101+2

Birzeit University
Mathematics Department
Math 234

Second Exam — Summer 2011/2012

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Question 1 (42 points). Mark each of the following by True of False

(1) (T) If A is a nonzero 3×2 matrix such that $Ax = 0$ has infinitely many solutions, then $\text{rank}(A) = 1$.

(2) (..T..) Let $S = \{v_1, v_2, \cdots, v_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

(3) (.F..) Let V be a vector space. If $v_1, v_2, v_3, v_4 \in V$ with $\text{span}(v_1, v_2, v_3, v_4) = V$, then v_1, v_2, v_3, v_4 are linearly independent.

(4) (..F..) If A is a 3×3-matrix with nullity $(A) = 0$, then A is singular.

(5) (..T..) If $L : P_3 \to P_2$ is the linear transformation defined by $L(ax^2 + bx + c) = (a-c)x + (b+c)$, then $x^2 - x + 1$ is in $\ker(L)$.

(6) (..T..) If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined by $L(x, y) = (x - y, x + y)$, then $(2, 3)$ is in $\text{Im}(L)$.

(7) (.F..) If $f_1, f_2, \cdots, f_n \in C^{n-1}[a, b]$ and $W[f_1, f_2, \cdots, f_n](x) = 0$ for all $x \in [a, b]$, then $f_1, f_2, \cdots, f_n$ are linearly independent.

(8) (.F..) If A is an $n \times n$ matrix, $\det(A) \neq 0$, then $\text{rank}(A) \neq n$.

(9) (.F..) If the columns of $A_{n \times n}$ are linearly dependent and $b \in \mathbb{R}^n$, then the system $Ax = b$ is inconsistent.

(10) (..F..) Let $S = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + y = 2\}$, then S is a subspace of $\mathbb{R}^2$.

(11) Let A be a 4×5-matrix, with $\text{rank}(A) = 3$. Mark each of the following by true or false

- (..T..) The rows of A are linearly dependent.
- (.F..) The columns of A form a spanning set for $\mathbb{R}^4$.
- (.T..) $\text{nullity}(A) = 2$.

(12) (..T..) If x_0 is a solution of the nonhomogeneous system $Ax = b$ and x_1 is a solution of the homogeneous system $Ax = 0$. Then $x_1 + x_0$ is a solution of the nonhomogeneous system $Ax = b$.

(13) (.F..) If $v_1, v_2, \cdots, v_n$ are linearly dependent vectors in a vector space V, and $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$, then the scalars $c_1, c_2, \cdots, c_n$ are not all zero.

(14) (..T..) If A is a 4×6 matrix with $Rank(A) = 4$, then the homogeneous system $Ax = 0$ has a nontrivial solution.

2 free variables

(15) (..F..) If S is a subset of a vector space V that doesn't contain the zero vector, then S may be a subspace of V.

Question 6 (8 points). If $A = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & -1 & 0 & 1 \\ 2 & 3 & -1 & 2 & 1 \\ 4 & 5 & -1 & 6 & 1 \end{pmatrix}$ and the row echelon form of A is $U = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) Find a basis for the row space of A.

(b) Find a basis for the column space of A.

(c) Find Rank(A), Nullity(A).

a) basis for the row space of $A = \{(1,1,0,2,0), (0,1,-1,-2,1)\}$.

b) 〃 〃 〃 column 〃 〃 〃 $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix} \right\}$.

c) rank = 2

Nullity = 5 - 2 = 3.

8

(16) (F) The transition matrix of two bases could be singular.

(17) (F) The coordinate vector of $2+6x$ with respect to the basis $[2x, 4]$ is $(\frac{1}{2}, 3)^T$

(18) (T) If $b \in \mathbf{R}^n$ is in the column space of an $n \times n$-matrix A, and the columns of A are linearly independent, then $Ax = b$ has a unique solution.

(19) (F) Let $L : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $L(x, y) = (x - y, 0, 2x + 3)$, then L is a linear transformation.

Question 2 (26 points). Circle the most correct answer

(1) Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + 2x_2 \\ x_3 \end{pmatrix}$. Then $\underline{\dim(\ker(L))}$ is

(a) 0

(b) 1

(c) 2

(d) 3

$x_1 - x_2 = 0$

$x_1 + 2x_2 = 0$

$x_3 = 0$

$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

(2) The coordinate vector of $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ with respect to the ordered basis $[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}]$ is

(a) $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

(c) $\begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$

$1 - 8 + 12 =$

(3) The dimension of the column space of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix}$ is

(a) 4

(b) 3

(c) 2

(d) 1

$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 8 \\ 0 & -1 & -5 & -8 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 8 \\ 0 & 0 & -\frac{5}{2} & -4 \end{pmatrix}$

(4) If A is a singular $n \times n$-matrix, then

$0 \le r(A) < n$

(a) $0 \le \text{rank}(A) \le n$
(b) $0 \le \text{rank}(A) < n$
(c) $0 < \text{rank}(A) < n$
(d) $0 < \text{rank}(A) \le n$

$n \le m$

(5) Let A be an $m \times n$ matrix. If the columns of A are linearly independent, then

L.I.

(a) $n \le m$
(b) $m \le n$
(c) $n = m$
(d) the columns of A form a basis for $\mathbf{R}^m$.

(6) If V is a vector space, $\dim(V) = n$ and $\{v_1, v_2, \cdots, v_n\}$ is a spanning set for V and $v_{n+1} \in V$, then

basis

(a) the set $\{v_1, v_2, \cdots, v_{n+1}\}$ is not a spanning set.
(b) $v_1, v_2, \cdots, v_{n+1}$ are linearly independent
(c) $v_1, v_2, \cdots, v_{n+1}$ are linearly dependent
(d) none

(7) If A is an $n \times n$-matrix and for each $b \in \mathbf{R}^n$ the system $Ax = b$ is consistent, then

$b \in C(A)$ $\quad \dim(C(A)) = n$

(a) A is nonsingular
(b) $\text{rank}(A) = n$
(c) $\text{nullity}(A) = 0$
(d) all of the above

(8) Let v_1, v_2, v_3 be linearly independent in a vector space V, $V \ne \text{span}(v_1, v_2, v_3)$, then

(a) $\dim(V) \ge 3$
(b) $\dim(V) \ge 4$
(c) $\dim(V) \le 3$
(d) $\dim(V) \le 4$

(9) $\dim(\text{span}(x^3 + x^2, x^2 + x, x + 1, 1))$ is

(a) 1
(b) 2
(c) 3
(d) 4

$0 = c_1(x^3+x^2) + c_2(x^2+x) + c_3(x+1) + c_4(1)$

$x^3: \ c_1 = 0$
$x^2: \ c_1 + c_2 = 0$
$x: \ c_2 + c_3 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$
$c_3 + c_4 = 0$

(10) One of the following sets is a subspace of P_4

(a) $\{f(x) \in P_4 : f(0) = 2\}$

(b) $\{f(x) \in P_4 : f(2) = 2\}$

(c) $\{f(x) \in P_4 : f(2) = 0\}$

(d) $\{f(x) \in P_4 : f(x) = x^3 + bx^2 + cx, b, c \in \mathbb{R}\}$

(11) The transition matrix from the ordered basis $[e_1, e_2]$ to the ordered basis $\left[\begin{pmatrix}3\\7\end{pmatrix}, \begin{pmatrix}-2\\-5\end{pmatrix}\right]$ is

$T_{S\to E} = (T_{E\to S})^{-1}$

(a) $\begin{pmatrix}-5 & 2\\-7 & 3\end{pmatrix}$

(b) $\begin{pmatrix}5 & -2\\7 & -3\end{pmatrix}$

(c) $\begin{pmatrix}-3 & 7\\-2 & 5\end{pmatrix}$

(d) $\begin{pmatrix}3 & -7\\2 & -5\end{pmatrix}$

$\begin{pmatrix}3 & -2\\7 & -5\end{pmatrix}$ $\begin{pmatrix}3 & -2\\7 & -5\end{pmatrix}^{-1}$

(12) let A be a 4×5-matrix, if the row echelon form of A has 2 nonzero rows, then nullity(A) is

(a) 2

(b) 3

(c) 5

(d) 6

(13) If A is a 4×3 matrix such that $N(A) = \{0\}$ and $b = (1, 2, 3, 4)^T$, then the system $Ax = b$ has

nonsingular

(a) no solution

(b) at most one solution

(c) exactly one solution

(d) infinitely many solutions.

rank(A)=3

4×3

$Ax = 0$ has only the zero sol.

$\Rightarrow$ columns of A are l.i.

three vectors in $\mathbb{R}^4$

$b \in \mathbb{R}^4$

$Ax=b$

26

Question 5 (

Question 3 (8 points).

Let $S = \left\{ p(x) \in P_3 : \int_0^1 p(x)\,dx = 0 \right\}$.

(a) Show that S is a subspace of P_3

1- $S \neq \phi$ since $0 \in S$. why

2- let $f(x), g(x) \in S \Rightarrow \int_0^1 f(x)dx = 0, \int_0^1 g(x)dx = 0$

$\Rightarrow \int_0^1 (f+g)(x).dx = \int_0^1 (f(x)+g(x)).dx = \int_0^1 f(x).dx + \int_0^1 g(x).dx = 0+0=0$

$\Rightarrow (f+g)(x) \in S.$

3- let $h(x) \in S$, α scalar $\Rightarrow \int_0^1 h(x).dx = 0$

$\int_0^1 (\alpha h)(x).dx = \int_0^1 \alpha h(x).dx = \alpha \int_0^1 h(x).dx = \alpha.0 = 0$

$\Rightarrow (\alpha h)(x) \in S \Rightarrow S$ is a subspace of P_3.

(b) Find a basis and dimension of S.

$S = \left\{ ax^2+bx+c : \int_0^1 (ax^2+bx+c).dx = 0 \right\}$

$\Rightarrow \int_0^1 (ax^2+bx+c)\,dx = \frac{a}{3} + \frac{b}{2} + c = 0 \Rightarrow c = -\frac{a}{3} - \frac{b}{2}$

$\Rightarrow S = \left\{ ax^2+bx-\frac{a}{3}-\frac{b}{2} : a,b \in \mathbb{R} \right\} = \left\{ a(x^2-\frac{1}{3}) + b(x-\frac{1}{2}) : a,b \right\}$

$\Rightarrow$ a sp. set for S is $\{x^2-\frac{1}{3}, x-\frac{1}{2}\}$

clear L.I. $\Rightarrow$ a basis for S is $\{x^2-\frac{1}{3}, x-\frac{1}{2}\}$

$\dim(S) = 2$.

Question 4 (5 points). Let V be a vector space, $v_1, v_2, v_3 \in V$ be linearly independent. If $w_1 = v_1 + v_2 + v_3$, $w_2 = v_2 + v_3$, $w_3 = v_3$, show that w_1, w_2, w_3 are linearly independent.

to show that w_1, w_2, w_3 are L.I. we should prove that $c_1w_1 + c_2w_2 + c_3w_3 = 0$ has only the zero sol. $(c_1=c_2=c_3=0)$.

$\Rightarrow c_1(v_1+v_2+v_3) + c_2(v_2+v_3) + c_3v_3 = 0$

$\Rightarrow c_1v_1 + (c_1+c_2)v_2 + (c_1+c_2+c_3)v_3 = 0$

since v_1,v_2,v_3 are L.I. $\Rightarrow c_1 = 0$

$c_1+c_2 = 0$

13

Question 5 (12 points). Let $A = \begin{pmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$. The reduced row echelon form of A is

$$U = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis for $R(A)$.

$$\{(1\ 0\ -2\ 1\ 0),\ (0\ 1\ 3\ 0\ 0),\ (0\ 0\ 0\ 0\ 1)\}$$

(b) Find a basis for $C(A)$.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 1 \\ 4 \end{pmatrix} \right\}$$

(c) Find a basis for $N(A)$.

$$\left(\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

c_3, c_4 free variables.
$c_3 = \alpha,\ c_4 = \beta$

$c_5 = 0,\ c_2 = -3\alpha,\ c_1 = 2\alpha - \beta \Rightarrow$ sol. for $N(A)$ is $\begin{pmatrix} 2\alpha-\beta \\ -3\alpha \\ \alpha \\ \beta \\ 0 \end{pmatrix}$

$= \alpha \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ basis for $N(A)$ is $\left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

sp. set and L.I.

(d) Find rank(A), nullity(A).

rank$(A) = \dim(C(A)) = \dim(R(A)) = 3$.

nullity$(A) = \dim(N(A)) = 2$ $(5 - \text{rank}(A))$

/12

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Question 6 (10 points). Let $B = [1, x, x^2]$ and $C = [2x-1, 2x+1, x^2]$ be ordered bases for P_3

(a) Find the transition matrix from B to C.

$T_{B\to C}$ (B in terms of C) $= (T_{C\to B})^{-1}$

$$2x-1 = (-1)\cdot(1) + 2\cdot(x) + 0\cdot(x^2)$$
$$2x+1 = 1\cdot(1) + 2\cdot(x) + 0\cdot(x^2)$$
$$x^2 = 0\cdot(1) + 0\cdot(x) + 1\cdot(x^2)$$
$$\Rightarrow T_{C\to B} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow T_{B\to C} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \Rightarrow \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 1/4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad \therefore T_{B\to C} = \begin{pmatrix} -1/2 & 1/4 & 0 \\ 1/2 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Find the coordinate vector of $p(x) = 2x^2 + 2x - 3$ with respect to the ordered basis C.

$2x^2+2x-3 = \alpha(2x-1) + \beta(2x+1) + \gamma(x^2)$

$2x^2+2x-3 = c_1(1) + c_2(x) + c_3(x^2)$

$\Rightarrow c_1 = -3,\ c_2 = 2,\ c_3 = 2$

$$[2x^2+2x-3]_C = T_{B\to C}\,[2x^2+2x-3]_B \qquad \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1/4 & 0 \\ 1/2 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} ✓$$

10