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BIRZEIT UNIVERSITY

Math. Dept.
Math-243

Second Exam

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Question 1 (7%). Mark each of the following statements by true or false. (4)

(1) If $f : X \to X$ is a bijection, then $f^{-1} : X \to X$ is a bijection. (T)

(2) If R is an equivalence relation, then R^{-1} is an equivalence relation (T)

(3) Let R be an equivalence relation on a set X, $x, y \in X$. If xRy, then $[x] \cap [y] \neq \phi$ (T)

(4) If $f : X \longrightarrow Y$ is an onto function, and B is a nonempty subset of Y, then $f^{-1}(B)$ is a nonempty subset of X. (T)

(5) Let $f : X \to Y$ be a function. If $a = b \in X$, then $f(a)$ need not equal $f(b)$ (T)

Question 2. Let A, B, C, D be sets, show that

$$(A \times C) \bigcap (B \times D) = (A \bigcap B) \times (C \bigcap D)$$

Let $(x,y) \in (A \times C) \cap (B \times D)$

$\Leftrightarrow (x,y) \in A \times C$ and $(x,y) \in B \times D$

$\Leftrightarrow x \in A$ and $y \in C$ and $x \in B$ and $y \in D$

$\Leftrightarrow x \in A$ and $x \in B$ and $y \in C$ and $y \in D$

$\Leftrightarrow x \in A \cap B$ and $y \in C \cap D$

$\Leftrightarrow (x,y) \in (A \cap B) \times (C \cap D)$

so $(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D)$

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Question 3 (10 points). Let $X = \{a, b, c, d, e\}$ and let
$E = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d)\}$

(a) Find all equivalence classes of E.

$[a] = [b] = [c] = \{a, b, c\}$

$[e] = [d] = \{e, d\}$

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(b) Find $\mathcal{P}_E$ (the partition induced by E)

$P_E = \{\{a, b, c\}, \{e, d\}\}$

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Question 4 (10%). Prove or disprove:
The relation $R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} | nm \geq 0\}$ is an equivalence relation

Yes, it's an equivalence relation

proof:- R is symmetric ~~because~~ $n \cdot n = n^2 \geq 0 \quad \forall n \in \mathbb{Z}$

$\rightarrow (n R n)$ that is $(n,n) \in R \quad \forall n \in \mathbb{Z} \quad (1)$

R is symmetric to show that suppose $(n, m) \in R$

$\rightarrow nm \geq 0$ but $nm = mn \rightarrow mn \geq 0$

$\rightarrow (m,n) \in R \quad (2)$

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R is transitive to show that suppose $(n,m) \in R$ and $(m,z) \in R$ (we need to show that $(n,z) \in R$), since $(n,m) \in R$ $nm \geq 0$ also since $(m,z) \in R$ $mz \geq 0 \rightarrow n \cdot m \cdot m \cdot z \geq 0$ (since $nm \geq 0$ and $mz \geq 0$) $\rightarrow n m^2 z \geq 0$ but $m^2 \geq 0$

$\rightarrow nz \geq 0 \rightarrow (n, z) \in R \quad (3)$

from (1) and (2) and (3) R is reflexive and symmetric and transitive so R is an equavelance relation.

Question 5 (10p). Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions suppose $f : A \to B$ and $g\, B \to C$ functions and that f ~~and g are~~ injective

1. If f and g are injective, show that $g \circ f : \overset{A}{\cancel{X}} \longrightarrow \overset{C}{\cancel{Z}}$ is injective.

let $c \in C \to \exists\, b \in B$ such that $g(b) = c$ (since g is injective)

$\to \exists\, a \in A$ s.t $f(a) = b$ (since f is injective)

$\to g(f(a)) = c$ (since $f(a) = b$)

$\to g \circ f\,(a) = c$ so we have found an element $a \in A$

such that $g \circ f(a) = c$ so $g \circ f$ is injective

injective not surjective

2. Show that if $g \circ f$ is injective, then f is injective.

Suppose $g \circ f$ is injective and let $c \in C$, $b \in B$

$\to \exists\, a \in A$ such that $g \circ f(a) = c \to g(f(a)) = c$

$\to g$ is injective since $\exists\, b \in B$ namely $b = f(a)$ such that

$g(b) = c$ ~~[illegible]~~ since g is injective

$\to \forall c \in C$ $\exists\, b \in B$ s.t $g(b) = c$

$\to b = f(a) \to f$ is injective

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(1) and (2) $\to$ $f(A\cap f^{-1}(B)) = f(A)\cap B$

Question 6 (12%). Prove the following

(a) Let $f: X \longrightarrow Y$ be a function, and let $A \subseteq X$, $B \subseteq Y$. Show that

$$f(A\cap f^{-1}(B)) = f(A)\cap B$$

let $x \in f(A\cap f^{-1}(B)) \Leftrightarrow \exists\, a \in A\cap f^{-1}(B)$ s.t $f(a) = x$

$\Leftrightarrow$ $a \in A$ and $a \in f^{-1}(B)$ and $f(a) = x$

$\Leftrightarrow$ $f(a) = x \in f(A)$ (since $a\in A$) and $\exists\, b \in B$ such that $f(a)=b$

$\Leftrightarrow$ $x \in f(A)$ and $f(a) = x = b \in B$

$\Leftrightarrow$ $x \in f(A)$ and $x \in B$

$\Leftrightarrow$ $x \in f(A)\cap B$ $\to$ $f(A\cap f^{-1}(B)) \subseteq f(A)\cap B$ (1)

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now let $x \in f(A)\cap B \to x \in f(A)$ and $x \in B$

$\to \exists\, a \in A$ s.t. $f(a) = x$ (since $x \in f(A)$)

$a = f^{-1}(x) \in f^{-1}(B)$ since $x \in B$

$\to$ $a \in (A\cap f^{-1}(B))$ and $f(a) = x$

$\to$ $x \in f(A\cap f^{-1}(B))$ $\to$ $f(A)\cap B \subseteq f(A\cap f^{-1}(B))$ (2)

(b) Let $a, b, c, d \in \mathbb{Z}$, $n \in \mathbb{N}$. If $a \equiv b \pmod n$, and $c \equiv d \pmod n$, prove that $a - c \equiv b - d \pmod n$

suppose $a \equiv b \pmod n$ and $c \equiv d$ mod n

$\to$ $a - b = k_1 n$ for some $k_1 \in \mathbb{Z}$ and $c - d = k_2 n$ for some $k_2 \in \mathbb{Z}$

$\to$ $(a-b)-(c-d) = (k_1-k_2)n$, $k_1-k_2 \in \mathbb{Z}$

$\to$ $(a-c)-(b-d) = (k_1-k_2)n$, $k_1-k_2 \in \mathbb{Z}$

$\to$ $(a-c) \equiv (b-d) \pmod n$

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(c) Let E be an equivalence relation on a set $X \neq \phi$, $x, y \in X$. If xRy, show that $E[x] = E[y]$

suppose E is an equivalence relation on a set $X \neq \phi$ and $x, y \in X$

and xRy, let $z \in E[x] \to (x, z) \in E$

$\to (z, x) \in E$ (since E is symmetric)

$xRy \to (x, y) \in E$, since $(z, x) \in E$ and $(x, y) \in E \to (z, y) \in E$ (since E is transitive)

$(z, y) \in E \to (y, z) \in E$ since E is symmetric

$(y, z) \in E \to z \in E[y]$ so we have shown that if

$z \in E[x]$ then $z \in E[y]$ that is $E[x] \subseteq E[y]$ (1)

now let $a \in E[y] \to (y, a) \in E$

we know that $xRy \to (x, y) \in E$

$(x, y) \in E$ and $(y, a) \in E \to (x, a) \in E$ (since E is transitive)

$(x, a) \in E \to a \in E[x]$ so we know have shown that if $a \in E[y]$ then $a \in E[x]$ that is $E[y] \subseteq E[x]$ (2)

(1) and (2) $\longrightarrow$ $E[x] = E[y]$

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