dim(V) = n$

(d) $dim(V) \geq n$

---

(8) If V is a vector space with $dim(V) = 10$ and the vectors $v_1, \ldots, v_k$ are linearly independent, then

(a) $k < 10$

(b) $k \leq 10$

(c) $k > 10$

(d) $k \geq 10$

(9) If $S = \{A \in \mathbb{R}^{2\times 2}\ ;\ A \text{ is lower triangular}\}$ and $T = \{A \in \mathbb{R}^{2\times 2}\ ;\ A \text{ is upper triangular}\}$, then a basis of $S \cap T$ is

(a) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$

---

(10) One of the following is a spanning set of $\mathbb{R}^3$

(a) $\{(1,1,1)^T, (3,3,3)^T, (1,0,0)^T\}$

(b) $\{(1,0,0)^T, (0,1,0)^T, (1,1,0)^T, (0,2,0)^T\}$

(c) $\{(1,1,1)^T, (1,2,1)^T, (1,0,0)^T\}$

(d) $\{(1,0,0)^T, (0,1,1)^T\}$

---

(11) If V is a vector space and $dim(V) = n > 0$, then

(a) Any collection of m vectors in V, where $m > n$, span V.

(b) Any collection of m vectors in V, where $m > n$, are linearly independent.

(c) Any collection of m vectors in V, where $m < n$, are linearly independent.

(d) Any collection of m vectors in V, where $m < n$, cannot span V.

---

(12) Let A be a 2×3 matrix with $b = a_1 - 2a_3$. If $\{(1,-1,1)^T\}$ is a basis for $N(A)$, then the solution set of the linear system $Ax = b$ is

(a) $\{(1,-2)^T + \alpha(1,-1,1)^T\ ;\ \alpha \in \mathbb{R}\}$

(b) $\{(1,0,-2)^T + \alpha(1,-1,1)^T\ ;\ \alpha \in \mathbb{R}\}$

(c) $\{(1,-1,1)^T + \alpha(1,0,-2)^T\ ;\ \alpha \in \mathbb{R}\}$

(d) $\{\alpha(1,0,-2)^T + \beta(1,-1,1)^T\ ;\ \alpha, \beta \in \mathbb{R}\}$

---

(13) One of the following statements is false.

(a) If S, T are subspaces of a vector space V, then $S \cup T$ is a subspace of V.

(b) If S, T are subspaces of a vector space V, then $S \cap T$ is a subspace of V.

(c) P_3 is a subspace of P_4

(d) P_4 is a subspace of $C[-1,2]$

---

(14) One of the following sets is a subspace of $C[-1,1]$

(a) $\{f(x) \in C[-1,1]\ ;\ f(1) = -1\}$

(b) $\{f(x) \in C[-1,1]\ ;\ f(1) = 0\}$

(c) $\{f(x) \in C[-1,1]\ ;\ f(1) = 1\}$

(d) $\{f(x) \in C[-1,1]\ ;\ f(1) = 0 \text{ or } f(-1) = 0\}$

(15) One of the following statements is false.

(a) If $f_1, f_2, f_3 \in C^2[a,b]$ and $W[f_1, f_2, f_3](x) \neq 0$ for some $x \in [a,b]$, then f_1, f_2, f_3 are linearly independent.

(b) If $f_1, f_2, f_3 \in C^2[a,b]$ and $W[f_1, f_2, f_3](x) \neq 0$ for all $x \in [a,b]$, then f_1, f_2, f_3 are linearly independent.

$\boxed{\text{(c)}}$ If $f_1, f_2, f_3 \in C^2[a,b]$ and $W[f_1, f_2, f_3](x) = 0$ for all $x \in [a,b]$, then f_1, f_2, f_3 are linearly dependent.

(Q3) [12 points] Find a **basis** and the **dimension** of the following subspaces.

(a) $S = \{(a,b,c,d)^T \ ; \ a + 2b = c = 0\}$

$$\left.\begin{array}{l} a+2b=0 \Rightarrow a=-2b \\ c=0 \end{array}\right\} \Rightarrow S = \{(-2b, b, 0, d)^T \mid b, d \in \mathbb{R}\}$$

$$= \{b(-2,1,0,0)^T + d(0,0,0,1)^T \mid b, d \in \mathbb{R}\} \quad \text{2pts.}$$

$$= \text{span}\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$$

$$\Rightarrow \text{Basis} = \left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\} \text{(2pts)}, \quad \dim(S) = 2 \ \text{(2pts)}$$

(b) $S = \{p(x) \in P_3 \ ; \ p(0) = 0 \text{ and } p'(1) = 0\}$

$p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b$

$$\left.\begin{array}{l} p(0)=0 \Rightarrow c=0 \\ p'(1)=0 \Rightarrow 2a+b=0 \Rightarrow b=-2a \end{array}\right\} \Rightarrow S = \{ax^2 - 2ax \mid a \in \mathbb{R}\}$$

$$= \text{span}(x^2 - 2x) \quad \text{2pts.}$$

$\Rightarrow$ Basis $= \{x^2 - 2x\}$ (2pts)

$\dim(S) = 1$ (2pts)

(Q4) [12 points] Let $V = \mathbb{R}^3$. Answer the following.

(a) Given $B = \{(1,0,1)^T, (1,1,1)^T\}$, extend this set to a basis of V. Justify your answer.

Take any $v \in \mathbb{R}^3$ such that $\begin{vmatrix} 1 & 1 & \\ 0 & 1 & v \\ 1 & 1 & \end{vmatrix} \neq 0$

any $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a \neq c$ will work.

For example, if $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ then $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis (4 pts)

because $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$ (2 pts)

(b) Given $B = \{(1,1,1)^T, (1,0,1)^T, (2,1,2)^T, (1,3,0)^T\}$, pare down this set to a basis of V. Justify your answer. v_1 v_2 v_3 v_4

Take any three vectors from B with $\det \neq 0$

possible choices:

$B = \{v_1, v_2, v_4\}$ basis because $|v_1 \ v_2 \ v_4| \neq 0$ (det = 1) (2 pts)

4 pts for basis

or $B = \{v_1, v_3, v_4\}$ basis because $|v_1 \ v_3 \ v_4| \neq 0$ (det = 1)

or $B = \{v_2, v_3, v_4\}$ basis because $|v_2 \ v_3 \ v_4| \neq 0$ (det = -1)

(Q5) [8 points] Use the Wronskian to test whether the vectors x, e^x, e^{-x} are linearly independent in $C[0,1]$

$$W[x, e^x, e^{-x}](x) = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^{+x} & e^{-x} \end{vmatrix} \quad (4pts)$$

$$= x(1+1) - 1(1-1) = 2x \neq 0 \text{ in } [0,1] \quad (2pts)$$

$\Rightarrow$ The vectors are L.I (2pts)

(Q6) [8 points] Let v_1, v_2, v_3, v_4 be linearly independent vectors in a vector space V. Show that the vectors $v_1 + v_3, v_3 + v_4, v_2 + v_3 + v_4$ are linearly independent.

Proof: Let $c_1(v_1+v_3) + c_2(v_3+v_4) + c_3(v_2+v_3+v_4) = 0$ (2pts)

$\Rightarrow c_1 v_1 + c_3 v_2 + (c_1 + c_2 + c_3) v_3 + (c_2 + c_3) v_4 = 0$ (2pts)

But v_1, v_2, v_3, v_4 are L.I

$$\Rightarrow \left.\begin{array}{l} c_1 = 0 \\ c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \\ c_2 + c_3 \end{array}\right\} (2pts) \Rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow \text{L.I} \ (2pts)$$