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MATH 234
SECOND HOUR EXAM

Student Name: Key I Student Number:

Instructor: Section:

Question 1. (1 point each) Answer by true or false:

1. F Any set of vectors containing the zero vector is linearly independent.
2. F The rank of a matrix A is the number of the nonzero rows of A.
3. T If A is an $m \times n$ matrix, then $\text{rank}(A) \leq m$.
4. T The set $\{1, \sin^2 x, \cos^2 x\}$ is linearly dependent in $C[0, \pi]$.
5. T Any basis of $\mathbb{R}^{2\times 4}$ must contain exactly eight vectors.
6. T If A is a 4×4 matrix with $a_2 = -a_4$, then $N(A) \neq \{0\}$.
7. T If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span a vector space V and $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_n$, then $V = \text{span}(\mathbf{v}_2, \ldots, \mathbf{v}_n)$.
8. T Every set of vectors that spans $\mathbb{R}^5$ must contain at least five vectors.
9. T The columns of a nonsingular 10×10 matrix form a basis for $\mathbb{R}^{10}$.
10. T The set $\{(x_1, x_2, x_3, x_4)^T \mid x_1 + x_3 = 0\}$ is a subspace of $\mathbb{R}^4$.
11. T If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent in $\mathbb{R}^3$, then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbb{R}^3$.
12. F The functions $f(x) = 3x$ and $g(x) = |-3x|$ are linearly independent in $C[-4, 0]$.
13. F In P_2, the coordinate vector of $12 + 6x$ with respect to the basis $\{x, 4\}$ is $(3, 6)^T$.
14. F If three vectors span a vector space V, then $dim(V) = 3$.
15. T The transition matrix corresponding to the change of basis in a vector space V is always nonsingular.
16. T If the system $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ does not belong to the column space of A.
17. T The vectors $(4, 2, 3)^T, (2, 3, 1)^T, (2, 5, 3)^T, (2, 0, 3)^T$ are linearly dependent in $\mathbb{R}^3$.
18. T If A is a 5×4 matrix and $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $\text{rank}(A) = 4$.
19. F In $\mathbb{R}^3$, every set containing more than three vectors can be reduced to form a basis for $\mathbb{R}^3$.
20. F $\mathbb{R}^2$ is a subspace of $\mathbb{R}^4$.
21. T P_2 is a subspace of P_4.

Question 2 (3 points each) Circle the most correct answer:

1. One of the following sets is a basis for P_3.

   (a) $\{x+2, x-1, x^2\}$
   (b) $\{x^2, x, 5, x+1\}$
   (c) $\{x^2+x+1, x^2-x\}$
   (d) $\{x^2+3, x^2+2, 1\}$

2. One of the following statements is false:

   (a) $\dim(\mathbb{R}^{2\times 3}) = 6$.
   (b) $\dim(C^6[-1,1]) = 6$.
   (c) $\dim(P_6) = 6$.
   (d) $\dim(\{0\}) = 0$.

3. If the set $\{u_1, u_2, \ldots, u_n\}$ is a spanning set of a vector space V, then

   (a) The set $\{u_1, u_2, \ldots, u_{n-1}\}$ is also a spanning set of V.
   (b) The set $\{u_1, u_2, \ldots, u_n, u\}$ is also a spanning set of V for any $u \in V$.
   (c) $dim(V) = n$
   (d) $dim(V) \geq n$

4. If V is a vector space with $dim(V) = 10$ and the vectors $v_1, \ldots, v_k$ are linearly independent, then

   (a) $k < 10$
   (b) $k \leq 10$
   (c) $k > 10$
   (d) $k \geq 10$

5. One of the following is a spanning set of $\mathbb{R}^3$

   (a) $\{(1,1,1)^T, (3,3,3)^T, (1,0,0)^T\}$
   (b) $\{(1,0,0)^T, (0,1,0)^T, (1,1,0)^T, (0,2,0)^T\}$
   (c) $\{(1,1,1)^T, (1,2,1)^T, (1,0,0)^T\}$
   (d) $\{(1,0,0)^T, (0,1,1)^T\}$

6. The vectors $1, x, x^2, x^2+x-1$

   (a) span P_3
   (b) span P_4
   (c) are linearly independent in P_3
   (d) are linearly independent in P_4

7. One of the following sets is a subspace of $C[-1,1]$

(a) $\{f(x) \in C[-1,1] \ ; \ f(1) = -1\}$
(b) $\{f(x) \in C[-1,1] \ ; \ f(1) = 0\}$
(c) $\{f(x) \in C[-1,1] \ ; \ f(1) = 1\}$
(d) $\{f(x) \in C[-1,1] \ ; \ f(1) = 0 \text{ or } f(-1) = 0\}$

8. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}. \ rank(A) = 3 \text{ if}$$

(a) $a \neq 0$ and $b \neq 0$
(b) $a = 0$ and $b = 0$
(c) $a \neq 0$ and $b = 0$
(d) $a = 0$ and $b \neq 0$

9. Consider the ordered basis $B = \{e_1, e_1 - e_2\}$ for $\mathbb{R}^2$. If $[v]_B = (1, -1)^T$, then $v =$

(a) $-e_2$.
(b) $(0, 1)^T$.
(c) $2e_1 + e_2$.
(d) $e_1 + e_2$.

10. Let A be a 3×3 matrix with $rank(A) = 2$, then

(a) A is singular.
(b) $Ax = 0$ has a nontrivial solution.
(c) Nullity of A is 1.
(d) All of the above.

11. Let V be a vector space such that $dim(V) = 5$, then

(a) Any five vectors in V are linearly independent.
(b) Any five vectors in V form a spanning set for V.
(c) Any basis for V has five vectors.
(d) All of the above.

12. Let $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ be the transition matrix from the basis $\{v_1, v_2\}$ to the basis $\{(1,1)^T, (1,-1)^T\}$. Then $\{v_1, v_2\} =$

(a) $\{(2,0)^T, (3,-1)^T\}$.
(b) $\{(1,0)^T, (3,-2)^T\}$.
(c) $\{(2,3)^T, (0,-1)^T\}$.
(d) $\{(1,-1)^T, (3,1)^T\}$.

13. One of the following vector spaces is infinite-dimensional.

(a) $\mathbb{R}$

(b) P_8

(c) $C[-2,2]$

(d) $\text{span}(x, e^x, xe^x)$

Question 3.(8 points) Find a **basis** and the **dimension** of the following subspaces:

1. $S = \{(a,b,c,d)^T \ ; \ b = -d\}$

$(a,b,c,-b)^T = a(1,0,0,0)^T + b(0,1,0,-1)^T + c(0,0,1,0)^T$

$\Rightarrow S = \text{span}\{(1,0,0,0)^T, (0,1,0,-1)^T, (0,0,1,0)^T\}$ (1)

lin ind. (check!) (1)

$\Rightarrow$ Basis for S is $\{(1,0,0,0)^T, (0,1,0,-1)^T, (0,0,1,0)^T$ (1)

$\dim S = 3$ (1)

2. $S = \{p(x) \in P_3 \ ; \ p''(x) = 0\}$

$p(x) = ax^2 + bx + c \rightarrow p'(x) = 2ax + b \rightarrow \boxed{p''(x) = 2a = 0}$ (1

$p \in S \Rightarrow p(x) = bx + c = b(x) + c(1)$

(1

$\Rightarrow S = \text{span}\{x, 1\}$

since $\{x,1\}$ is lin ind. (1 $\Rightarrow \{x,1\}$ form a basis for S

$\dim(S) = 2$ (1

Question 4.(12 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Row echelon form : $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(٤) ارسم -1
تصميم -2
ضبط سطر -3

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Reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1. Find a basis for the row space of A

$\{(1,1,1,1), (0,1,0,0), (0,0,1,1)\}$ 2

2. Find a basis for the column space of A

$\{(1,1,-1,1)^T, (1,2,-1,1)^T, (1,1,1,1)^T\}$ 2

3. Find a basis for the null space of A

let $x_4 = t \Rightarrow x_3 = -t,\ x_2 = 0,\ x_1 = 0$ 2

$\Rightarrow \vec{x} = (0,0,-t,t)^T = t\underbrace{(0,0,-1,1)^T}_{\text{Basis}}$

4. What is the rank and nullity of A?

rank = 3 2

nullity = 1

Question 5.(8 points) Let $E = [3x + 6, 9]$ and $F = [2x + 1, x - 4]$ be two ordered bases of P_2

1. Find the transition matrix corresponding to the change of basis from E to F
2. Use part (1) to find the coordinate vector of $3x + 15$ with respect to the basis F

1. $$\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$$

4 2

2. $3x + 15 = 1(3x + 6) + 1(9)$

$$\Rightarrow [3x+15]_F = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

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SECOND HOUR EXAM

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Question 2 (3 points each) Circle the most correct answer:

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   (a) Any five vectors in V form a spanning set for V.
   (b) Any five vectors in V are linearly independent.
   (c) Any basis for V has five vectors.
   (d) All of the above.

2. One of the following vector spaces is infinite-dimensional.
   (a) $\text{span}(x, e^x, xe^x)$
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   (d) $C[-2, 2]$

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10. The vectors $1, x, x^2, x^2 + x - 1$

   (a) are linearly independent in P_3
   (b) are linearly independent in P_4
   (c) span P_3
   (d) span P_4

11. One of the following sets is a basis for P_3.

   (a) $\{x^2, x, 5, x+1\}$
   (b) $\{x^2 + x + 1, x^2 - x\}$
   (c) $\{x^2 + 3, x^2 + 2, 1\}$
   (d) $\{x + 2, x - 1, x^2\}$

12. One of the following statements is false:

   (a) $\dim(P_6) = 6$.
   (b) $\dim(\mathbb{R}^{2\times 3}) = 6$.
   (c) $\dim(C^6[-1,1]) = 6$.
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