Birzeit University
Mathematics Department
Math234
Second Exam (KEY)

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Name:... **Number:**..............
Section:(3) **Date: 04/07/2020**

Exercise 1 [40 marks]. Answer by **true** or **false**.

1. **(F)** If the set $\{v_1, v_2, ..., v_k\}$ spans P_4, then $k = 4$.
2. **(F)** If u, v, w are nonzero vectors in $\mathbb{R}^2$, then $w \in span(u, v)$.
3. **(T)** If A is a 4×4 matrix with $a_2 + a_4 = 0$,, then $N(A) \neq \{0\}$.
4. **(T)** If the vectors u_1, u_2, u_3, u_4 span $\mathbb{R}^{2\times 2}$, then they are linearly independent.
5. **(F)** The coordinate vector of $q(x) = 4 + 6x$ with respect to the basis $[2x, 2]$ is $(2, 3)^T$.
6. **(T)** The transition matrix of two basis is nonsingular.
7. **(T)** If $dim\ V = n < +\infty$, then an n linearly independent set of vectors in V is a basis for V.
8. **(T)** Let $W = \{(x, y, x + y + 2z)^T : x, y, z \in \mathbb{R}\}$, then $\{(1, 0, 1)^T, (0, 0, 1)^T, (0, 1, 1)^T\}$ is a basis for W.
9. **(T)** Let $S = Span\ \{v_1, v_2, v_3, v_4\}$ and suppose that $v_1 = v_2 + v_3,\ v_3 = v_2 - v_4$, then $S = Span\ \{v_3, v_4\}$.
10. **(F)** Let $S = \{ax^2 + ax : a \in \mathbb{R}\}$, then $\{x^2, x\}$ is a basis for S.
11. **(T)** If $f_1, ...f_n$ are linearly dependent, then Wronskian$(f_1, ...f_n)$=0.
12. **(F)** The set $S = \{(x, y) : y = x + 3\}$ is a subspace of $\mathbb{R}^2$.
13. **(T)** The dimension of the subspace $W = \{A \in \mathbb{R}^{2\times 2} :\ A\ is\ symmetric\}$ is 3.
14. **(F)** If V is a vector space with $dim(V) = 4$ and $\{v_1, v_2, v_3, v_4\} \subseteq V$, then $span\{v_1, v_2, v_3, v_4\} = V$.
15. **(F)** If A is a singular $n \times n$ matrix, then $rank(A) = n$.
16. **(T)** If S is a subspace of a vector space V, then S is a vector space.
17. **(T)** If $\{v_1, v_2, v_3\}$ are vectors in a vector space V and $Span\ \{v_1, v_2\} = Span\ \{v_1, v_2, v_3\}$, then $\{v_1, v_2, v_3\}$ are linearly dependent.
18. **(F)** If A is a 3×3 matrix and $rank(A) = 2$, then A is nonsingular.
19. **(T)** If A is an $m \times n$ matrix, then A and A^T have the same rank.
20. **(T)** If A is an 3×4 matrix, then $rank(A) \leq 3$.

Exercise 2 [20 marks]. Circle the correct answer.

1. let $S = \{p \in P_3 : p(0) = 0\}$. One of the following is a basis for S.

   (a) $\{1, x, x^2\}$

   (b) $\{x, x^2\}$

   (c) $\{x^2 + x\}$

   (d) $\{x^2 + 1\}$

2. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

   (a) A is in REF

   (b) nullity(A)=2

   (c) rank(A)=2

   (d) $\det(A) \neq 0$

3. The vectors $\{2, x, \sin x\}$ in $C[0, 2\pi]$ are

   (a) Linearly independent

   (b) Linearly dependent

   (c) A basis for $C[0, 2\pi]$

   (d) A spanning set for $C[0, 2\pi]$

4. Consider the ordered basis $E = \{e_1, e_1 - e_2\}$ for $\mathbb{R}^2$. If $[v]_E = (1, -1)^T$, then $v =$

   (a) $-e_2$

   (b) $2e_1 + e_2$

   (c) $e_1 + e_2$

   (d) $(0, 1)^T$

5. If the reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $a_2 = (2, 2)^T$, then $A =$

   (a) $\begin{bmatrix} 4 & 2 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

6. The transition matrix from the basis $E = [1, -x]$ to the basis $F = [-1, x-1]$ of P_2 is

(a) $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

7. If $A = \begin{bmatrix} 1 & -3 & 2 & 3 \\ -6 & 6 & -4 & -5 \end{bmatrix}$, then

(a) rank(A)=1, nullity(A)=3.

(b) rank(A)=3, nullity(A)=1.

(c) rank(A)=4, nullity(A)=0.

(d) rank(A)=nullity(A)=2.

8. The dimension of the vector space spanned by $\{1 - x - x^2,\ 1 + x + x^2,\ 2 - x,\ 2x - 4\}$ is

(a) 1

(b) 2

(c) 3

(d) 4

9. One of the following sets is a subspace of P_4

(a) $\{f(x) \in P_4 : f(0) = 1\}$

(b) $\{f(x) \in P_4 : f(1) = 1\}$

(c) $\{f(x) \in P_4 : f(1) = 0\}$

(d) $\{f(x) \in P_4 : f(0) = 0,\ f'''(0) = 6\}$

10. If A is a 4×3 matrix such that $N(A) = \{0\}$, and $b = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, then

(a) It is possible that $Ax = b$ has infinitely many solutions

(b) The system $Ax = b$ has exactly one solution.

(c) The system $Ax = b$ has at most one solution.

(d) The system $Ax = b$ has no solution

Exercise 3 [8 marks]. Let $V = \mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$, and W_2 set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.

a. Find $W_1 \cap W_2$.

b. Find a basis and dimention of $W_1 \cap W_2$.

Exercise 4 [12 marks]. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions and V_o be the subset of odd functions.

a. Prove that V_e and V_o are subspaces of V. **(Do only one case)**.

b. Prove that $V_e \cap V_o = \{0\}$.

c. Prove that $V = V_e + V_o$.

Good Luck

Exercise #3. $V_e = \{ f : \mathbb{R} \to \mathbb{R} : f(-x) = f(x) \}$

$V_o = \{ f : \mathbb{R} \to \mathbb{R} : f(-x) = -f(x) \}$

(a) V_e is a subspace (V_o is similar).

(i) $0(-x) = 0(x) \Rightarrow 0 \in V_e \therefore V_e \neq \Phi$.

(ii) Let $f, g \in V_e$, then $f(-x) = f(x)$, $g(-x) = g(x)$.

$$(f+g)(-x) = f(-x) + g(-x)$$
$$= f(x) + g(x)$$
$$= (f+g)(x).$$

$\therefore f + g \in V_e$.

(iii) $\forall f \in V_e$, $\alpha \in \mathbb{R}$, we have

$$(\alpha f)(-x) = \alpha f(-x)$$
$$= \alpha f(x) \quad \text{since } f \in V_e$$
$$= (\alpha f)(x)$$

$\therefore \alpha f \in V_e$.

(b) $V_e \cap V_o = \{0\}$.

Let $f \in V_e \cap V_o$, then $f \in V_e$ and $f \in V_o$

$\Rightarrow f(-x) = f(x)$ and $f(-x) = -f(x)$

$\Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0$

(c) $V = V_e + V_o$

Let $f \in V$, then $f(x) = \left(\frac{f(x)+f(-x)}{2}\right) + \left(\frac{f(x)-f(-x)}{2}\right)$

$\in V_e$ $\in V_o$

Exercise #4. $V = \mathbb{R}^{2\times 2}$

$W_1 = \left\{ A \in \mathbb{R}^{2\times 2} : A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \right\}$

$W_2 = \left\{ B \in \mathbb{R}^{2\times 2} : B = \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \right\}$

a) $W_1 \cap W_2 = \left\{ C \in \mathbb{R}^{2\times 2} : C = \begin{bmatrix} \alpha & -\alpha \\ -\alpha & \beta \end{bmatrix} \right\}$

b) $W_1 \cap W_2 = \left\{ C \in \mathbb{R}^{2\times 2} : C = \alpha \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$= \text{Span}\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

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A basis for $W_1 \cap W_2$ is

$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\dim W_1 \cap W_2 = 2$