MATH 234
EXAM 2

Student Name: Key Student Number:

Instructor: Section:

Question 1 (4 points each) Circle the most correct answer.

1. One of the following is true:

   (a) Elementary row operations do not change the null space of a matrix.
   (b) In P_3, every set of 4 polynomials can be reduced to form a basis for P_3.
   (c) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is not a spanning set for R^n.

2. One of the following is false:

   (a) Every set of vectors spanning R^3 contains at least 3 vectors.
   (b) If Wronskian$(f_1, f_2, ..., f_n) = 0$, then $f_1, f_2, ..., f_n$ are linearly independent.
   (c) If A is a nonzero 5×4 matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then rank$(A) = 4$.

3. One of the following is true:

   (a) The column vectors of any matrix A form a basis for the column space of A.
   (b) The dimensions of the column spaces of two row equivalent matrices are equal.
   (c) If U is the reduced row echelon form of A then A and U have the same column space.

4. One of the following is false

   (a) The dimension of $W = \{p(x) \in P_3 \,|\, p(1) = 0\}$ is 2.
   (b) If W is three dimensional subspace of R^3 then $W = R^3$.
   (c) R^2 is a subspace of R^3.

5. One of the following is true

   (a) If $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a spanning set for a vector space V, then $\{\mathbf{v}_2, ..., \mathbf{v}_n\}$ are linearly independent.
   (b) If span$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$= span$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}\}$ then $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ are linearly independent.
   (c) If $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a vector space V, and $\alpha \neq 0$ then $\{\alpha\,\mathbf{v}_1, \alpha\,\mathbf{v}_2, ..., \alpha\,\mathbf{v}_n\}$ is a basis for V.

6. One of the following is a spanning set for R^3

(a) $\{(1,2,3)^T,\ (2,1,1)^T,\ (4,5,7)^T,\ (1,-1,-2)^T\}$
(b) $\{(1,2,3)^T,\ (2,5,4)^T,\ (0,0,0)^T\}$
(c) $\{(1,2,3)^T,\ (2,5,1)^T\}$
(d) $\{(1,0,0)^T,\ (0,1,0)^T,\ (1,1,2)^T\}$

7. Let $W = \{A \in R^{2\times 2} \mid A \text{ is upper triangular}\}$ then $\dim(W)$ is

(a) 1
(b) 2
(c) 3
(d) 4

8. If A is a 3×3 nonsingular matrix then

(a) Nullity of A is 0.
(b) Rank of A is 3
(c) Column space of A is R^3
(d) All of the above are true.

9. Suppose that the coordinate vector of $\mathbf{v} \in R^3$ with respect to the basis $S = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is $\mathbf{v} = 2\,\mathbf{b}_1 + 4\,\mathbf{b}_2 - 5\,\mathbf{b}_3$, where $\mathbf{b}_1 = (1,0,0)^T, \mathbf{b}_2 = (0,1,0)^T, \mathbf{b}_3 = (1,1,2)^T$ then the coordinate vector of $\mathbf{v}$ with respect to the standard basis is

(a) $(7,9,5)^T$
(b) $(-3,-1,-10)^T$
(c) $(7, \frac{13}{2}, \frac{5}{2})^T$
(d) $(0, \frac{13}{2}, \frac{5}{2})^T$

10. One of the following is linearly dependent in $C[-\pi,\pi]$

(a) $\{(1+x)^2, x^2+2x, 5\}$
(b) $\{1, \sin x, \sin 2x\}$
(c) $\{\cos x, x\}$
(d) $\{x, x^2\}$

11. Let $\{v_1, v_2, v_3\}$ be a spanning set for a nonzero vector space V and suppose that $v_1 - v_2 + 2\,v_3 = 0$, then the dimension of V is

(a) 1
(b) 2 or 3
(c) 2
(d) 1 or 2

a
b
b
b
b

12. Let $S = \{ax^2 + ax + b \,|\, a, b \in R\}$. Then a basis for S is

(a) $x^2 + x, 1$
(b) $x^2 + 1, x$
(c) $x^2, x + 1$
(d) $x^2, 1$

13. Let A be an $m \times n$ matrix. If the columns of A span R^m, then

(a) $n \leq m$
(b) $m \leq n$
(c) $n = m$
(d) The columns of A form a basis for R^m

14. Let $\mathbf{v}_1 = (1, 2)^T$, $\mathbf{v}_2 = (2, 3)^T$ and let $S = \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$. Find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S is the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$

(a) $\mathbf{w}_1 = (1, 5)^T$ and $\mathbf{w}_2 = (9, 4)^T$
(b) $\mathbf{w}_1 = (5, 1)^T$ and $\mathbf{w}_2 = (9, 4)^T$
(c) $\mathbf{w}_1 = (5, 9)^T$ and $\mathbf{w}_2 = (1, 4)^T$
(d) $\mathbf{w}_1 = (1, 0)^T$ and $\mathbf{w}_2 = (0, 1)^T$

15. If $p(x) = 3x^2 + x + 2$, then the coordinate vector of $p(x)$ with respect to $F = [1, 1 + x, 1 + x + x^2]$ of P_3 is

(a) $(-2, 1, -2)^T$
(b) $(1, -2, 3)^T$
(c) $(2, -1, 2)^T$
(d) $(3, -2, 1)^T$

16. Let $S = \text{Span}(x^2 - 2x + 1,\ x^2 + 1,\ -2x,\ 3x^2 - 4x + 3)$. Then the $\dim(S)$ is

(a) 1
(b) 2
(c) 3
(d) 4

Question 2 (12 points) Let $A = \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{bmatrix}$

1. Find a basis for the null space of A.

$$A \longrightarrow \begin{bmatrix} 1 & 3 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2)

Let $x_2 = t$, $x_5 = s \Rightarrow x_4 = -s$, $x_3 = 0$, $x_1 = -3t - s$

2)

$$\begin{bmatrix} -3t-s \\ t \\ 0 \\ -s \\ s \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Basis for the null space.

2. Find a basis for the row space of A.

3) $\{(1,3,1,3,4)^{T}, (0,0,1,1,1)^{T}, (0,0,0,1,1)^{T}\}$

3. Find a basis for the column space of A.

3) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

4. What is the rank and nullity of A.

2)

Rank $= 3$

Nullity $= 2$

Question 3 (16 points) Let L be the linear transformation $L : R^2 \to R^3$ defined by $L(\mathbf{x}) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$.

Let $E = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2\}$ be two ordered bases for R^3 and R^2, where

$\mathbf{u}_1 = (1, 0, -1)^T$, $\mathbf{u}_2 = (1, 2, 1)^T$, $\mathbf{u}_3 = (-1, 1, 1)^T$ and

$\mathbf{b}_1 = (1, -1)^T$, $\mathbf{b}_2 = (2, 1)^T$.

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1. Find the Ker(L) and its dimension.

$\ker(L) = \{\vec{0}\}$

$\dim(\ker(L)) = 0$

4)

2. Find a basis for the range of L.

$$\begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Basis for the range is

5)

3. Find the matrix representing L with respect to the ordered bases E and F.

$$\left[\begin{array}{ccc|cc} 1 & 1 & -1 & 0 & 3 \\ 0 & 2 & 1 & 2 & 1 \\ -1 & 1 & 1 & 5 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & \frac{1}{2} & \frac{-25}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{11}{2} \\ 0 & 0 & 1 & 1 & -10 \end{array}\right]$$

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{25}{2} \\ \frac{1}{2} & \frac{11}{2} \\ 1 & -10 \end{bmatrix}$$

3)

4. Use the matrix in part (c) to find $L(1, 5)^T$.

Let $\vec{x} = (1,5)^T$

$[\vec{x}]_F = (-3, 2)^T \Rightarrow [L(\vec{x})]_E = A \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{53}{2} \\ \frac{19}{2} \\ -23 \end{bmatrix}$

So, $L(1,5)^T = -\frac{53}{2}\vec{u}_1 + \frac{19}{2}\vec{u}_2 - 23\vec{u}_3 = (6, -4, 13)^T$

Question 4 (12 points)

6

1. Let x_1, x_2 and x_3 be linearly independent vectors in R^n and let A be a nonsingular $n \times n$ matrix. prove that if $y_1 = Ax_1$, $y_2 = Ax_2$, $y_3 = Ax_3$, then y_1, y_2 and y_3 are linearly independent.

Suppose $c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 = \vec{0}$

$$c_1(A\vec{x}_1) + c_2(A\vec{x}_2) + c_3(A\vec{x}_3) = \vec{0}$$

$$A^{-1}\left(A(c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3) = \vec{0} \right)$$

$\Rightarrow c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0} \Rightarrow c_1 = c_2 = c_3 = 0$ since $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are lin. ind

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2. Let S be the set of symmetric 2×2 matrices. Find a basis for S and the dimension of S.

Let $A \in S$ then $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basis

$\dim(S) = 3$