SEVENTH EDITION

LINEAR ALGEBRA WITH APPLICATIONS

Instructor's Solutions Manual

Steven J. Leon

PREFACE

This solutions manual is designed to accompany the seventh edition of *Linear Algebra with Applications* by Steven J. Leon. The answers in this manual supplement those given in the answer key of the textbook. In addition this manual contains the complete solutions to all of the nonroutine exercises in the book.

At the end of each chapter of the textbook there are two chapter tests (A and B) and a section of computer exercises to be solved using MATLAB. The questions in each Chapter Test A are to be answered as either *true* or *false*. Although the true-false answers are given in the Answer Section of the textbook, students are required to explain or prove their answers. This manual includes explanations, proofs, and counterexamples for all Chapter Test A questions. The chapter tests labelled B contain workout problems. The answers to these problems are not given in the Answers to Selected Exercises Section of the textbook, however, they are provided in this manual. Complete solutions are given for all of the nonroutine Chapter Test B exercises.

In the MATLAB exercises most of the computations are straightforward. Consequently they have not been included in this solutions manual. On the other hand, the text also includes questions related to the computations. The purpose of the questions is to emphasize the significance of the computations. The solutions manual does provide the answers to most of these questions. There are some questions for which it is not possible to provide a single answer. For example, aome exercises involve randomly generated matrices. In these cases the answers may depend on the particular random matrices that were generated.

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CHAPTER 1

SECTION 1

2. (d) $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 & 1 \\ 0 & 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

5. (a) $\begin{aligned} 3x_1 + 2x_2 &= 8 \\ x_1 + 5x_2 &= 7 \end{aligned}$

(b) $\begin{aligned} 5x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 - 4x_3 &= 0 \end{aligned}$

(c) $\begin{aligned} 2x_1 + x_2 + 4x_3 &= -1 \\ 4x_1 - 2x_2 + 3x_3 &= 4 \\ 5x_1 + 2x_2 + 6x_2 &= -1 \end{aligned}$

(d) $\begin{aligned} 4x_1 - 3x_2 + x_3 + 2x_4 &= 4 \\ 3x_1 + x_2 - 5x_3 + 6x_4 &= 5 \\ x_1 + x_2 + 2x_3 + 4x_4 &= 8 \\ 5x_1 + x_2 + 3x_3 - 2x_4 &= 7 \end{aligned}$

9. Given the system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ -m_2x_1 + x_2 &= b_2 \end{aligned}$$

one can eliminate the variable x_2 by subtracting the first row from the second. One then obtains the equivalent system

$$\begin{aligned} -m_1x_1 + x_2 &= b_1 \\ (m_1 - m_2)x_1 &= b_2 - b_1 \end{aligned}$$

(a) If $m_1 \neq m_2$, then one can solve the second equation for x_1

$$x_1 = \frac{b_2 - b_1}{m_1 - m_2}$$

One can then plug this value of x_1 into the first equation and solve for x_2. Thus, if $m_1 \neq m_2$, there will be a unique ordered pair (x_1, x_2) that satisfies the two equations.

(b) If $m_1 = m_2$, then the x_1 term drops out in the second equation

$$0 = b_2 - b_1$$

This is possible if and only if $b_1 = b_2$.

(c) If $m_1 \neq m_2$, then the two equations represent lines in the plane with different slopes. Two nonparallel lines intersect in a point. That point will be the unique solution to the system. If $m_1 = m_2$ and $b_1 = b_2$, then both equations represent the same line and consequently every point on that line will satisfy both equations. If $m_1 = m_2$ and $b_1 \neq b_2$, then the equations represent parallel lines. Since parallel lines do not intersect, there is no point on both lines and hence no solution to the system.

10. The system must be consistent since $(0, 0)$ is a solution.

11. A linear equation in 3 unknowns represents a plane in three space. The solution set to a 3×3 linear system would be the set of all points that lie on all three planes. If the planes are parallel or one plane is parallel to the line of intersection of the other two, then the solution set will be empty. The three equations could represent the same plane or the three planes could all intersect in a line. In either case the solution set will contain infinitely many points. If the three planes intersect in a point then the solution set will contain only that point.

SECTION 2

2. (b) The system is consistent with a unique solution $(4, -1)$.

4. (b) x_1 and x_3 are lead variables and x_2 is a free variable.
(d) x_1 and x_3 are lead variables and x_2 and x_4 are free variables.
(f) x_2 and x_3 are lead variables and x_1 is a free variable.

5. (l) The solution is $(0, -1.5, -3.5)$.

6. (c) The solution set consists of all ordered triples of the form $(0, -\alpha, \alpha)$.

7. A homogeneous linear equation in 3 unknowns corresponds to a plane that passes through the origin in 3-space. Two such equations would correspond to two planes through the origin. If one equation is a multiple of the other, then both represent the same plane through the origin and every point on that plane will be a solution to the system. If one equation is not a multiple of the other, then we have two distinct planes that intersect in a line through the origin. Every point on the line of intersection will be a solution to the linear system. So in either case the system must have infinitely many solutions.

In the case of a nonhomogeneous 2×3 linear system, the equations correspond to planes that do not both pass through the origin. If one equation is a multiple of the other, then both represent the same plane and there are infinitely many solutions. If the equations represent planes that are parallel, then they do not intersect and hence the system will not have any solutions. If the equations represent distinct planes that are not parallel, then they must intersect in a line and hence there will be infinitely many solutions. So the only possibilities for a nonhomogeneous 2×3 linear system are 0 or infinitely many solutions.

9. (a) Since the system is homogeneous it must be consistent.

14. At each intersection the number of vehicles entering must equal the number of vehicles leaving in order for the traffic to flow. This condition leads to the following system of equations

$$\begin{aligned} x_1 + a_1 &= x_2 + b_1 \\ x_2 + a_2 &= x_3 + b_2 \\ x_3 + a_3 &= x_4 + b_3 \\ x_4 + a_4 &= x_1 + b_4 \end{aligned}$$

If we add all four equations we get

$$x_1 + x_2 + x_3 + x_4 + a_1 + a_2 + a_3 + a_4 = x_1 + x_2 + x_3 + x_4 + b_1 + b_2 + b_3 + b_4$$

and hence

$$a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$$

15. If (c_1, c_2) is a solution, then

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 &= 0 \\ a_{21}c_1 + a_{22}c_2 &= 0 \end{aligned}$$

Multiplying both equations through by α, one obtains

$$\begin{aligned} a_{11}(\alpha c_1) + a_{12}(\alpha c_2) &= \alpha \cdot 0 = 0 \\ a_{21}(\alpha c_1) + a_{22}(\alpha c_2) &= \alpha \cdot 0 = 0 \end{aligned}$$

Thus $(\alpha c_1, \alpha c_2)$ is also a solution.

16. (a) If $x_4 = 0$ then x_1, x_2, and x_3 will all be 0. Thus if no glucose is produced then there is no reaction. $(0, 0, 0, 0)$ is the trivial solution in the sense that if there are no molecules of carbon dioxide and water, then there will be no reaction.
(b) If we choose another value of x_4, say $x_4 = 2$, then we end up with solution $x_1 = 12$, $x_2 = 12$, $x_3 = 12$, $x_4 = 2$. Note the ratios are still 6:6:6:1.

SECTION 3

1. (e) $\begin{pmatrix} 8 & -15 & 11 \\ 0 & -4 & -3 \\ -1 & -6 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & -10 & 15 \\ 5 & -1 & 4 \\ 8 & -9 & 6 \end{pmatrix}$

2. (d) $\begin{pmatrix} 36 & 10 & 56 \\ 10 & 3 & 16 \end{pmatrix}$

5. (a) $5A = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$

$$2A + 3A = \begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} + \begin{pmatrix} 9 & 12 \\ 3 & 3 \\ 6 & 21 \end{pmatrix} = \begin{pmatrix} 15 & 20 \\ 5 & 5 \\ 10 & 35 \end{pmatrix}$$

(b) $6A = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$

$$3(2A) = 3\begin{pmatrix} 6 & 8 \\ 2 & 2 \\ 4 & 14 \end{pmatrix} = \begin{pmatrix} 18 & 24 \\ 6 & 6 \\ 12 & 42 \end{pmatrix}$$

(c) $A^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}$

$$(A^T)^T = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 7 \end{pmatrix}^T = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & 7 \end{pmatrix} = A$$

6. (a) $A + B = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} = B + A$

(b) $3(A+B) = 3\begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 12 & 18 \\ 0 & 15 & 3 \end{pmatrix}$

$$\begin{aligned} 3A + 3B &= \begin{pmatrix} 12 & 3 & 18 \\ 6 & 9 & 15 \end{pmatrix} + \begin{pmatrix} 3 & 9 & 0 \\ -6 & 6 & -12 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 12 & 18 \\ 0 & 15 & 3 \end{pmatrix} \end{aligned}$$

(c) $(A+B)^T = \begin{pmatrix} 5 & 4 & 6 \\ 0 & 5 & 1 \end{pmatrix}^T = \begin{pmatrix} 5 & 0 \\ 4 & 5 \\ 6 & 1 \end{pmatrix}$

$$A^T + B^T = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 6 & 5 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ 3 & 2 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 4 & 5 \\ 6 & 1 \end{pmatrix}$$

7. (a) $3(AB) = 3\begin{pmatrix} 5 & 14 \\ 15 & 42 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$

$$(3A)B = \begin{pmatrix} 6 & 3 \\ 18 & 9 \\ -6 & 12 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$$

$$A(3B) = \begin{pmatrix} 2 & 1 \\ 6 & 3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 6 & 12 \\ 3 & 18 \end{pmatrix} = \begin{pmatrix} 15 & 42 \\ 45 & 126 \\ 0 & 48 \end{pmatrix}$$

(b) $$(AB)^T = \begin{pmatrix} 5 & 14 \\ 15 & 42 \\ 0 & 16 \end{pmatrix}^T = \begin{pmatrix} 5 & 15 & 0 \\ 14 & 42 & 16 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 2 & 6 & -2 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 15 & 0 \\ 14 & 42 & 16 \end{pmatrix}$$

8. (a) $$(A+B)+C = \begin{pmatrix} 0 & 5 \\ 1 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

$$A+(B+C) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 3 & 8 \end{pmatrix}$$

(b) $$(AB)C = \begin{pmatrix} -4 & 18 \\ -2 & 13 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 24 & 14 \\ 20 & 11 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -4 & -1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 24 & 14 \\ 20 & 11 \end{pmatrix}$$

(c) $$A(B+C) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 24 \\ 7 & 17 \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} -4 & 18 \\ -2 & 13 \end{pmatrix} + \begin{pmatrix} 14 & 6 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 24 \\ 7 & 17 \end{pmatrix}$$

(d) $$(A+B)C = \begin{pmatrix} 0 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 17 & 8 \end{pmatrix}$$

$$AC + BC = \begin{pmatrix} 14 & 6 \\ 9 & 4 \end{pmatrix} + \begin{pmatrix} -4 & -1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 17 & 8 \end{pmatrix}$$

9. Let

$$D = (AB)C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

It follows that

$$\begin{aligned}
d_{11} &= (a_{11}b_{11} + a_{12}b_{21})c_{11} + (a_{11}b_{12} + a_{12}b_{22})c_{21} \\
&= a_{11}b_{11}c_{11} + a_{12}b_{21}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{22}c_{21} \\
d_{12} &= (a_{11}b_{11} + a_{12}b_{21})c_{12} + (a_{11}b_{12} + a_{12}b_{22})c_{22} \\
&= a_{11}b_{11}c_{12} + a_{12}b_{21}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{22}c_{22} \\
d_{21} &= (a_{21}b_{11} + a_{22}b_{21})c_{11} + (a_{21}b_{12} + a_{22}b_{22})c_{21} \\
&= a_{21}b_{11}c_{11} + a_{22}b_{21}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{22}c_{21} \\
d_{22} &= (a_{21}b_{11} + a_{22}b_{21})c_{12} + (a_{21}b_{12} + a_{22}b_{22})c_{22} \\
&= a_{21}b_{11}c_{12} + a_{22}b_{21}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{22}c_{22}
\end{aligned}$$

If we set

$$E = A(BC) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}$$

then it follows that

$$\begin{aligned}
e_{11} &= a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) \\
&= a_{11}b_{11}c_{11} + a_{11}b_{12}c_{21} + a_{12}b_{21}c_{11} + a_{12}b_{22}c_{21} \\
e_{12} &= a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\
&= a_{11}b_{11}c_{12} + a_{11}b_{12}c_{22} + a_{12}b_{21}c_{12} + a_{12}b_{22}c_{22} \\
e_{21} &= a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) \\
&= a_{21}b_{11}c_{11} + a_{21}b_{12}c_{21} + a_{22}b_{21}c_{11} + a_{22}b_{22}c_{21} \\
e_{22} &= a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \\
&= a_{21}b_{11}c_{12} + a_{21}b_{12}c_{22} + a_{22}b_{21}c_{12} + a_{22}b_{22}c_{22}
\end{aligned}$$

Thus

$$d_{11} = e_{11} \qquad d_{12} = e_{12} \qquad d_{21} = e_{21} \qquad d_{22} = e_{22}$$

and hence

$$(AB)C = D = E = A(BC)$$

12.

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^4 = O$. If $n > 4$, then

$$A^n = A^{n-4}A^4 = A^{n-4}O = O$$

13. (b) $\mathbf{x} = (2, 1)^T$ is a solution since $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$. There are no other solutions since the echelon form of A is strictly triangular.
(c) The solution to $A\mathbf{x} = \mathbf{c}$ is $\mathbf{x} = (-\frac{5}{2}, -\frac{1}{4})^T$. Therefore $\mathbf{c} = -\frac{5}{2}\mathbf{a}_1 - \frac{1}{4}\mathbf{a}_2$.

15. If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$ then

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{pmatrix} = I$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left[\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right] = \begin{pmatrix} \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} & 0 \\ 0 & \dfrac{a_{11}a_{22} - a_{12}a_{21}}{d} \end{pmatrix} = I$$

Therefore

$$\frac{1}{d}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} = A^{-1}$$

16. Since

$$A^{-1}A = AA^{-1} = I$$

it follows from the definition that A^{-1} is nonsingular and its inverse is A.

17. Since

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I \end{aligned}$$

it follows that

$$(A^{-1})^T = (A^T)^{-1}$$

18. If $A\mathbf{x} = A\mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, then A must be singular, for if A were nonsingular then we could multiply by A^{-1} and get

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}A\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

19. For $m = 1$,

$$(A^1)^{-1} = A^{-1} = (A^{-1})^1$$

Assume the result holds in the case $m = k$, that is,

$$(A^k)^{-1} = (A^{-1})^k$$

It follows that

$$(A^{-1})^{k+1}A^{k+1} = A^{-1}(A^{-1})^k A^k A = A^{-1}A = I$$

and

$$A^{k+1}(A^{-1})^{k+1} = AA^k(A^{-1})^k A^{-1} = AA^{-1} = I$$

Therefore

$$(A^{-1})^{k+1} = (A^{k+1})^{-1}$$

and the result follows by mathematical induction.

20. **(a)** $(A+B)^2 = (A+B)(A+B) = (A+B)A+(A+B)B = A^2+BA+AB+B^2$
In the case of real numbers $ab + ba = 2ab$, however, with matrices $AB + BA$ is generally not equal to $2AB$.

(b)

$$\begin{aligned} (A+B)(A-B) &= (A+B)(A-B) \\ &= (A+B)A - (A+B)B \\ &= A^2 + BA - AB - B^2 \end{aligned}$$

In the case of real numbers $ab - ba = 0$, however, with matrices $AB - BA$ is generally not equal to O.

21. If we replace a by A and b by the identity matrix, I, then both rules will work, since

$$(A+I)^2 = A^2 + IA + AI + B^2 = A^2 + AI + AI + B^2 = A^2 + 2AI + B^2$$

and

$$(A+I)(A-I) = A^2 + IA - AI - I^2 = A^2 + A - A - I^2 = A^2 - I^2$$

22. There are many possible choices for A and B. For example, one could choose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

More generally if

$$A = \begin{pmatrix} a & b \\ ca & cb \end{pmatrix} \qquad B = \begin{pmatrix} db & eb \\ -da & -ea \end{pmatrix}$$

then $AB = O$ for any choice of the scalars a, b, c, d, e.

23. To construct nonzero matrices A, B, C with the desired properties, first find nonzero matrices C and D such that $DC = O$ (see Exercise 22). Next, for any nonzero matrix A, set $B = A + D$. It follows that

$$BC = (A + D)C = AC + DC = AC + O = AC$$

24. A 2×2 symmetric matrix is one of the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Thus

$$A^2 = \begin{pmatrix} a^2 + b^2 & ab + bc \\ ab + bc & b^2 + c^2 \end{pmatrix}$$

If $A^2 = O$, then its diagonal entries must be 0.

$$a^2 + b^2 = 0 \qquad \text{and} \qquad b^2 + c^2 = 0$$

Thus $a = b = c = 0$ and hence $A = O$.

25. For most pairs of symmetric matrices A and B the product AB will not be symmetric. For example

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 5 & 4 \end{pmatrix}$$

See Exercise 27 for a characterization of the conditions under which the product will be symmetric.

26. (a) A^T is an $n \times m$ matrix. Since A^T has m columns and A has m rows, the multiplication A^TA is possible. The multiplication AA^T is possible since A has n columns and A^T has n rows.

(b) $(A^TA)^T = A^T(A^T)^T = A^TA$
$(AA^T)^T = (A^T)^TA^T = AA^T$

27. Let A and B be symmetric $n \times n$ matrices. If $(AB)^T = AB$ then

$$BA = B^TA^T = (AB)^T = AB$$

Conversely if $BA = AB$ then

$$(AB)^T = B^TA^T = BA = AB$$

28. If A is skew-symmetric then $A^T = -A$. Since the (j, j) entry of A^T is a_{jj} and the (j, j) entry of $-A$ is $-a_{jj}$, it follows that is $a_{jj} = -a_{jj}$ for each j and hence the diagonal entries of A must all be 0.

29. **(a)**

$$\begin{aligned} B^T &= (A + A^T)^T = A^T + (A^T)^T = A^T + A = B \\ C^T &= (A - A^T)^T = A^T - (A^T)^T = A^T - A = -C \end{aligned}$$

(b) $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

31. The search vector is $\mathbf{x} = (1, 0, 1, 0, 1, 0)^T$. The search result is given by the vector

$$\mathbf{y} = A^T\mathbf{x} = (1, 2, 2, 1, 1, 2, 1)^T$$

The ith entry of $\mathbf{y}$ is equal to the number of search words in the title of the ith book.

34. If $\alpha = a_{21}/a_{11}$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ 0 & b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \alpha a_{11} & \alpha a_{12} + b \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & \alpha a_{12} + b \end{pmatrix}$$

The product will equal A provided

$$\alpha a_{12} + b = a_{22}$$

Thus we must choose

$$b = a_{22} - \alpha a_{12} = a_{22} - \frac{a_{21}a_{12}}{a_{11}}$$

SECTION 4

2. (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, type I

(b) The given matrix is not an elementary matrix. Its inverse is given by

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$, type III

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, type II

5. (c) Since

$$C = FB = FEA$$

where F and E are elementary matrices, it follows that C is row equivalent to A.

6. (b) $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

The product $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is lower triangular.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

7. A can be reduced to the identity matrix using three row operations

$$\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elementary matrices corresponding to the three row operations are

$$E_1 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$E_3E_2E_1A = I$$

and hence

$$A = E_1^{-1}E_3^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

and $A^{-1} = E_3E_2E_1$.

8. (b) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

9. (a) $\begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

10. (e) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

12. (b) $XA + B = C$
$X = (C - B)A^{-1}$
$= \begin{pmatrix} 8 & -14 \\ -13 & 19 \end{pmatrix}$

(d) $XA + C = X$
$XA - XI = -C$
$X(A - I) = -C$
$X = -C(A - I)^{-1}$
$= \begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$

13. (a) If E is an elementary matrix of type I or type II then E is symmetric. Thus $E^T = E$ is an elementary matrix of the same type. If E is the elementary matrix of type III formed by adding α times the ith row of the identity matrix to the jth row, then E^T is the elementary matrix of type III formed from the identity matrix by adding α times the jth row to the ith row.

(b) In general the product of two elementary matrices will not be an elementary matrix. Generally the product of two elementary matrices will be a matrix formed from the identity matrix by the performance of two row operations. For example, if

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

then E_1 and E_2 are elementary matrices, but

$$E_1E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is not an elementary matrix.

14. If $T = UR$, then

$$t_{ij} = \sum_{k=1}^{n} u_{ik} r_{kj}$$

Since U and R are upper triangular

$$\begin{aligned} u_{i1} &= u_{i2} = \cdots = u_{i,i-1} = 0 \\ r_{j+1,j} &= r_{j+2,j} = \cdots - r_{nj} = 0 \end{aligned}$$

If $i > j$, then

$$\begin{aligned} t_{ij} &= \sum_{k=1}^{j} u_{ik} r_{kj} + \sum_{k=j+1}^{n} u_{ik} r_{kj} \\ &= \sum_{k=1}^{j} 0\, r_{kj} + \sum_{k=j+1}^{n} u_{ik} 0 \\ &= 0 \end{aligned}$$

Therefore T is upper triangular.

If $i = j$, then

$$\begin{aligned} t_{jj} = t_{ij} &= \sum_{k=1}^{i-1} u_{ik} r_{kj} + u_{jj} r_{jj} + \sum_{k=j+1}^{n} u_{ik} r_{kj} \\ &= \sum_{k=1}^{i-1} 0\, r_{kj} + u_{jj} r_{jj} + \sum_{k=j+1}^{n} u_{ik} 0 \\ &= u_{jj} r_{jj} \end{aligned}$$

Therefore

$$t_{jj} = u_{jj} r_{jj} \qquad j = 1, \ldots, n$$

15. If we set $\mathbf{x} = (2, 1 - 4)^T$, then

$$A\mathbf{x} = 2\mathbf{a}_1 + 1\mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

Thus $\mathbf{x}$ is a nonzero solution to the system $A\mathbf{x} = \mathbf{0}$. But if a homogeneous system has a nonzero solution, then it must have infinitely many solutions. In particular, if c is any scalar, then $c\mathbf{x}$ is also a solution to the system since

$$A(c\mathbf{x}) = cA\mathbf{x} = c\mathbf{0} = \mathbf{0}$$

Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$ it follows that the matrix A must be singular. (See Theorem 1.4.2)

16. If $\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$, then

$$\mathbf{a}_1 - 3\mathbf{a}_2 + 2\mathbf{a}_3 = \mathbf{0}$$

Therefore $\mathbf{x} = (1, -3, 2)^T$ is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$. It follows form Theorem 1.4.2 that A must be singular.

17. If $\mathbf{x}_0 \neq \mathbf{0}$ and $A\mathbf{x}_0 = B\mathbf{x}_0$, then $C\mathbf{x}_0 = \mathbf{0}$ and it follows from Theorem 1.4.2 that C must be singular.

18. If B is singular, then it follows from Theorem 1.4.2 that there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. If $C = AB$, then

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

Thus, by Theorem 1.4.2, C must also be singular.

19. (a) If U is upper triangular with nonzero diagonal entries, then using row operation II, U can be transformed into an upper triangular matrix with 1's on the diagonal. Row operation III can then be used to eliminate all of the entries above the diagonal. Thus U is row equivalent to I and hence is nonsingular.

(b) The same row operations that were used to reduce U to the identity matrix will transform I into U^{-1}. Row operation II applied to I will just change the values of the diagonal entries. When the row operation III steps referred to in part (a) are applied to a diagonal matrix, the entries above the diagonal are filled in. The resulting matrix, U^{-1}, will be upper triangular.

20. Since A is nonsingular it is row equivalent to I. Hence there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_1 A = I$$

It follows that

$$A^{-1} = E_k \cdots E_1$$

and

$$E_k \cdots E_1 B = A^{-1} B = C$$

The same row operations that reduce A to I, will transform B to C. Therefore the reduced row echelon form of $(A \mid B)$ will be $(I \mid C)$.

21. (a) If the diagonal entries of D_1 are $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the diagonal entries of D_2 are $\beta_1, \beta_2, \ldots, \beta_n$, then $D_1 D_2$ will be a diagonal matrix with diagonal entries $\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n$ and $D_2 D_1$ will be a diagonal matrix with diagonal entries $\beta_1\alpha_1, \beta_2\alpha_2, \ldots, \beta_n\alpha_n$. Since the two have the same diagonal entries it follows that $D_1 D_2 = D_2 D_1$.

(b)

$$\begin{aligned} AB &= A(a_0 I + a_1 A + \cdots + a_k A^k) \\ &= a_0 A + a_1 A^2 + \cdots + a_k A^{k+1} \\ &= (a_0 I + a_1 A + \cdots + a_k A^k)A \\ &= BA \end{aligned}$$

22. If A is symmetric and nonsingular, then

$$(A^{-1})^T = (A^{-1})^T (AA^{-1}) = ((A^{-1})^T A^T)A^{-1} = A^{-1}$$

23. If A is row equivalent to B then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 B$$

Each of the E_i's is invertible and E_i^{-1} is also an elementary matrix (Theorem 1.4.1). Thus

$$B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$$

and hence B is row equivalent to A.

24. (a) If A is row equivalent to B, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$A = E_k E_{k-1} \cdots E_1 B$$

Since B is row equivalent to C, there exist elementary matrices $H_1, H_2, \ldots, H_j$ such that

$$B = H_j H_{j-1} \cdots H_1 C$$

Thus

$$A = E_k E_{k-1} \cdots E_1 H_j H_{j-1} \cdots H_1 C$$

and hence A is row equivalent to C.

(b) If A and B are nonsingular $n \times n$ matrices then A and B are row equivalent to I. Since A is row equivalent to I and I is row equivalent to B it follows from part (a) that A is row equivalent to B.

25. If U is any row echelon form of A then A can be reduced to U using row operations, so A is row equivalent to U. If B is row equivalent to A then it follows from the result in Exercise 24(a) that B is row equivalent to U.

26. If B is row equivalent to A, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} \cdots E_1 A$$

Let $M = E_k E_{k-1} \cdots E_1$. The matrix M is nonsingular since each of the E_i's is nonsingular.

Conversely suppose there exists a nonsingular matrix M such that $B = MA$. Since M is nonsingular it is row equivalent to I. Thus there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$M = E_k E_{k-1} \cdots E_1 I$$

It follows that

$$B = MA = E_k E_{k-1} \cdots E_1 A$$

Therefore B is row equivalent to A.

27. (a) The system $V\mathbf{c} = \mathbf{y}$ is given by

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}$$

Comparing the ith row of each side, we have

$$c_1 + c_2 x_i + \cdots + c_{n+1} x_i^n = y_i$$

Thus

$$p(x_i) = y_i \qquad i = 1, 2, \ldots, n+1$$

(b) If $x_1, x_2, \ldots, x_{n+1}$ are distinct and $V\mathbf{c} = \mathbf{0}$, then we can apply part (a) with $\mathbf{y} = \mathbf{0}$. Thus if $p(x) = c_1 + c_2 x + \cdots + c_{n+1} x^n$, then

$$p(x_i) = 0 \qquad i = 1, 2, \ldots, n+1$$

The polynomial $p(x)$ has $n+1$ roots. Since the degree of $p(x)$ is less than $n+1$, $p(x)$ must be the zero polynomial. Hence

$$c_1 = c_2 = \cdots = c_{n+1} = 0$$

Since the system $V\mathbf{c} = \mathbf{0}$ has only the trivial solution, the matrix V must be nonsingular.

SECTION 5

2. $B = A^T A = \begin{pmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{pmatrix} (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \begin{pmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & & & \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{pmatrix}$

5. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} (1 \;\; 2 \;\; 3) = \begin{pmatrix} 6 & 0 & 1 \\ 11 & -1 & 4 \end{pmatrix}$

(c) Let

$$A_{11} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A_{21} = (0 \quad 0) \qquad A_{22} = (1 \quad 0)$$

The block multiplication is performed as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{pmatrix} = \begin{pmatrix} A_{11}A_{11}^T + A_{12}A_{12}^T & A_{11}A_{21}^T + A_{12}A_{22}^T \\ A_{21}A_{11}^T + A_{22}A_{12}^T & A_{21}A_{21}^T + A_{22}A_{22}^T \end{pmatrix}$$
$$= \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

6. (a)

$$\begin{aligned} XY^T &= \mathbf{x}_1\mathbf{y}_1^T + \mathbf{x}_2\mathbf{y}_2^T + \mathbf{x}_3\mathbf{y}_3^T \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} + \begin{pmatrix} 20 & 5 \\ 12 & 3 \end{pmatrix} \end{aligned}$$

(b) Since $\mathbf{y}_i\mathbf{x}_i^T = (\mathbf{x}_i\mathbf{y}_i^T)^T$ for $j = 1, 2, 3$, the outer product expansion of YX^T is just the transpose of the outer product expansion of XY^T. Thus

$$\begin{aligned} YX^T &= \mathbf{y}_1\mathbf{x}_1^T + \mathbf{y}_2\mathbf{x}_2^T + \mathbf{y}_3\mathbf{x}_3^T \\ &= \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 20 & 12 \\ 5 & 3 \end{pmatrix} \end{aligned}$$

7. It is possible to perform both block multiplications. To see this suppose A_{11} is a $k \times r$ matrix, A_{12} is a $k \times (n-r)$ matrix, A_{21} is an $(m-k) \times r$ matrix and A_{22} is $(m-k) \times (n-r)$. It is possible to perform the block multiplication of AA^T since the matrix multiplication $A_{11}A_{11}^T$, $A_{11}A_{21}^T$, $A_{12}A_{12}^T$, $A_{12}A_{22}^T$, $A_{21}A_{11}^T$, $A_{21}A_{21}^T$, $A_{22}A_{12}^T$, $A_{22}A_{22}^T$ are all possible. It is possible to perform the block multiplication of A^TA since the matrix multiplications $A_{11}^TA_{11}$, $A_{11}^TA_{12}$, $A_{21}^TA_{21}$, $A_{21}^TA_{11}$, $A_{12}^TA_{12}$, $A_{22}^TA_{21}$, $A_{22}^TA_{22}$ are all possible.

8. $AX = A(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = (A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_r)$
$B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$
$AX = B$ if and only if the column vectors of AX and B are equal

$$A\mathbf{x}_j = \mathbf{b}_j \qquad j = 1, \ldots, r$$

9. (a) Since D is a diagonal matrix, its jth column will have d_{jj} in the jth row and the other entries will all be 0. Thus $\mathbf{d}_j = d_{jj}\mathbf{e}_j$ for $j = 1, \ldots, n$.

(b)

$$\begin{aligned} AD &= A(d_{11}\mathbf{e}_1,\ d_{22}\mathbf{e}_2, \ldots, d_{nn}\mathbf{e}_n) \\ &= (d_{11}A\mathbf{e}_1,\ d_{22}A\mathbf{e}_2, \ldots, d_{nn}A\mathbf{e}_n) \\ &= (d_{11}\mathbf{a}_1,\ d_{22}\mathbf{a}_2, \ldots, d_{nn}\mathbf{a}_n) \end{aligned}$$

10. (a)

$$U\Sigma = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 \\ O \end{pmatrix} = U_1\Sigma_1 + U_2O = U_1\Sigma_1$$

(b) If we let $X = U\Sigma$, then

$$X = U_1\Sigma_1 = (\sigma_1\mathbf{u}_1, \sigma_2\mathbf{u}_2, \ldots, \sigma_n\mathbf{u}_n)$$

and it follows that

$$A = U\Sigma V^T = XV^T = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \sigma_2\mathbf{u}_2\mathbf{v}_2^T + \cdots + \sigma_n\mathbf{u}_n\mathbf{v}_n^T$$

11.

$$\begin{pmatrix} A_{11}^{-1} & C \\ O & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix} = \begin{pmatrix} I & A_{11}^{-1}A_{12} + CA_{22} \\ O & I \end{pmatrix}$$

If

$$A_{11}^{-1}A_{12} + CA_{22} = O$$

then

$$C = -A_{11}^{-1}A_{12}A_{22}^{-1}$$

Let

$$B = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}$$

Since $AB = BA = I$ it follows that $B = A^{-1}$.

12. Let $\mathbf{0}$ denote the zero vector in R^n. If A is singular then there exists a vector $\mathbf{x}_1 \neq \mathbf{0}$ such that $A\mathbf{x}_1 = \mathbf{0}$. If we set

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix}$$

then

$$M\mathbf{x} = \begin{pmatrix} A & C \\ O & B \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A\mathbf{x}_1 + C\mathbf{0} \\ O\mathbf{x}_1 + B\mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

By Theorem 1.4.2, M must be singular. Similarly, if B is singular then there exists a vector $\mathbf{x}_2 \neq \mathbf{0}$ such that $B\mathbf{x}_2 = \mathbf{0}$. So if we set

$$\mathbf{x} = \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{pmatrix}$$

then $\mathbf{x}$ is a nonzero vector and $M\mathbf{x}$ is equal to the zero vector.

15. The block form of S^{-1} is given by

$$S^{-1} = \begin{pmatrix} I & -A \\ O & I \end{pmatrix}$$

It follows that

$$\begin{aligned} S^{-1}MS &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & O \\ B & O \end{pmatrix} \begin{pmatrix} I & A \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} I & -A \\ O & I \end{pmatrix} \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} O & O \\ B & BA \end{pmatrix}$$

16. The block multiplication of the two factors yields

$$\begin{pmatrix} I & O \\ B & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & C \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ BA_{11} & BA_{12} + C \end{pmatrix}$$

If we equate this matrix with the block form of A and solve for B and C we get

$$B = A_{21}A_{11}^{-1} \qquad \text{and} \qquad C = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

To check that this works note that

$$\begin{aligned} BA_{11} &= A_{21}A_{11}^{-1}A_{11} = A_{21} \\ BA_{12} + C &= A_{21}A_{11}^{-1}A_{12} + A_{22} - A_{21}A_{11}^{-1}A_{12} = A_{22} \end{aligned}$$

and hence

$$\begin{pmatrix} I & O \\ B & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ O & C \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A$$

17. In order for the block multiplication to work we must have

$$XB = S \qquad \text{and} \qquad YM = T$$

Since both B and M are nonsingular, we can satisfy these conditions by choosing $X = SB^{-1}$ and $Y = TM^{-1}$.

18. (a)

$$BC = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} (c) = \begin{pmatrix} b_1 c \\ b_2 c \\ \vdots \\ b_n c \end{pmatrix} = c\mathbf{b}$$

(b)

$$\begin{aligned} A\mathbf{x} &= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \mathbf{a}_1(x_1) + \mathbf{a}_2(x_2) + \cdots + \mathbf{a}_n(x_n) \end{aligned}$$

(c) It follows from parts (a) and (b) that

$$\begin{aligned} A\mathbf{x} &= \mathbf{a}_1(x_1) + \mathbf{a}_2(x_2) + \cdots + \mathbf{a}_n(x_n) \\ &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \end{aligned}$$

19. If $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in R^n$, then

$$\mathbf{a}_j = A\mathbf{e}_j = \mathbf{0} \quad \text{for} \quad j = 1, \ldots, n$$

and hence A must be the zero matrix.

20. If

$$B\mathbf{x} = C\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in R^n$$

then

$$(B - C)\mathbf{x} = \mathbf{0} \quad \text{for all} \quad \mathbf{x} \in R^n$$

It follows from Exercise 19 that

$$\begin{aligned} B - C &= O \\ B &= C \end{aligned}$$

21. (a)

$$\begin{pmatrix} A^{-1} & \mathbf{0} \\ -\mathbf{c}^T A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{a} \\ \mathbf{c}^T & \beta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} A^{-1} & \mathbf{0} \\ -\mathbf{c}^T A^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ b_{n+1} \end{pmatrix}$$

$$\begin{pmatrix} I & A^{-1}\mathbf{a} \\ \mathbf{0}^T & -\mathbf{c}^T A^{-1}\mathbf{a} + \beta \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} A^{-1}\mathbf{b} \\ -\mathbf{c}^T A^{-1}\mathbf{b} + b_{n+1} \end{pmatrix}$$

(b) If

$$\mathbf{y} = A^{-1}\mathbf{a} \qquad \text{and} \qquad \mathbf{z} = A^{-1}\mathbf{b}$$

then

$$(-\mathbf{c}^T\mathbf{y} + \beta)x_{n+1} = -\mathbf{c}^T\mathbf{z} + b_{n+1}$$

$$x_{n+1} = \frac{-\mathbf{c}^T\mathbf{z} + b_{n+1}}{-\mathbf{c}^T\mathbf{y} + \beta} \qquad (\beta - \mathbf{c}^T\mathbf{y} \neq 0)$$

and

$$\mathbf{x} + x_{n+1}A^{-1}\mathbf{a} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b} - x_{n+1}A^{-1}\mathbf{a} = \mathbf{z} - x_{n+1}\mathbf{y}$$

MATLAB EXERCISES

1. In parts (a), (b), (c) it should turn out that $A1 = A4$ and $A2 = A3$. In part (d) $A1 = A3$ and $A2 = A4$. Exact equality will not occur in parts (c) and (d) because of roundoff error.

2. The solution $\mathbf{x}$ obtained using the \ operation will be more accurate and yield the smaller residual vector. The computation of $\mathbf{x}$ is also more efficient since the solution is computed using Gaussian elimination with partial pivoting and this involves less arithmetic than computing the inverse matrix and multiplying it times $\mathbf{b}$.

3. (a) Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, it follows from Theorem 1.4.2 that A is singular.

(b) The columns of B are all multiples of $\mathbf{x}$. Indeed,

$$B = (\mathbf{x}, 2\mathbf{x}, 3\mathbf{x}, 4\mathbf{x}, 5\mathbf{x}, 6\mathbf{x})$$

and hence

$$AB = (A\mathbf{x}, 2A\mathbf{x}, 3A\mathbf{x}, 4A\mathbf{x}, 5A\mathbf{x}, 6A\mathbf{x}) = O$$

(c) If $D = B + C$, then

$$AD = AB + AC = O + AC = AC$$

4. By construction B is upper triangular whose diagonal entries are all equal to 1. Thus B is row equivalent to I and hence B is nonsingular. If one changes B by setting $b_{10,1} = -1/256$ and computes $B\mathbf{x}$, the result is the zero vector. Since $\mathbf{x} \neq \mathbf{0}$, the matrix B must be singular.

5. (a) Since A is nonsingular its reduced row echelon form is I. If $E_1, \ldots, E_k$ are elementary matrices such that $E_k \cdots E_1 A = I$, then these same matrices can be used to transform $(A \;\; \mathbf{b})$ to its reduced row echelon form U. It follows then that

$$U = E_k \cdots E_1 (A \;\; \mathbf{b}) = A^{-1}(A \;\; \mathbf{b}) = (I \;\; A^{-1}\mathbf{b})$$

Thus, the last column of U should be equal to the solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$.

(b) After the third column of A is changed, the new matrix A is now singular. Examining the last row of the reduced row echelon form of the augmented matrix $(A \; \mathbf{b})$, we see that the system is inconsistent.

(c) The system $A\mathbf{x} = \mathbf{c}$ is consistent since $\mathbf{y}$ is a solution. There is a free variable x_3, so the system will have infinitely many solutions.

(f) The vector $\mathbf{v}$ is a solution since

$$A\mathbf{v} = A(\mathbf{w} + 3\mathbf{z}) = A\mathbf{w} + 3A\mathbf{z} = \mathbf{c}$$

For this solution the free variable $x_3 = v_3 = 3$. To determine the general solution just set $\mathbf{x} = \mathbf{w} + t\mathbf{z}$. This will give the solution corresponding to $x_3 = t$ for any real number t.

6. (c) There will be no walks of even length from V_i to V_j whenever $i + j$ is odd.

(d) There will be no walks of length k from V_i to V_j whenever $i + j + k$ is odd.

(e) The conjecture is still valid for the graph containing the additional edges.

(f) If the edge $\{V_6, V_8\}$ is included, then the conjecture is no longer valid. There is now a walk of length 1 V_6 to V_8 and $i + j + k = 6 + 8 + 1$ is odd.

8. The change in part (b) should not have a significant effect on the survival potential for the turtles. The change in part (c) will effect the $(2, 2)$ and $(3, 2)$ of the Leslie matrix. The new values for these entries will be $l_{22} = 0.9540$ and $l_{32} = 0.0101$. With these values the Leslie population model should predict that the survival period will double but the turtles will still eventually die out.

9. (b) $\mathbf{x1} = \mathbf{c} - V\mathbf{x2}$.

10. (b)

$$A^{2k} = \begin{pmatrix} I & kB \\ kB & I \end{pmatrix}$$

This can be proved using mathematical induction. In the case $k = 1$

$$A^2 = \begin{pmatrix} O & I \\ I & B \end{pmatrix} \begin{pmatrix} O & I \\ I & B \end{pmatrix} = \begin{pmatrix} I & B \\ B & I \end{pmatrix}$$

If the result holds for $k = m$

$$A^{2m} = \begin{pmatrix} I & mB \\ mB & I \end{pmatrix}$$

then

$$\begin{aligned} A^{2m+2} &= A^2A^{2m} \\ &= \begin{pmatrix} I & B \\ B & I \end{pmatrix} \begin{pmatrix} I & mB \\ mB & I \end{pmatrix} \\ &= \begin{pmatrix} I & (m+1)B \\ (m+1)B & I \end{pmatrix} \end{aligned}$$

It follows by mathematical induction that the result holds for all positive integers k.

(b)

$$A^{2k+1} = AA^{2k} = \begin{pmatrix} O & I \\ I & B \end{pmatrix} \begin{pmatrix} I & kB \\ kB & I \end{pmatrix} = \begin{pmatrix} kB & I \\ I & (k+1)B \end{pmatrix}$$

11. (a) By construction the entries of A were rounded to the nearest integer. The matrix $B = A^TA$ must also have integer entries and it is symmetric since

$$B^T = (A^TA)^T = A^T(A^T)^T = A^TA = B$$

(b)

$$\begin{aligned} LDL^T &= \begin{pmatrix} I & O \\ E & I \end{pmatrix} \begin{pmatrix} B_{11} & O \\ O & F \end{pmatrix} \begin{pmatrix} I & E^T \\ O & I \end{pmatrix} \\ &= \begin{pmatrix} B_{11} & B_{11}E^T \\ EB_{11} & EB_{11}E^T + F \end{pmatrix} \end{aligned}$$

where

$$E = B_{21}B_{11}^{-1} \quad \text{and} \quad F = B_{22} - B_{21}B_{11}^{-1}B_{12}$$

It follows that

$$\begin{aligned} B_{11}E^T &= B_{11}(B_{11}^{-1})^TB_{21}^T = B_{11}B_{11}^{-1}B_{12} = B_{12} \\ EB_{11} &= B_{21}B_{11}^{-1}B_{11} = B_{21} \end{aligned}$$

$$\begin{aligned} EB_{11}E^T + F &= B_{21}E^T + B_{22} - B_{21}B_{11}^{-1}B_{12} \\ &= B_{21}B_{11}^{-1}B_{12} + B_{22} - B_{21}B_{11}^{-1}B_{12} \\ &= B_{22} \end{aligned}$$

Therefore

$$LDL^T = B$$

CHAPTER TEST A

1. The statement is false in general. If the row echelon form has free variables and the linear system is consistent, then there will be infinitely many solutions. However, it is possible to have an inconsistent system whose coefficient matrix will reduce to an echelon form with free variables. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then A involves one free variable, but the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

2. The statement is true since the zero vector will always be a solution.

3. The statement is true. A matrix A is nonsingular if and only if it is row equivalent to the I (the identity matrix). A will be row equivalent to I if and only if its reduced row echelon form is I.

4. The statement is false in general. For example, if $A = I$ and $B = -I$, the matrices A and B are both nonsingular, but $A + B = O$ is singular.

5. The statement is false in general. If A and B are nonsingular, then AB must also be nonsingular, however, $(AB)^{-1}$ is equal to $B^{-1}A^{-1}$ rather than $A^{-1}B^{-1}$. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad (AB)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

however,

$$A^{-1}B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Note that

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = (AB)^{-1}$$

6. The statement is false in general.

$$(A - B)^2 = A^2 - BA - AB + B^2 \neq A^2 - 2AB + B^2$$

since in general $BA \neq AB$. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$(A - B)^2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

however,

$$A^2 - 2AB + B^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

7. The statement is false in general. If A is nonsingular and $AB = AC$, then we can multiply both sides of the equation by A^{-1} and conclude that $B = C$. However, if A is singular, then it is possible to have $AB = AC$ and $B \neq C$. For example, if

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

then

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \\ AC &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \end{aligned}$$

8. The statement is false. An elementary matrix is a matrix that is constructed by performing exactly one elementary row operation on the identity matrix. The product of two elementary matrices will be a matrix formed by performing *two* elementary row operations on the identity matrix. For example,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

are elementary matrices, however,

$$E_1E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

is not an elementary matrix.

9. The statement is true. The row vectors of A are $x_1\mathbf{y}^T, x_2\mathbf{y}^T, \ldots, x_n\mathbf{y}^T$. Note, all of the row vectors are multiples of $\mathbf{y}^T$. Since $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors, at least one of these row vectors must be nonzero. However, if any nonzero row is picked as a pivot row, then since all of the other rows are multiples of the pivot row, they will all be eliminated in the first step of the reduction process. The resulting row echelon form will have exactly one nonzero row.

10. The statement is true. If $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, then $\mathbf{x} = (1, 1, 1)^T$ is a solution to $A\mathbf{x} = \mathbf{b}$, since

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 = \mathbf{b}$$

If $\mathbf{a}_2 = \mathbf{a}_3$, then we can also express $\mathbf{b}$ as a linear combination

$$\mathbf{b} = \mathbf{a}_1 + 0\mathbf{a}_2 + 2\mathbf{a}_3$$

Thus $\mathbf{y} = (1, 0, 2)^T$ is also a solution to the system. However, if there is more than one solution, then the echelon form of A must involve a free variable. A consistent system with a free variable must have infinitely many solutions.

CHAPTER TEST B

1.

$$\left(\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 7 & 7 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rrrr|r} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 3 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{rrrr|r} 1 & -1 & 0 & -7 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The free variables are x_2 and x_4. If we set $x_2 = a$ and $x_4 = b$, then

$$x_1 = 4 + a + 7b \qquad \text{and} \qquad x_3 = -1 - 3b$$

and hence the solution set consists of all vectors of the form

$$\mathbf{x} = \begin{pmatrix} 4 + a + 7b \\ a \\ -1 - 3b \\ b \end{pmatrix}$$

2. **(a)** A linear equation in 3 unknowns corresponds to a plane in 3-space.

(b) Given 2 equations in 3 unknowns, each equation corresponds to a plane. If one equation is a multiple of the other then the equations represent the same plane and any point on the that plane will be a solution to the system. If the two planes are distinct then they are either parallel or they intersect in a line. If they are parallel they do not intersect, so the system will have no solutions. If they intersect in a line then there will be infinitely many solutions.

(c) A homogeneous linear system is always consistent since it has the trivial solution $\mathbf{x} = \mathbf{0}$. It follows from part (b) then that a homogeneous system of 2 equations in 3 unknowns must have infinitely many solutions. Geometrically the 2 equations represent planes that both pass through the origin, so if the planes are distinct they must intersect in a line.

3. **(a)** If the system is consistent and there are two distinct solutions there must be a free variable and hence there must be infinitely many solutions. In fact all vectors of the form $\mathbf{x} = \mathbf{x}_1 + c(\mathbf{x}_1 - \mathbf{x}_2)$ will be solutions since

$$A\mathbf{x} = A\mathbf{x}_1 + c(A\mathbf{x}_1 - A\mathbf{x}_2) = \mathbf{b} + c(\mathbf{b} - \mathbf{b}) = \mathbf{b}$$

(b) If we set $\mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2$ then $\mathbf{z} \neq \mathbf{0}$ and $A\mathbf{z} = \mathbf{0}$. Therefore it follows from Theorem 1.4.2 that A must be singular.

4. **(a)** The system will be consistent if and only if the vector $\mathbf{b} = (3, 1)^T$ can be written as a linear combination of the column vectors of A. Linear combinations of the column vectors of A are vectors of the form

$$c_1 \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} + c_2 \begin{pmatrix} \beta \\ 2\beta \end{pmatrix} = (c_1\alpha + c_2\beta) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Since $\mathbf{b}$ is not a multiple of $(1, 2)^T$ the system must be inconsistent.

(b) To obtain a consistent system choose $\mathbf{b}$ to be a multiple of $(1, 2)^T$. If this is done the second row of the augmented matrix will zero out in the elimination process and you will end up with one equation in 2 unknowns. The reduced system will have infinitely many solutions.

5. (a) To transform A to B you need to interchange the second and third rows of A. The elementary matrix that does this is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) To transform A to C using a column operation you need to subtract twice the second column of A from the first column. The elementary matrix that does this is

$$F = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6. If $\mathbf{b} = 3\mathbf{a}_1 + \mathbf{a}_2 + 4\mathbf{a}_3$ then $\mathbf{b}$ is a linear combination of the column vectors of A and it follows from the consistency theorem that the system $A\mathbf{x} = \mathbf{b}$ is consistent. In fact $\mathbf{x} = (3, 1, 4)^T$ is a solution to the system.

7. If $\mathbf{a}_1 - 3\mathbf{a}_2 + 2\mathbf{a}_3 = \mathbf{0}$ then $\mathbf{x} = (1, -3, 2)^T$ is a solution to $A\mathbf{x} = \mathbf{0}$. It follows from Theorem 1.4.2 that A must be singular.

8. If

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

then

$$A\mathbf{x} = \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = B\mathbf{x}$$

9. In general the product of two symmetric matrices is not necessarily symmetric. For example if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

then A and B are both symmetric but their product

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 4 & 10 \end{pmatrix}$$

is not symmetric.

10. If E and F are elementary matrices then they are both nonsingular and their inverses are elementary matrices of the same type. If $C = EF$ then C is a product of nonsingular matrices, so C is nonsingular and $C^{-1} = F^{-1}E^{-1}$.

11.

$$A^{-1} = \begin{pmatrix} I & O & O \\ O & I & O \\ O & -B & I \end{pmatrix}$$

12. (a) The column partition of A and the row partition of B must match up, so k must be equal to 5. There is really no restriction on r, it can be any integer in the range $1 \leq r \leq 9$. In fact $r = 10$ will work when B has block structure

$$\begin{pmatrix} B_{11} \\ B_{21} \end{pmatrix}$$

(b) The (2,2) block of the product is given by $A_{21}B_{12} + A_{22}B_{22}$

Chapter 2

SECTION 1

1. (c) $\det(A) = -3$

7. Given that $a_{11} = 0$ and $a_{21} \neq 0$, let us interchange the first two rows of A and also multiply the third row through by $-a_{21}$. We end up with the matrix

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ -a_{21}a_{31} & -a_{21}a_{32} & -a_{21}a_{33} \end{pmatrix}$$

Now if we add a_{31} times the first row to the third, we obtain the matrix

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{31}a_{22} - a_{21}a_{32} & a_{31}a_{23} - a_{21}a_{33} \end{pmatrix}$$

This matrix will be row equivalent to I if and only if

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{31}a_{22} - a_{21}a_{32} & a_{31}a_{23} - a_{21}a_{33} \end{vmatrix} \neq 0$$

Thus the original matrix A will be row equivalent to I if and only if

$$a_{12}a_{31}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22} + a_{13}a_{21}a_{32} \neq 0$$

8. **Theorem 2.1.3.** If A is an $n \times n$ triangular matrix then the determinant of A equals the product of the diagonal elements of A.

Proof: The proof is by induction on n. In the case $n = 1$, $A = (a_{11})$ and $\det(A) = a_{11}$. Assume the result holds for all $k \times k$ triangular matrices and let A be a $(k+1)\times(k+1)$ lower triangular matrix. (It suffices to prove the theorem for lower triangular matrices since $\det(A^T) = \det(A)$.) If $\det(A)$ is expanded by cofactors using the first row of A we get

$$\det(A) = a_{11}\det(M_{11})$$

where M_{11} is the $k\times k$ matrix obtained by deleting the first row and column of A. Since M_{11} is lower triangular we have

$$\det(M_{11}) = a_{22}a_{33}\cdots a_{k+1,k+1}$$

and consequently

$$\det(A) = a_{11}a_{22}\cdots a_{k+1,k+1}$$

9. If the ith row of A consists entirely of 0's then

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = 0$$

If the ith column of A consists entirely of 0's then

$$\det(A) = \det(A^T) = 0$$

10. In the case $n = 1$, if A is a matrix of the form

$$\begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

then $\det(A) = ab - ab = 0$. Suppose that the result holds for $(k+1)\times(k+1)$ matrices and that A is a $(k+2)\times(k+2)$ matrix whose ith and jth rows are identical. Expand $\det(A)$ by factors along the mth row where $m \neq i$ and $m \neq j$.

$$\det(A) = a_{m1}\det(M_{m1}) + a_{m2}\det(M_{m2}) + \cdots + a_{m,k+2}\det(M_{m,k+2}).$$

Each M_{ms}, $1 \le s \le k+2$, is a $(k+1)\times(k+1)$ matrix having two rows that are identical. Thus by the induction hypothesis

$$\det(M_{ms}) = 0 \qquad (1 \le s \le k+2)$$

and consequently $\det(A) = 0$.

11. (a) In general $\det(A+B) \neq \det(A) + \det(B)$. For example if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\det(A) + \det(B) = 0 + 0 = 0$$

and

$$\det(A+B) = \det(I) = 1$$

(b)

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

and hence

$$\begin{aligned}\det(AB) &= (a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}) \\ &\quad -(a_{21}b_{11}a_{11}b_{12} + a_{21}b_{11}a_{12}b_{22} + a_{22}b_{21}a_{11}b_{12} + a_{22}b_{21}a_{12}b_{22}) \\ &= a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} - a_{21}b_{11}a_{12}b_{22} - a_{22}b_{21}a_{11}b_{12}\end{aligned}$$

On the other hand

$$\begin{aligned}\det(A)\det(B) &= (a_{11}a_{22} - a_{21}a_{12})(b_{11}b_{22} - b_{21}b_{12}) \\ &= a_{11}a_{22}b_{11}b_{22} + a_{21}a_{12}b_{21}b_{12} - a_{21}a_{12}b_{11}b_{22} - a_{11}a_{22}b_{21}b_{12}\end{aligned}$$

Therefore $\det(AB) = \det(A)\det(B)$

(c) In part (b) it was shown that for any pair of 2×2 matrices, the determinant of the product of the matrices is equal to the product of the determinants. Thus if A and B are 2×2 matrices, then

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

12. (a) If $d = \det(A + B)$, then

$$\begin{aligned}d &= (a_{11} + b_{11})(a_{22} + b_{22}) - (a_{21} + b_{21})(a_{12} + b_{12}) \\ &= a_{11}a_{22} + a_{11}b_{22} + b_{11}a_{22} + b_{11}b_{22} - a_{21}a_{12} - a_{21}b_{12} - b_{21}a_{12} - b_{21}b_{12} \\ &= (a_{11}a_{22} - a_{21}a_{12}) + (b_{11}b_{22} - b_{21}b_{12}) + (a_{11}b_{22} - b_{21}a_{12}) + (b_{11}a_{22} - a_{21}b_{12}) \\ &= \det(A) + \det(B) + \det(C) + \det(D)\end{aligned}$$

(b) If

$$B = EA = \begin{pmatrix} \alpha a_{21} & \alpha a_{22} \\ \beta a_{11} & \beta a_{12} \end{pmatrix}$$

then

$$C = \begin{pmatrix} a_{11} & a_{12} \\ \beta a_{11} & \beta a_{12} \end{pmatrix} \qquad D = \begin{pmatrix} \alpha a_{21} & \alpha a_{22} \\ a_{21} & a_{22} \end{pmatrix}$$

and hence

$$\det(C) = \det(D) = 0$$

It follows from part (a) that

$$\det(A + B) = \det(A) + \det(B)$$

13. Expanding $\det(A)$ by cofactors using the first row we get

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12})$$

If the first row and column of M_{12} are deleted the resulting matrix will be the matrix B obtained by deleting the first two rows and columns of A. Thus if $\det(M_{12})$ is expanded along the first column we get

$$\det(M_{12}) = a_{21}\det(B)$$

Since $a_{21} = a_{12}$ we have

$$\det(A) = a_{11}\det(M_{11}) - a_{12}^2\det(B)$$

SECTION 2

5. To transform the matrix A into the matrix αA one must perform row operation II n times. Each time row operation II is performed the value of the determinant is changed by a factor of α. Thus

$$\det(\alpha A) = \alpha^n \det(A)$$

Alternatively, one can show this result holds by noting that $\det(\alpha I)$ is equal to the product of its diagonal entries. Thus, $\det(\alpha I) = \alpha^n$ and it follows that

$$\det(\alpha A) = \det(\alpha IA) = \det(\alpha I)\det(A) = \alpha^n \det(A)$$

6. Since

$$\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$$

it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

8. If E is an elementary matrix of type I or II then E is symmetric, so $E^T = E$. If E is an elementary matrix of type III formed from the identity matrix by adding c times its ith row to its jth row, then E^T will be the elementary matrix of type III formed from the identity matrix by adding c times its jth row to its ith row

9. (b) 18; (d) -6; (f) -3

10. Row operation III has no effect on the value of the determinant. Thus if B can be obtained from A using only row operation III, then $\det(B) = \det(A)$. Row operation I has the effect of changing the sign of the determinant. If B is obtained from A using only row operations I and III, then $\det(B) = \det(A)$ if row operation I has been applied an even number of times and $\det(B) = -\det(A)$ if row operation I has been applied an odd number of times.

11. Since $\det(A^2) = \det(A)^2$ it follows that $\det(A^2)$ must be a nonnegative real number. (We are assuming the entries of A are all real numbers.) If $A^2 + I = O$ then $A^2 = -I$ and hence $\det(A^2) = \det(-I)$. This is not possible if n is odd, since for n odd, $\det(-I) = -1$. On the other hand it is possible for $A^2 + I = O$ when n is even. For example when $n = 2$, if we take

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then it is easily verified that $A^2 + I = O$.

12. (a) Row operation III has no effect on the value of the determinant. Thus

$$\det(V) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix}$$

and hence

$$\begin{aligned}\det(V) &= (x_2 - x_1)(x_3^2 - x_1^2) - (x_3 - x_1)(x_2^2 - x_1^2)\\ &= (x_2 - x_1)(x_3 - x_1)[(x_3 + x_1) - (x_2 + x_1)]\\ &= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)\end{aligned}$$

(b) The determinant will be nonzero if and only if no two of the x_i values are equal. Thus V will be nonsingular if and only if the three points x_1, x_2, x_3 are distinct.

14. Since

$$\det(AB) = \det(A)\det(B)$$

it follows that $\det(AB) \neq 0$ if and only if $\det(A)$ and $\det(B)$ are both nonzero. Thus AB is nonsingular if and only if A and B are both nonsingular.

15. If $AB = I$, then $\det(AB) = 1$ and hence by Exercise 14 both A and B are nonsingular. It follows then that

$$B = IB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}I = A^{-1}$$

Thus to show that a square matrix A is nonsingular it suffices to show that there exists a matrix B such that $AB = I$. We need not check whether or not $BA = I$.

16. If A is a $n \times n$ skew symmetric matrix, then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$

Thus if n is odd then

$$\begin{aligned}\det(A) &= -\det(A)\\ 2\det(A) &= 0\end{aligned}$$

and hence A must be singular.

17. If A_{nn} is nonzero and one subtracts $c = \det(A)/A_{nn}$ from the (n, n) entry of A, then the resulting matrix, call it B, will be singular. To see this look at the cofactor expansion of the B along its last row.

$$\begin{aligned}\det(B) &= b_{n1}B_{n1} + \cdots + b_{n,n-1}B_{n,n-1} + b_{nn}B_{nn}\\ &= a_{n1}A_{n1} + \cdots + A_{n,n-1}A_{n,n-1} + (a_{nn} - c)A_{nn}\\ &= \det(A) - cA_{nn}\\ &= 0\end{aligned}$$

18.

$$X = \begin{pmatrix} x_1 & x_2 & x_3\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix} \qquad Y = \begin{pmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ y_1 & y_2 & y_3\end{pmatrix}$$

Since X and Y both have two rows the same it follows that $\det(X) = 0$ and $\det(Y) = 0$. Expanding $\det(X)$ along the first row, we get

$$\begin{aligned}0 &= x_1X_{11} + x_2X_{12} + x_3X_{13}\\ &= x_1z_1 + x_2z_2 + x_3z_3\end{aligned}$$

$$= \mathbf{x}^T\mathbf{z}$$

Expanding $\det(Y)$ along the third row, we get

$$\begin{aligned} 0 &= y_1Y_{31} + y_2Y_{32} + y_3Y_{33} \\ &= y_1z_1 + y_2z_2 + y_3z_3 \\ &= \mathbf{y}^T\mathbf{z}. \end{aligned}$$

19. Prove: Evaluating an $n \times n$ matrix by cofactors requires $(n! - 1)$ additions and

$$\sum_{k=1}^{n-1} \frac{n!}{k!}$$

multiplications.

Proof: The proof is by induction on n. In the case $n = 1$ no additions and multiplications are necessary. Since $1! - 1 = 0$ and

$$\sum_{k=1}^{0} \frac{n!}{k!} = 0$$

the result holds when $n = 1$. Let us assume the result holds when $n = m$. If A is an $(m+1) \times (m+1)$ matrix then

$$\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) \pm \cdots \pm a_{1,m+1}\det(M_{1,m+1})$$

Each M_{1j} is an $m \times m$ matrix. By the induction hypothesis the calculation of $\det(M_{1j})$ requires $(m! - 1)$ additions and

$$\sum_{k=1}^{m-1} \frac{m!}{k!}$$

multiplications. The calculation of all $m+1$ of these determinants requires $(m+1)(m!-1)$ additions and

$$\sum_{k=1}^{m-1} \frac{(m+1)!}{k!}$$

multiplications. The calculation of $\det(A)$ requires an additional $m+1$ multiplications and an additional m additions. Thus the number of additions necessary to compute $\det(A)$ is

$$(m+1)(m! - 1) + m = (m+1)! - 1$$

and the number of multiplications needed is

$$\sum_{k=1}^{m-1} \frac{(m+1)!}{k!} + (m+1) = \sum_{k=1}^{m-1} \frac{(m+1)!}{k!} + \frac{(m+1)!}{m!} = \sum_{k=1}^{m} \frac{(m+1)!}{k!}$$

20. In the elimination method the matrix is reduced to triangular form and the determinant of the triangular matrix is calculated by multiplying its diagonal elements. At the first step of the reduction process the first row is multiplied by $m_{i1} = -a_{i1}/a_{11}$ and then added to the ith row. This requires 1 division,

$n-1$ multiplications and $n-1$ additions. However, this row operation is carried out for $i = 2, \ldots, n$. Thus the first step of the reduction requires $n-1$ divisions, $(n-1)^2$ multiplications and $(n-1)^2$ additions. At the second step of the reduction this same process is carried out on the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and first column of the matrix obtained from step 1. The second step of the elimination process requires $n-2$ divisions, $(n-2)^2$ multiplications, and $(n-2)^2$ additions. After $n-1$ steps the reduction to triangular form will be complete. It will require:

$$\begin{aligned}(n-1) + (n-2) + \cdots + 1 &= \frac{n(n-1)}{2} \quad \text{divisions}\\ (n-1)^2 + (n-2)^2 + \cdots + 1^2 &= \frac{n(2n-1)(n-1)}{6} \quad \text{multiplications}\\ (n-1)^2 + (n-2)^2 + \cdots + 1^2 &= \frac{n(2n-1)(n-1)}{6} \quad \text{additions}\end{aligned}$$

It takes $n-1$ additional multiplications to calculate the determinant of the triangular matrix. Thus the calculation $\det(A)$ by the elimination method requires:

$$\frac{n(n-1)}{2} + \frac{n(2n-1)(n-1)}{6} + (n-1) = \frac{(n-1)(n^2+n+3)}{3}$$

multiplications and divisions and $\frac{n(2n-1)(n-1)}{6}$ additions.

SECTION 3

1. (b) $\det(A) = 10$, $\operatorname{adj} A = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, $A^{-1} = \frac{1}{10}\operatorname{adj} A$

(d) $\det(A) = 1$, $A^{-1} = \operatorname{adj} A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

6. $A \operatorname{adj} A = O$

7. The solution of $I\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{b}$. It follows from Cramer's rule that

$$b_j = x_j = \frac{\det(B_j)}{\det(I)} = \det(B_j)$$

8. If $\det(A) = \alpha$ then $\det(A^{-1}) = 1/\alpha$. Since $\operatorname{adj} A = \alpha A^{-1}$ we have

$$\det(\operatorname{adj} A) = \det(\alpha A^{-1}) = \alpha^n \det(A^{-1}) = \alpha^{n-1} = \det(A)^{n-1}$$

10. If A is nonsingular then $\det(A) \neq 0$ and hence

$$\operatorname{adj} A = \det(A)A^{-1}$$

is also nonsingular. It follows that

$$(\operatorname{adj} A)^{-1} = \frac{1}{\det(A)}(A^{-1})^{-1} = \det(A^{-1})A$$

Also

$$\operatorname{adj} A^{-1} = \det(A^{-1})(A^{-1})^{-1} = \det(A^{-1})A$$

11. If $A = O$ then $\operatorname{adj} A$ is also the zero matrix and hence is singular. If A is singular and $A \neq O$ then

$$A \operatorname{adj} A = \det(A)I = 0I = O$$

If $\mathbf{a}^T$ is any nonzero row vector of A then

$$\mathbf{a}^T \operatorname{adj} A = \mathbf{0}^T \qquad \text{or} \qquad (\operatorname{adj} A)^T \mathbf{a} = \mathbf{0}$$

By Theorem 1.4.2, $(\operatorname{adj} A)^T$ is singular. Since

$$\det(\operatorname{adj} A) = \det[(\operatorname{adj} A)^T] = 0$$

it follows that adj A is singular.

12. If $\det(A) = 1$ then

$$\operatorname{adj} A = \det(A)A^{-1} = A^{-1}$$

and hence

$$\operatorname{adj}(\operatorname{adj} A) = \operatorname{adj}(A^{-1})$$

It follows from Exercise 10 that

$$\operatorname{adj}(\operatorname{adj} A) = \det(A^{-1})A = \frac{1}{\det(A)}A = A$$

13. The (j, i) entry of Q^T is q_{ij}. Since

$$Q^{-1} = \frac{1}{\det(Q)} \operatorname{adj} Q$$

its (j, i) entry is $Q_{ij}/\det(Q)$. If $Q^{-1} = Q^T$, then

$$q_{ij} = \frac{Q_{ij}}{\det(Q)}$$

MATLAB EXERCISES

2. The magic squares generated by MATLAB have the property that they are nonsingular when n is odd and singular when n is even.

3. (a) The matrix B is formed by interchanging the first two rows of A.
$\det(B) = -\det(A)$.

(b) The matrix C is formed by multiplying the third row of A by 4.
$\det(C) = 4\det(A)$.

(c) The matrix D is formed from A by adding 4 times the fourth row of A to the fifth row.
$\det(D) = \det(A)$.

5. The matrix U is very ill-conditioned. In fact it is singular with respect to the machine precision used by MATLAB. So in general one could not expect to get even a single digit of accuracy in the computed values of $\det(U^T)$ and $\det(UU^T)$. On the other hand, since U is upper triangular, the computed value of $\det(U)$ is the product of its diagonal entries. This value should be accurate to the machine precision.

6. (a) Since $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, the matrix must be singular. However, there may be no indication of this if the computations are done in floating point arithmetic. To compute the determinant MATLAB does Gaussian elimination to reduce the matrix to upper triangular form U and then multiplies the diagonal entries of U. In this case the product $u_{11}u_{22}u_{33}u_{44}u_{55}$ has magnitude on the order of 10^{14}. If the computed value of u_{66} has magnitude of the order 10^{-k} and $k \leq 14$, then MATLAB will round the result to a nonzero integer. (MATLAB knows that if you started with an integer matrix, you should end up with an integer value for the determinant.) In general if the determinant is computed in floating point arithmetic, then you cannot expect it to be a reliable indicator of whether or not a matrix is nonsingular.

(c) Since A is singular, $B = AA^T$ should also be singular. Hence the exact value of $\det(B)$ should be 0.

CHAPTER TEST A

1. The statement is true since

$$\det(AB) = \det(A)\det(B) = \det(B)\det(A) = \det(BA)$$

2. The statement is false in general. For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then $\det(A + B) = \det(I) = 1$ while $\det(A) + \det(B) = 0 + 0 = 0$.

3. The statement is false in general. For example, if $A = I$, (the 2×2 identity matrix), then $\det(2A) = 4$ while $2\det(A) = 2$.

4. The statement is true. For any matrix C, $\det(C^T) = \det(C)$, so in particular for $C = AB$ we have

$$\det((AB)^T) = \det(AB) = \det(A)\det(B)$$

5. The statement is false in general. For example if

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$

then $\det(A) = \det(B) = 8$, however, $A \neq B$.

6. The statement is true. For a product of two matrices we know that

$$\det(AB) = \det(A)\det(B)$$

Using this it is easy to see that the determinant of a product of k matrices is the product of the determinants of the matrices, i.e,

$$\det(A_1A_2\cdots A_k) = \det(A_1)\det(A_2)\cdots\det(A_k)$$

(This can be proved formally using mathematical induction.) In the special case that $A_1 = A_2 = \cdots = A_k$ we have

$$\det(A^k) = \det(A)^k$$

7. The statement is true. A triangular matrix T is nonsingular if and only if

$$\det(T) = t_{11}t_{22}\cdots t_{nn} \neq 0$$

Thus T is nonsingular if and only if all of its diagonal entries are nonzero.

8. The statement is true. If $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, then it follows from Theorem 1.4.2 that A must be singular. If A is singular then $\det(A) = 0$.

9. The statement is false in general. For example, if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and B is the 2×2 identity matrix, then A and B are row equivalent, however, their determinants are not equal.

10. The statement is true. If $A^k = O$, then

$$\det(A)^k = \det(A^k) = \det(O) = 0$$

So $\det(A) = 0$, and hence A must be singular.

CHAPTER TEST B

1. **(a)** $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det(A) = \frac{1}{8} \cdot 4 = \frac{1}{2}$

(b) $\det(B^{-1}A^T) = \det(B^{-1})\det(A^T) = \frac{1}{\det(B)}\det(A) = \frac{1}{6} \cdot 4 = \frac{2}{3}$

(c) $\det(EA^2) = -\det(A^2) = -\det(A)^2 = -16$

2. **(a)**

$$\begin{aligned}\det(A) &= x\begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -1 & x \end{vmatrix} + \begin{vmatrix} 1 & x \\ -1 & -1 \end{vmatrix} \\ &= x(x^2-1) - (x-1) + (-1+x) \\ &= x(x-1)(x+1)\end{aligned}$$

(b) The matrix will be singular if x equals 0, 1, or -1.

3. **(a)**

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix} \quad (l_{21} = l_{31} = l_{41} = 1)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix} \qquad (l_{32} = 2,\ l_{42} = 3)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (l_{43} = 3)$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) $\det(A) = \det(LU) = \det(L)\det(U) = 1 \cdot 1 = 1$

4. If A is nonsingular then $\det(A) \neq 0$ and it follows that

$$\det(A^TA) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2 > 0$$

Therefore A^TA must be nonsingular.

5. If $B = S^{-1}AS$, then

$$\begin{aligned} \det(B) = \det(S^{-1}AS) &= \det(S^{-1})\det(A)\det(S) \\ &= \frac{1}{\det(S)}\det(A)\det(S) = \det(A) \end{aligned}$$

6. If A is singular then $\det(A) = 0$ and if B is singular then $\det(B)$ so if one of the matrices is singular then

$$\det(C) = \det(AB) = \det(A)\det(B) = 0$$

Therefore the matrix C must be singular.

7. The determinant of $A - \lambda I$ will equal 0 if and only if $A - \lambda I$ is singular. By Theorem 1.4.2, $A - \lambda I$ is singular if and only if there exists a nonzero vector $\mathbf{x}$ such that $(A - \lambda I)\mathbf{x} = \mathbf{0}$. It follows then that $\det(A - \lambda I) = 0$ if and only if $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$.

8. If $A = \mathbf{x}\mathbf{y}^T$ then all of the rows of A are multiples of $\mathbf{y}^T$. In fact $\mathbf{a}(i,:) = x_i\mathbf{y}^T$ for $j = 1, \ldots, n$. It follows that if U is any row echelon form of A then U can have at most one nonzero row. Since A is row equivalent to U and $\det(U) = 0$, it follows that $\det(A) = 0$.

9. Let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Since $\mathbf{x}$ and $\mathbf{y}$ are distinct it follows that $\mathbf{z} \neq \mathbf{0}$. Since

$$A\mathbf{z} = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$$

it follows from Theorem 1.4.2 that A must be singular and hence $\det(A) = 0$.

10. If A has integer entries then adj A will have integer entries. So if $|\det(A)| = 1$ then

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}\,A = \pm\,\text{adj}\,A$$

and hence A^{-1} must also have integer entries.

CHAPTER 3

SECTION 1

3. To show that C is a vector space we must show that all eight axioms are satisfied.

A1. $$\begin{aligned}(a+bi)+(c+di) &= (a+c)+(b+d)i\\ &= (c+a)+(d+b)i\\ &= (c+di)+(a+bi)\end{aligned}$$

A2. $$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= [(x_1+x_2i)+(y_1+y_2i)]+(z_1+z_2i)\\ &= (x_1+y_1+z_1)+(x_2+y_2+z_2)i\\ &= (x_1+x_2i)+[(y_1+y_2i)+(z_1+z_2i)]\\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z})\end{aligned}$$

A3. $(a+bi)+(0+0i)=(a+bi)$

A4. If $\mathbf{z}=a+bi$ then define $-\mathbf{z}=-a-bi$. It follows that

$$\mathbf{z}+(-\mathbf{z})=(a+bi)+(-a-bi)=0+0i=\mathbf{0}$$

A5. $$\begin{aligned}\alpha[(a+bi)+(c+di)] &= (\alpha a+\alpha c)+(\alpha b+\alpha d)i\\ &= \alpha(a+bi)+\alpha(c+di)\end{aligned}$$

A6. $$\begin{aligned}(\alpha+\beta)(a+bi) &= (\alpha+\beta)a+(\alpha+\beta)bi\\ &= \alpha(a+bi)+\beta(a+bi)\end{aligned}$$

A7. $$\begin{aligned}(\alpha\beta)(a+bi) &= (\alpha\beta)a+(\alpha\beta)bi\\ &= \alpha(\beta a+\beta bi)\end{aligned}$$

A8. $1 \cdot (a+bi) = 1 \cdot a + 1 \cdot bi = a + bi$

4. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be arbitrary elements of $R^{m\times n}$.

A1. Since $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for each i and j it follows that $A+B = B+A$.

A2. Since

$$(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$$

for each i and j it follows that

$$(A+B)+C = A+(B+C)$$

A3. Let O be the $m \times n$ matrix whose entries are all 0. If $M = A + O$ then

$$m_{ij} = a_{ij} + 0 = a_{ij}$$

Therefore $A + O = A$.

A4. Define $-A$ to be the matrix whose ijth entry is $-a_{ij}$. Since

$$a_{ij} + (-a_{ij}) = 0$$

for each i and j it follows that

$$A + (-A) = O$$

A5. Since

$$\alpha(a_{ij} + b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$$

for each i and j it follows that

$$\alpha(A+B) = \alpha A + \alpha B$$

A6. Since

$$(\alpha + \beta)a_{ij} = \alpha a_{ij} + \beta a_{ij}$$

for each i and j it follows that

$$(\alpha+\beta)A = \alpha A + \beta A$$

A7. Since

$$(\alpha\beta)a_{ij} = \alpha(\beta a_{ij})$$

for each i and j it follows that

$$(\alpha\beta)A = \alpha(\beta A)$$

A8. Since

$$1 \cdot a_{ij} = a_{ij}$$

for each i and j it follows that

$$1A = A$$

5. Let f, g and h be arbitrary elements of $C[a,b]$.

A1. For all x in $[a,b]$

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x).$$

Therefore

$$f + g = g + f$$

A2. For all x in $[a, b]$,

$$\begin{aligned}[(f+g)+h](x) &= (f+g)(x)+h(x)\\ &= f(x)+g(x)+h(x)\\ &= f(x)+(g+h)(x)\\ &= [f+(g+h)](x)\end{aligned}$$

Therefore

$$[(f+g)+h] = [f+(g+h)]$$

A3. If $z(x)$ is identically 0 on $[a, b]$, then for all x in $[a, b]$

$$(f+z)(x) = f(x)+z(x) = f(x)+0 = f(x)$$

Thus

$$f+z = f$$

A4. Define $-f$ by

$$(-f)(x) = -f(x) \quad \text{for all } x \text{ in } [a, b]$$

Since

$$(f+(-f))(x) = f(x)-f(x) = 0$$

for all x in $[a, b]$ it follows that

$$f+(-f) = z$$

A5. For each x in $[a, b]$

$$\begin{aligned}[\alpha(f+g)](x) &= \alpha f(x)+\alpha g(x)\\ &= (\alpha f)(x)+(\alpha g)(x)\end{aligned}$$

Thus

$$\alpha(f+g) = \alpha f+\alpha g$$

A6. For each x in $[a, b]$

$$\begin{aligned}[(\alpha+\beta)f](x) &= (\alpha+\beta)f(x)\\ &= \alpha f(x)+\beta f(x)\\ &= (\alpha f)(x)+(\beta f)(x)\end{aligned}$$

Therefore

$$(\alpha+\beta)f = \alpha f+\beta f$$

A7. For each x in $[a, b]$,

$$[(\alpha\beta)f](x) = \alpha\beta f(x) = \alpha[\beta f(x)] = [\alpha(\beta f)](x)$$

Therefore

$$(\alpha\beta)f = \alpha(\beta f)$$

A8. For each x in $[a, b]$

$$1f(x) = f(x)$$

Therefore

$$1f = f$$

6. The proof is exactly the same as in Exercise 5.

9. (a) If $\mathbf{y} = \beta\mathbf{0}$ then

$$\mathbf{y} + \mathbf{y} = \beta\mathbf{0} + \beta\mathbf{0} = \beta(\mathbf{0} + \mathbf{0}) = \beta\mathbf{0} = \mathbf{y}$$

and it follows that

$$\begin{aligned}(\mathbf{y} + \mathbf{y}) + (-\mathbf{y}) &= \mathbf{y} + (-\mathbf{y})\\ \mathbf{y} + [\mathbf{y} + (-\mathbf{y})] &= \mathbf{0}\\ \mathbf{y} + \mathbf{0} &= \mathbf{0}\\ \mathbf{y} &= \mathbf{0}\end{aligned}$$

(b) If $\alpha\mathbf{x} = \mathbf{0}$ and $\alpha \neq 0$ then it follows from part (a), A7 and A8 that

$$\mathbf{0} = \frac{1}{\alpha}\mathbf{0} = \frac{1}{\alpha}(\alpha\mathbf{x}) = \left(\frac{1}{\alpha}\alpha\right)\mathbf{x} = 1\mathbf{x} = \mathbf{x}$$

10. Axiom 6 fails to hold.

$$\begin{aligned}(\alpha + \beta)\mathbf{x} &= ((\alpha + \beta)x_1,\ (\alpha + \beta)x_2)\\ \alpha\mathbf{x} + \beta\mathbf{x} &= ((\alpha + \beta)x_1, 0)\end{aligned}$$

12. A1. $x \oplus y = x \cdot y = y \cdot x = y \oplus x$

A2. $(x \oplus y) \oplus z = x \cdot y \cdot z = x \oplus (y \oplus z)$

A3. Since $x \oplus 1 = x \cdot 1 = x$ for all x, it follows that 1 is the zero vector.

A4. Let

$$-x = -1 \circ x = x^{-1} = \frac{1}{x}$$

It follows that

$$x \oplus (-x) = x \cdot \frac{1}{x} = 1 \quad \text{(the zero vector).}$$

Therefore $\frac{1}{x}$ is the additive inverse of x for the operation $\oplus$.

A5. $\alpha \circ (x \oplus y) = (x \oplus y)^{\alpha} = (x \cdot y)^{\alpha} = x^{\alpha} \cdot y^{\alpha}$
$\alpha \circ x \oplus \alpha \circ y = x^{\alpha} \oplus y^{\alpha} = x^{\alpha} \cdot y^{\alpha}$

A6. $(\alpha + \beta) \circ x = x^{(\alpha+\beta)} = x^{\alpha} \cdot x^{\beta}$
$\alpha \circ x \oplus \beta \circ x = x^{\alpha} \oplus x^{\beta} = x^{\alpha} \cdot x^{\beta}$

A7. $(\alpha\beta) \circ x = x^{\alpha\beta}$
$\alpha \circ (\beta \circ x) = \alpha \circ x^{\beta} = (x^{\beta})^{\alpha} = x^{\alpha\beta}$

A8. $1 \circ x = x^1 = x$

Since all eight axioms hold, R^+ is a vector space under the operations of $\circ$ and $\oplus$.

13. The system is not a vector space. Axioms A3, A4, A5, A6 all fail to hold.

14. Axioms 6 and 7 fail to hold. To see this consider the following example. If $\alpha = 1.5$, $\beta = 1.8$ and $x = 1$, then

$$(\alpha + \beta) \circ x = [\![3.3]\!] \cdot 1 = 3$$

and

$$\alpha \circ x + \beta \circ x = [\![1.5]\!] \cdot 1 + [\![1.8]\!] \cdot 1 = 1 \cdot 1 + 1 \cdot 1 = 2$$

So Axiom 6 fails. Furthermore,

$$(\alpha\beta) \circ x = [\![2.7]\!] \cdot 1 = 2$$

and

$$\alpha \circ (\beta \circ x) = [\![1.5]\!]([\![1.8]\!] \cdot 1) = 1 \cdot (1 \cdot 1) = 1$$

so Axiom 7 also fails to hold.

15. If $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are arbitrary elements of S, then for each n

$$a_n + b_n = b_n + a_n$$

and

$$a_n + (b_n + c_n) = (a_n + b_n) + c_n$$

Hence

$$\begin{aligned} \{a_n\} + \{b_n\} &= \{b_n\} + \{a_n\} \\ \{a_n\} + (\{b_n\} + \{c_n\}) &= (\{a_n\} + \{b_n\}) + \{c_n\} \end{aligned}$$

so Axioms 1 and 2 hold.

The zero vector is just the sequence $\{0, 0, \ldots\}$ and the additive inverse of $\{a_n\}$ is the sequence $\{-a_n\}$. The last four axioms all hold since

$$\begin{aligned} &\alpha(a_n + b_n) = \alpha a_n + \alpha b_n \\ &(\alpha + \beta)a_n = \alpha a_n + \beta a_n \\ &\alpha\beta a_n = \alpha(\beta a_n) \\ &1a_n = a_n \end{aligned}$$

for each n. Thus all eight axioms hold and hence S is a vector space.

16. If

$$\begin{aligned} p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1} &\leftrightarrow \mathbf{a} = (a_1, a_2, \ldots, a_n)^T \\ q(x) = b_1 + b_2 x + \cdots + b_n x^{n-1} &\leftrightarrow \mathbf{b} = (b_1, b_2, \ldots, b_n)^T \end{aligned}$$

then

$$\begin{aligned} \alpha p(x) &= \alpha a_1 + \alpha a_2 x + \cdots + \alpha a_n x^{n-1} \\ \alpha \mathbf{a} &= (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)^T \end{aligned}$$

and

$$\begin{aligned} (p+q)(x) &= (a_1 + b_1) + (a_2 + b_2)x + \cdots + (a_n + b_n)x^{n-1} \\ \mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n)^T \end{aligned}$$

Thus

$$\alpha p \leftrightarrow \alpha \mathbf{a} \qquad \text{and} \qquad p + q \leftrightarrow \mathbf{a} + \mathbf{b}$$

SECTION 2

7. $C^n[a,b]$ is a nonempty subset of $C[a,b]$. If $f \in C^n[a,b]$, then $f^{(n)}$ is continuous. Any scalar multiple of a continuous function is continuous. Thus for any scalar α, the function

$$(\alpha f)^{(n)} = \alpha f^{(n)}$$

is also continuous and hence $\alpha f \in C^n[a,b]$. If f and g are vectors in $C^n[a,b]$ then both have continuous nth derivatives and their sum will also have a continuous nth derivative. Thus $f+g \in C^n[a,b]$ and therefore $C^n[a,b]$ is a subspace of $C[a,b]$.

8. (a) If $B \in S_1$, then $AB = BA$. It follows that

$$A(\alpha B) = \alpha AB = \alpha BA = (\alpha B)A$$

and hence $\alpha B \in S_1$.

If B and C are in S_1, then

$$AB = BA \qquad \text{and} \qquad AC = CA$$

thus

$$A(B+C) = AB + AC = BA + CA = (B+C)A$$

and hence $B + C \in S_1$. Therefore S_1 is a subspace of $R^{2\times 2}$.

(b) If $B \in S_2$, then $AB \neq BA$. However, for the scalar 0, we have

$$0B = O \notin S_2$$

Therefore S_2 is not a subspace. (Also, S_2 is not closed under addition.)

(c) If $B \in S_3$, then $BA = O$. It follows that

$$(\alpha B)A = \alpha(BA) = \alpha O = O$$

Therefore, $\alpha B \in S_3$. If B and C are in S_3, then

$$BA = O \qquad \text{and} \qquad CA = O$$

It follows that

$$(B+C)A = BA + CA = O + O = O$$

Therefore $B + C \in S_3$ and hence S_3 is a subspace of $R^{2\times 2}$.

11 (a) $\mathbf{x} \in \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2)$ if and only if there exist scalars c_1 and c_2 such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{x}$$

Thus $\mathbf{x} \in \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2)$ if and only if the system $X\mathbf{c} = \mathbf{x}$ is consistent. To determine whether or not the system is consistent we can compute the row echelon form of the augmented matrix $(X\,|\,\mathbf{x})$.

$$\left(\begin{array}{rr|r} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

The system is inconsistent and therefore $\mathbf{x} \notin \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2)$.

(b)

$$\left(\begin{array}{rr|r} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{array}\right) \rightarrow \left(\begin{array}{rr|r} 1 & -3 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

The system is consistent and therefore $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$.

12. **(a)** Since the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ span V, any vector $\mathbf{v}$ in V can be written as a linear combination $\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$. If we add a vector $\mathbf{x}_{k+1}$ to our spanning set, then we can write $\mathbf{v}$ as a letter combination of the vectors in this augmented set since

$$\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + 0\mathbf{v}_{k+1}$$

So the new set of $k+1$ vectors will still be a spanning set.

(b) If one of the vectors, say $\mathbf{x}_k$, is deleted from the set then we may or may not end up with a spanning set. It depends on whether $\mathbf{x}_k$ is in $\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$. If $\mathbf{x}_k \notin \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}\}$ cannot be a spanning set. On the other hand if $\mathbf{x}_k \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$, then

$$\text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$$

and hence the $k-1$ vectors will span the entire vector space.

13. If $A = (a_{ij})$ is any element of $R^{2\times 2}$, then

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_{22} \end{pmatrix} \\ &= a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22} \end{aligned}$$

15. If $\{a_n\} \in S_0$, then $a_n \to 0$ as $n \to \infty$. If α is any scalar, then $\alpha a_n \to 0$ as $n \to \infty$ and hence $\{\alpha a_n\} \in S_0$. If $\{b_n\}$ is also an element of S_0, then $b_n \to 0$ as $n \to \infty$ and it follows that

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0 + 0 = 0$$

Therefore $\{a_n + b_n\} \in S_0$, and it follows that S_0 is a subspace of S.

16. Let $S \neq \{0\}$ be a subspace of R^1 and let $\mathbf{a}$ be an arbitrary element of R^1. If $\mathbf{s}$ is a nonzero element of S, then we can define a scalar α to be the real number a/s. Since S is a subspace it follows that

$$\alpha\mathbf{s} = \frac{a}{s}\mathbf{s} = \mathbf{a}$$

is an element of S. Therefore $S = R^1$.

17. (a) implies (b).

If $N(A) = \{\mathbf{0}\}$, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. By Theorem 1.4.2, A must be nonsingular.

(b) implies (c).

If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x} = A^{-1}\mathbf{b}$. Thus $A^{-1}\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$.

(c) implies (a).

If the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$, then in particular for $\mathbf{b} = \mathbf{0}$ the solution $\mathbf{x} = \mathbf{0}$ must be unique. Therefore $N(A) = \{\mathbf{0}\}$.

18. Let α be a scalar and let $\mathbf{x}$ and $\mathbf{y}$ be elements of $U \cap V$. The vectors $\mathbf{x}$ and $\mathbf{y}$ are elements of both U and V. Since U and V are subspaces it follows that

$$\alpha\mathbf{x} \in U \quad \text{and} \quad \mathbf{x} + \mathbf{y} \in U$$

$$\alpha\mathbf{x} \in V \quad \text{and} \quad \mathbf{x} + \mathbf{y} \in V$$

Therefore

$$\alpha\mathbf{x} \in U \cap V \quad \text{and} \quad \mathbf{x} + \mathbf{y} \in U \cap V$$

Thus $U \cap V$ is a subspace of W.

19. $S \cup T$ is not a subspace of R^2.

$$S \cup T = \{(s,t)^T \mid s = 0 \text{ or } t = 0\}$$

The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are both in $S \cup T$, however, $\mathbf{e}_1 + \mathbf{e}_2 \notin S \cup T$.

20. If $\mathbf{z} \in U + V$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in U$ and $\mathbf{v} \in V$. Since U and V are subspaces it follows that

$$\alpha\mathbf{u} \in U \quad \text{and} \quad \alpha\mathbf{v} \in V$$

for all scalars α. Thus

$$\alpha\mathbf{z} = \alpha\mathbf{u} + \alpha\mathbf{v}$$

is an element of $U + V$. If $\mathbf{z}_1$ and $\mathbf{z}_2$ are elements of $U + V$, then

$$\mathbf{z}_1 = \mathbf{u}_1 + \mathbf{v}_1 \quad \text{and} \quad \mathbf{z}_2 = \mathbf{u}_2 + \mathbf{v}_2$$

where $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$. Since U and V are subspaces it follows that

$$\mathbf{u}_1 + \mathbf{u}_2 \in U \quad \text{and} \quad \mathbf{v}_1 + \mathbf{v}_2 \in V$$

Thus

$$\mathbf{z}_1 + \mathbf{z}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

is an element of $U + V$. Therefore $U + V$ is a subspace of W.

21. **(a)** The distributive law does not work in general. For a counterexample, consider the vector space R^2. If we set $\mathbf{y} = \mathbf{e}_1 + \mathbf{e}_2$ and let

$$S = \text{Span}(\mathbf{e}_1), \quad T = \text{Span}(\mathbf{e}_2), \quad U = \text{Span}(\mathbf{y})$$

then

$$T + U = R^2, \quad S \cap T = \{\mathbf{0}\}, \quad S \cap U = \{\mathbf{0}\}$$

and hence

$$\begin{aligned} S \cap (T + U) &= S \cap R^2 = S \\ (S \cap T) + (S \cap U) &= \{\mathbf{0}\} + \{\mathbf{0}\} = \{\mathbf{0}\} \end{aligned}$$

(b) This distributive law also does not work in general. For a counterexample we can use the same subspaces S, T, and U of R^2 that were used in part (a). Since

$$T \cap U = \{\mathbf{0}\} \qquad \text{and} \qquad S + U = R^2$$

it follows that

$$\begin{aligned} S + (T \cap U) &= S + \{\mathbf{0}\} = S \\ (S + T) \cap (S + U) &= R^2 \cap R^2 = R^2 \end{aligned}$$

SECTION 3

5. (a) If $\mathbf{x}_{k+1} \in \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$, then the new set of vectors will be linearly dependent. To see this suppose that

$$\mathbf{x}_{k+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

If we set $c_{k+1} = -1$, then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1} = \mathbf{0}$$

with at least one of the coefficients, namely c_{k+1}, being nonzero. On the other hand if $\mathbf{x}_{k+1} \notin \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$ and

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_{k+1}\mathbf{x}_{k+1} = \mathbf{0}$$

then $c_{k+1} = 0$ (otherwise we could solve for $\mathbf{x}_{k+1}$ in terms of the other vectors). But then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k + c_k\mathbf{x}_k = \mathbf{0}$$

and it follows from the independence of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ that all of the c_i coefficients are zero and hence that $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$ are linearly independent. Thus if $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent and we add a vector $\mathbf{x}_{k+1}$ to the collection, then the new set of vectors will be linearly independent if and only if $\mathbf{x}_{k+1} \notin \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$

(b) Suppose that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent. To test whether or not $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}$ are linearly independent consider the equation

$$(1) \qquad c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1} = \mathbf{0}$$

If $c_1, c_2, \ldots, c_{k-1}$ work in equation (1), then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1} + 0\mathbf{x}_k = \mathbf{0}$$

and it follows from the independence of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ that

$$c_1 = c_2 = \cdots = c_{k-1} = 0$$

and hence $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}$ must be linearly independent.

7. (a) $W(\cos \pi x, \sin \pi x) = \pi$. Since the Wronskian is not identically zero the vectors are linearly independent.

(b) $W(x, e^x, e^{2x}) = 2(x-1)e^{3x} \not\equiv 0$

(c) $W(x^2, \ln(1+x^2), 1+x^2) = \dfrac{-8x^3}{(1+x^2)^2} \not\equiv 0$

(d) To see that x^3 and $|x|^3$ are linearly independent suppose

$$c_1x^3 + c_2|x|^3 \equiv 0$$

on $[-1,1]$. Setting $x=1$ and $x=-1$ we get

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned}$$

The only solution to this system is $c_1 = c_2 = 0$. Thus x^3 and $|x|^3$ are linearly independent.

8. The vectors are linearly dependent since

$$\cos x - 1 + 2\sin^2 \frac{x}{2} \equiv 0$$

on $[-\pi, \pi]$.

10. (a) If

$$c_1(2x) + c_2|x| = 0$$

for all x in $[-1,1]$, then in particular we have

$$\begin{aligned} -2c_1 + c_2 &= 0 \qquad (x=-1) \\ 2c_1 + c_2 &= 0 \qquad (x=1) \end{aligned}$$

and hence $c_1 = c_2 = 0$. Therefore $2x$ and $|x|$ are linearly independent in $C[-1,1]$.

(b) For all x in $[0,1]$

$$1 \cdot 2x + (-2)|x| = 0$$

Therefore $2x$ and $|x|$ are linearly dependent in $C[0,1]$.

11. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. If one of the vectors, say $\mathbf{v}_1$, is the zero vector then set

$$c_1 = 1, \quad c_2 = c_3 = \cdots = c_n = 0$$

Since

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

and $c_1 \neq 0$, it follows that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

12. If $\mathbf{v}_1 = \alpha\mathbf{v}_2$, then

$$1\mathbf{v}_1 - \alpha\mathbf{v}_2 = \mathbf{0}$$

and hence $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. Conversely, if $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly dependent, then there exists scalars c_1, c_2, not both zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

If say $c_1 \neq 0$, then

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\,\mathbf{v}_2$$

13. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a linearly independent set of vectors and suppose there is a subset, say $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of linearly dependent vectors. This would imply that there exist scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

but then

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

This contradicts the original assumption that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

14. If $\mathbf{x} \in N(A)$ then $A\mathbf{x} = \mathbf{0}$. Partitioning A into columns and $\mathbf{x}$ into rows and performing the block multiplication, we get

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2, \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Since $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly independent it follows that

$$x_1 = x_2 = \cdots = x_n = 0$$

Therefore $\mathbf{x} = \mathbf{0}$ and hence $N(A) = \{\mathbf{0}\}$.

15. If

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_k\mathbf{y}_k = \mathbf{0}$$

then

$$\begin{aligned} c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_kA\mathbf{x}_k &= \mathbf{0} \\ A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) &= \mathbf{0} \end{aligned}$$

Since A is nonsingular it follows that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}$$

and since $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent it follows that

$$c_1 = c_2 = \cdots = c_k = 0$$

Therefore $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are linearly independent.

16. Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V we can write

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

If we set $c_{n+1} = -1$ then $c_{n+1} \neq 0$ and

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{v} = \mathbf{0}$$

Thus $\mathbf{v}_1, \ldots, \mathbf{v}_n$, $\mathbf{v}$ are linearly dependent.

17. If $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ were a spanning set for V then we could write

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Setting $c_1 = -1$, we would have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

which would contradict the linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

SECTION 4

3. (a) Since

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$$

it follows that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent and hence form a basis for R^2.

(b) It follows from Theorem 3.4.1 that any set of more than two vectors in R^2 must be linearly dependent.

5. (a) Since

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix} = 0$$

it follows that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent.

(b) If $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$, then

$$\begin{aligned} 2c_1 + 3c_2 &= 0 \\ c_1 - c_2 &= 0 \\ 3c_1 + 4c_2 &= 0 \end{aligned}$$

and the only solution to this system is $c_1 = c_2 = 0$. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

8 (a) Since the dimension of R^3 is 3, it takes at least three vectors to span R^3. Therefore $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot possibly span R^3.

(b) The matrix X must be nonsingular or satisfy an equivalent condition such as $\det(X) \neq 0$.

(c) If $\mathbf{x}_3 = (a, b, c)^T$ and $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ then

$$\det(X) = \begin{vmatrix} 1 & 3 & a \\ 1 & -1 & b \\ 1 & 4 & c \end{vmatrix} = 5a - b - 4c$$

If one chooses a, b, and c so that

$$5a - b - 4c \neq 0$$

then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ will be a basis for R^3.

9. (a) If $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent then they span a 2-dimensional subspace of R^3. A 2-dimensional subspace of R^3 corresponds to a plane through the origin in 3-space.

(b) If $\mathbf{b} = A\mathbf{x}$ then

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

so $\mathbf{b}$ is in $\text{Span}(\mathbf{a}_1, \mathbf{a}_2)$ and hence the dimension of $\text{Span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b})$ is 2.

10. We must find a subset of three vectors that are linearly independent. Clearly $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, but

$$\mathbf{x}_3 = \mathbf{x}_2 - \mathbf{x}_1$$

so $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent. Consider next the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_4$. If $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4)$ then

$$\det(X) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 4 & 4 \end{vmatrix} = 0$$

so these three vectors are also linearly dependent. Finally if use $\mathbf{x}_5$ and form the matrix $X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_5)$ then

$$\det(X) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -2$$

so the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_5$ are linearly independent and hence form a basis for R^3.

16. $\dim U = 2$. The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for U.
$\dim V = 2$. The set $\{\mathbf{e}_2, \mathbf{e}_3\}$ is a basis for V.
$\dim U \cap V = 1$. The set $\{\mathbf{e}_2\}$ is a basis for $U \cap V$.
$\dim U + V = 3$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $U + V$.

17. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be a basis for U and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis for V. It follows from Theorem 3.4.1 that $\mathbf{u}_1, \mathbf{u}_2$, $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. Thus there exist scalars c_1, c_2, c_3, c_4 not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{v}_1 + c_4\mathbf{v}_2 = \mathbf{0}$$

Let

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = -c_3\mathbf{v}_1 - c_4\mathbf{v}_2$$

The vector $\mathbf{x}$ is an element of $U \cap V$. We claim $\mathbf{x} \neq \mathbf{0}$, for if $\mathbf{x} = \mathbf{0}$, then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0} = -c_3\mathbf{v}_1 - c_4\mathbf{v}_2$$

and by the linear independence of $\mathbf{u}_1$ and $\mathbf{u}_2$ and the linear independence of $\mathbf{v}_1$ and $\mathbf{v}_2$ we would have

$$c_1 = c_2 = c_3 = c_4 = 0$$

contradicting the definition of the c_i's.

18. Let U and V be subspaces of R^n with the property that $U \cap V = \{\mathbf{0}\}$. If either $U = \{\mathbf{0}\}$ or $V = \{\mathbf{0}\}$ the result is obvious, so assume that both subspaces are nontrivial with $\dim U = k > 0$ and $\dim V = r > 0$. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis for U and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis for V. The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_r$ span $U + V$. We claim that these vectors form a basis for $U + V$ and hence that $\dim U + \dim V = k + r$. To show this we must show that the vectors are linearly independent. Thus we must show that if

$$c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k + c_{k+1}\mathbf{v}_1 + \cdots + c_{k+r}\mathbf{v}_r = \mathbf{0} \tag{2}$$

then $c_1 = c_2 = \cdots = c_{k+r} = 0$. If we set

$$\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k \qquad \text{and} \qquad \mathbf{v} = c_{k+1}\mathbf{v}_1 + \cdots + c_{k+r}\mathbf{v}_r$$

then equation (2) becomes

$$\mathbf{u} + \mathbf{v} = \mathbf{0}$$

This implies $\mathbf{u} = -\mathbf{v}$ and hence that both $\mathbf{u}$ and $\mathbf{v}$ are in both in $U \cap V = \{\mathbf{0}\}$. Thus we have

$$\begin{aligned}\mathbf{u} &= c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k = \mathbf{0}\\ \mathbf{v} &= c_{k+1}\mathbf{v}_1 + \cdots + c_{k+r}\mathbf{v}_r = \mathbf{0}\end{aligned}$$

So, by the independence of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and the independence of $\mathbf{v}_1, \ldots, \mathbf{v}_r$ it follows that

$$c_1 = c_2 = \cdots = c_{k+r} = 0$$

SECTION 5

11. The transition matrix from E to F is $U^{-1}V$. To compute $U^{-1}V$, note that

$$U^{-1}(U \mid V) = (I \mid U^{-1}V)$$

and hence $(I \mid U^{-1}V)$ and $(U \mid V)$ are row equivalent. Thus $(I \mid U^{-1}V)$ is the reduced row echelon form of $(U \mid V)$.

SECTION 6

1. (a) The reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $(1, \ 0, \ 2)$ and $(0, \ 1, \ 0)$ form a basis for the row space. The first and second columns of the original matrix form a basis for the column space:

$$\mathbf{a}_1 = (1, \ 2, \ 4)^T \quad \text{and} \quad \mathbf{a}_2 = (3, \ 1, \ 7)^T$$

The reduced row echelon form involves one free variable and hence the nullspace will have dimension 1. Setting $x_3 = 1$, we get $x_1 = -2$ and $x_2 = 0$. Thus $(-2, \ 0, \ 1)^T$ is a basis for the nullspace.

(b) The reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Clearly then, the set

$$\{(1, \ 0, \ 0, \ -10/7), \ (0, \ 1, \ 0, \ -2/7), \ (0, \ 0, \ 1, \ 0)\}$$

is a basis for the row space. Since the reduced row echelon form of the matrix involves one free variable the nullspace will have dimension 1. Setting the free variable $\mathbf{x}_4 = 1$ we get

$$x_1 = 10/7, \quad x_2 = 2/7, \quad x_3 = 0$$

Thus $\{(10/7,\ 2/7,\ 0,\ 1)^T\}$ is a basis for the nullspace. The dimension of the column space equals the rank of the matrix which is 3. Thus the column space must be R^3 and we can take as our basis the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(c) The reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -0.65 \\ 0 & 1 & 0 & 1.05 \\ 0 & 0 & 1 & 0.75 \end{pmatrix}$$

The set $\{(1,\ 0,\ 0,\ -0.65), (0,\ 1,\ 0,\ 1.05), (0,\ 0,\ 1,\ 0,\ 0.75)\}$ is a basis for the row space. The set $\{(0.65,\ -1.05,\ -0.75,\ 1)^T\}$ is a basis for the nullspace. As in part (b) the column space is R^3 and we can take $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as our basis.

3 (b) The reduced row echelon form of A is given by

$$U = \begin{pmatrix} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The lead variables correspond to columns 1, 3, and 6. Thus $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_6$ form a basis for the column space of A. The remaining column vectors satisfy the following dependency relationships.

$$\begin{aligned} \mathbf{a}_2 &= 2\mathbf{a}_1 \\ \mathbf{a}_4 &= 5\mathbf{a}_1 - \mathbf{a}_3 \\ \mathbf{a}_5 &= -3\mathbf{a}_1 + 2\mathbf{a}_3 \end{aligned}$$

4. (c) consistent, (d) inconsistent, (f) consistent

6. There will be exactly one solution. The condition that $\mathbf{b}$ is in the column space of A guarantees that the system is consistent. If the column vectors are linearly independent, then there is at most one solution. Thus the two conditions together imply exactly one solution.

7. (a) Since $N(A) = \{\mathbf{0}\}$

$$A\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$, and hence $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. The column vectors cannot span R^m since there are only n vectors and $n < m$.

(b) If $\mathbf{b}$ is not in the column space of A, then the system must be inconsistent and hence there will be no solutions. If $\mathbf{b}$ is in the column space of A, then the system will be consistent, so there will be at least one solution. By part (a), the column vectors are linearly independent, so there cannot be more than one solution. Thus, if $\mathbf{b}$ is in the column space of A, then the system will have exactly one solution.

9. (a) If A and B are row equivalent, then they have the same row space and consequently the same rank. Since the dimension of the column space equals the rank it follows that the two column spaces will have the same dimension.

(b) If A and B are row equivalent, then they will have the same row space, however, their column spaces are in general not the same. For example if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then A and B are row equivalent but the column space of A is equal to $\text{Span}(\mathbf{e}_1)$ while the column space of B is $\text{Span}(\mathbf{e}_2)$.

10. The column vectors of A and U satisfy the same dependency relations. By inspection one can see that

$$\mathbf{u}_3 = 2\mathbf{u}_1 + \mathbf{u}_2 \qquad \text{and} \qquad \mathbf{u}_4 = \mathbf{u}_1 + 4\mathbf{u}_2$$

Therefore

$$\mathbf{a}_3 = 2\mathbf{a}_1 + \mathbf{a}_2 = \begin{pmatrix} -6 \\ 10 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 11 \\ 1 \end{pmatrix}$$

and

$$\mathbf{a}_4 = \mathbf{a}_1 + 4\mathbf{a}_2 = \begin{pmatrix} -3 \\ 5 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 16 \\ -12 \\ 28 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -7 \\ 30 \\ -3 \end{pmatrix}$$

11. If A is 5×8 with rank 5, then the column space of A will be R^5. So by the Consistency Theorem, the system $A\mathbf{x} = \mathbf{b}$ will be consistent for any $\mathbf{b}$ in R^5. Since A has 8 columns, its reduced row echelon form will involve 3 free variables. A consistent system with free variables must have infinitely many solutions.

12. If U is the reduced row echelon form of A then the given conditions imply that

$$\mathbf{u}_1 = \mathbf{e}_1,\ \mathbf{u}_2 = \mathbf{e}_2,\ \mathbf{u}_3 = \mathbf{u}_1 + 2\mathbf{u}_2,\ \mathbf{u}_4 = \mathbf{e}_3,\ \mathbf{u}_5 = 2\mathbf{u}_1 - \mathbf{u}_2 + 3\mathbf{u}_4$$

Therefore

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

13. **(a)** Since A is 5×3 with rank 3, its nullity is 0. Therefore $N(A) = \{\mathbf{0}\}$.

(b) If

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

then

$$\begin{aligned} c_1 A\mathbf{x}_1 + c_2 A\mathbf{x}_2 + c_3 A\mathbf{x}_3 &= \mathbf{0} \\ A(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3) &= \mathbf{0} \end{aligned}$$

and it follows that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$ is in $N(A)$. However, we know from part (a) that $N(A) = \{\mathbf{0}\}$. Therefore

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$$

Since $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent it follows that $c_1 = c_2 = c_3 = 0$ and hence $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are linearly independent.

(c) Since $\dim R^5 = 5$ it takes 5 linearly independent vectors to span the vector space. The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ do not span R^5 and hence cannot form a basis for R^5.

14. Given A is $m \times n$ with rank n and $\mathbf{y} = A\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. If $\mathbf{y} = \mathbf{0}$ then

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

But this would imply that the columns vectors of A are linearly dependent. Since A has rank n we know that its column vectors must be linearly independent. Therefore $\mathbf{y}$ cannot be equal to $\mathbf{0}$.

15. If the system $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the column space of A. Therefore the column space of $(A \mid \mathbf{b})$ will equal the column space of A. Since the rank of a matrix is equal to the dimension of the column space it follows that the rank of $(A \mid \mathbf{b})$ equals the rank of A.

Conversely if $(A \mid \mathbf{b})$ and A have the same rank, then $\mathbf{b}$ must be in the column space of A. If $\mathbf{b}$ were not in the column space of A, then the rank of $(A \mid \mathbf{b})$ would equal $\operatorname{rank}(A) + 1$.

16. (a) If $\mathbf{x} \in N(A)$, then

$$BA\mathbf{x} = B\mathbf{0} = \mathbf{0}$$

and hence $\mathbf{x} \in N(BA)$. Thus $N(A)$ is a subspace of $N(BA)$. On the other hand, if $\mathbf{x} \in N(BA)$, then

$$B(A\mathbf{x}) = BA\mathbf{x} = \mathbf{0}$$

and hence $A\mathbf{x} \in N(B)$. But $N(B) = \{\mathbf{0}\}$ since B is nonsingular. Therefore $A\mathbf{x} = \mathbf{0}$ and hence $\mathbf{x} \in N(A)$. Thus BA and A have the same nullspace. It follows from the Rank-Nullity Theorem that

$$\begin{aligned} \operatorname{rank}(A) &= n - \dim N(A) \\ &= n - \dim N(BA) \\ &= \operatorname{rank}(BA) \end{aligned}$$

(b) By part (a), left multiplication by a nonsingular matrix does not alter the rank. Thus

$$\begin{aligned} \operatorname{rank}(A) = \operatorname{rank}(A^T) &= \operatorname{rank}(C^T A^T) \\ &= \operatorname{rank}((AC)^T) \\ &= \operatorname{rank}(AC) \end{aligned}$$

17. Corollary 3.6.4. An $n \times n$ matrix A is nonsingular if and only if the column vectors of A form a basis for R^n.

Proof: It follows from Theorem 3.6.3 that the column vectors of A form a basis for R^n if and only if for each $\mathbf{b} \in R^n$ the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. We claim $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in R^n$ if and only if A is nonsingular. If A is nonsingular then $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution to $A\mathbf{x} = \mathbf{b}$. Conversely, if for each $\mathbf{b} \in R^n$, $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$. Thus it follows from Theorem 1.4.2 that A is nonsingular.

18. If $N(A - B) = R^n$ then the nullity of $A - B$ is n and consequently the rank of $A - B$ must be 0. Therefore

$$\begin{aligned} A - B &= O \\ A &= B \end{aligned}$$

19. (a) The column space of B will be a subspace of $N(A)$ if and only if

$$A\mathbf{b}_j = \mathbf{0} \quad \text{for} \quad j = 1, \ldots, n$$

However, the jth column of AB is

$$AB\mathbf{e}_j = A\mathbf{b}_j, \qquad j = 1, \ldots, n$$

Thus the column space of B will be a subspace of $N(A)$ if and only if all the column vectors of AB are $\mathbf{0}$ or equivalently $AB = O$.

(b) Suppose that A has rank r and B has rank k and $AB = O$. By part (a) the column space of B is a subspace of $N(A)$. Since $N(A)$ has dimension $n - r$, it follows that the dimension of the column space of B must be less than or equal to $n - r$. Therefore

$$\operatorname{rank}(A) + \operatorname{rank}(B) = r + k \le r + (n - r) = n$$

20. Let $\mathbf{x}_0$ be a particular solution to $A\mathbf{x} = \mathbf{b}$. If $\mathbf{y} = \mathbf{x}_0 + \mathbf{z}$, where $\mathbf{z} \in N(A)$, then

$$A\mathbf{y} = A\mathbf{x}_0 + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

and hence $\mathbf{y}$ is also a solution.

Conversely, if $\mathbf{x}_0$ and $\mathbf{y}$ are both solutions to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{z} = \mathbf{y} - \mathbf{x}_0$, then

$$A\mathbf{z} = A\mathbf{y} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

and hence $\mathbf{z} \in N(A)$.

21. (a) Since

$$A = \mathbf{x}\mathbf{y}^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \mathbf{y}^T = \begin{pmatrix} x_1\mathbf{y}^T \\ x_2\mathbf{y}^T \\ \vdots \\ x_m\mathbf{y}^T \end{pmatrix}$$

the rows of A are all multiples of $\mathbf{y}^T$. Thus $\{\mathbf{y}^T\}$ is a basis for the row space of A. Since

$$A = \mathbf{x}\mathbf{y}^T = \mathbf{x}(y_1, y_2, \ldots, y_n)$$

$$= (y_1\mathbf{x}, y_2\mathbf{x}, \dots, y_n\mathbf{x})$$

it follows that the columns of A are all multiples of $\mathbf{x}$ and hence $\{\mathbf{x}\}$ is a basis for the column space of A.

(b) Since A has rank 1, the nullity of A is $n-1$.

22. (a) If $\mathbf{c}$ is a vector in the column space of C, then

$$\mathbf{c} = AB\mathbf{x}$$

for some $\mathbf{x} \in R^r$. Let $\mathbf{y} = B\mathbf{x}$. Since $\mathbf{c} = A\mathbf{y}$, it follows that $\mathbf{c}$ is in the column space of A and hence the column space of C is a subspace of the column space of A.

(b) If $\mathbf{c}^T$ is a row vector of C, then $\mathbf{c}$ is in the column space of C^T. But $C^T = B^TA^T$. Thus, by part (a), $\mathbf{c}$ must be in the column space of B^T and hence $\mathbf{c}^T$ must be in the row space of B.

(c) It follows from part (a) that $\text{rank}(C) \leq \text{rank}(A)$ and it follows from part (b) that $\text{rank}(C) \leq \text{rank}(B)$. Therefore

$$\text{rank}(C) \leq \min\{\text{rank}(A), \text{rank}(B)\}$$

23 (a) In general a matrix E will have linearly independent column vectors if and only if $E\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. To show that C has linearly independent column vectors we will show that $C\mathbf{x} \neq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$ and hence that $C\mathbf{x} = \mathbf{0}$ has only the trivial solution. Let $\mathbf{x}$ be any nonzero vector in R^r and let $\mathbf{y} = B\mathbf{x}$. Since B has linearly independent column vectors it follows that $\mathbf{y} \neq \mathbf{0}$. Similarly since A has linearly independent column vectors, $A\mathbf{y} \neq \mathbf{0}$. Thus

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{y} \neq \mathbf{0}$$

(b) If A and B both have linearly independent row vectors, then B^T and A^T both have linearly independent column vectors. Since $C^T = B^TA^T$, it follows from part (a) that the column vectors of C^T are linearly independent, and hence the row vectors of C must be linearly independent.

24. (a) If the column vectors of B are linearly dependent then $B\mathbf{x} = \mathbf{0}$ for some nonzero vector $\mathbf{x} \in R^r$. Thus

$$C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$$

and hence the column vectors of C must be linearly dependent.

(b) If the row vectors of A are linearly dependent then the column vectors of A^T must be linearly dependent. Since $C^T = B^TA^T$, it follows from part (a) that the column vectors of C^T must be linearly dependent. If the column vectors of C^T are linearly dependent, then the row vectors of C must be linearly dependent.

25. (a) Let C denote the right inverse of A and let $\mathbf{b} \in R^m$. If we set $\mathbf{x} = C\mathbf{b}$ then

$$A\mathbf{x} = AC\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$$

Thus if A has a right inverse then $A\mathbf{x} = \mathbf{b}$ will be consistent for each $\mathbf{b} \in R^m$ and consequently the column vectors of A will span R^m.

(b) No set of less than m vectors can span R^m. Thus if $n < m$, then the column vectors of A cannot span R^m and consequently A cannot have a right inverse. If $n \geq m$ then a right inverse is possible.

27. Let B be an $n \times m$ matrix. Since

$$DB = I_m$$

if and only if

$$B^T D^T = I_m^T = I_m$$

it follows that D is a left inverse for B if and only if D^T is a right inverse for B^T.

28. If the column vectors of B are linearly independent, then the row vectors of B^T are linearly independent. Thus B^T has rank m and consequently the column space of B^T is R^m. By Exercise 26, B^T has a right inverse and consequently B must have a left inverse.

29. Let B be an $n \times m$ matrix. If B has a left inverse, then B^T has a right inverse. It follows from Exercise 25 that the column vectors of B^T span R^m. Thus the rank of B^T is m. The rank of B must also be m and consequently the column vectors of B must be linearly independent.

30. Let $\mathbf{u}(1,:), \mathbf{u}(2,:), \ldots, \mathbf{u}(k,:)$ be the nonzero row vectors of U. If

$$c_1\mathbf{u}(1,:) + c_2\mathbf{u}(2,:) + \cdots + c_k\mathbf{u}(k,:) = \mathbf{0}^T$$

then we claim

$$c_1 = c_2 = \cdots = c_k = 0$$

This is true since the leading nonzero entry in $\mathbf{u}(i,:)$ is the only nonzero entry in its column. Let us refer to the column containing the leading nonzero entry of $\mathbf{u}(i,:)$ as $j(i)$. Thus if

$$\mathbf{y}^T = c_1\mathbf{u}(1,:) + c_2\mathbf{u}(2,:) + \cdots + c_k\mathbf{u}(k,:) = \mathbf{0}^T$$

then

$$0 = y_{j(i)} = c_i, \qquad i = 1, \ldots, k$$

and it follows that the nonzero row vectors of U are linearly independent.

MATLAB EXERCISES

1. (a) The column vectors of U will be linearly independent if and only if the rank of U is 4.

(d) The matrices S and T should be inverses.

2. (a) Since

$$r = \text{dim of row space} \leq m$$

and

$$r = \text{dim of column space} \leq n$$

it follows that

$$r \leq \min(m, n)$$

(c) All the rows of A are multiples of $\mathbf{y}^T$ and all of the columns of A are multiples of $\mathbf{x}$. Thus the rank of A is 1.

(d) Since X and Y^T were generated randomly, both should have rank 2 and consequently we would expect that their product should also have rank 2.

3. (a) The column space of C is a subspace of the column space of B. Thus A and B must have the same column space and hence the same rank. Therefore we would expect the rank of A to be 4.

(b) The first four columns of A should be linearly independent and hence should form a basis for the column space of A. The first four columns of the reduced row echelon form of A should be the same as the first four columns of the 8×8 identity matrix. Since the rank is 4, the last four rows should consist entirely of 0's.

(c) If U is the reduced row echelon form of B, then $U = MB$ where M is a product of elementary matrices. If B is an $n \times n$ matrix of rank n, then $U = I$ and $M = B^{-1}$. In this case it follows that the reduced row echelon form of $(B \quad BX)$ will be

$$B^{-1}(B \quad BX) = (I \quad X)$$

If B is $m \times n$ of rank n and $n < m$, then its reduced row echelon form is given by

$$U = MB = \begin{pmatrix} I \\ O \end{pmatrix}$$

It follows that the reduced row echelon form of $(B \quad BX)$ will be

$$MB(I \quad X) = \begin{pmatrix} I \\ O \end{pmatrix}(I \quad X) = \begin{pmatrix} I & X \\ O & O \end{pmatrix}$$

4. (d) The vectors $C\mathbf{y}$ and $\mathbf{b} + c\mathbf{u}$ are equal since

$$C\mathbf{y} = (A + \mathbf{u}\mathbf{v}^T)\mathbf{y} = A\mathbf{y} + c\mathbf{u} = \mathbf{b} + c\mathbf{u}$$

The vectors $C\mathbf{z}$ and $(1 + d)\mathbf{u}$ are equal since

$$C\mathbf{z} = (A + \mathbf{u}\mathbf{v}^T)\mathbf{z} = A\mathbf{z} + d\mathbf{u} = \mathbf{u} + d\mathbf{u}$$

It follows that

$$C\mathbf{x} = C(\mathbf{y} - e\mathbf{z}) = \mathbf{b} + c\mathbf{u} - e(1 + d)\mathbf{u} = \mathbf{b}$$

The rank one update method will fail if $d = -1$. In this case

$$C\mathbf{z} = (1 + d)\mathbf{u} = \mathbf{0}$$

Since $\mathbf{z}$ is nonzero, the matrix C must be singular.

CHAPTER TEST A

1. The statement is true. If S is a subspace of a vector space V, then it is nonempty and it is closed under the operations of V. To show that S, with the operations of addition and scalar multiplication from V, forms a vector space we must show that the eight vector space axioms are satisfied. Since S is closed under scalar multiplication, it follows from Theorem 3.1.1 that if $\mathbf{x}$ is any vector in S, then $\mathbf{0} = 0\mathbf{x}$ is a vector in S and $-1\mathbf{x}$ is the additive inverse of $\mathbf{x}$. So axioms A3 and A4 are satisfied. The remaining six axioms hold for all vectors in V and hence hold for all vectors in S. Thus S is a vector space.
2. The statement is false. The elements of R^3 are 3×1 matrices. Vectors that are in R^2 cannot be in vectors in R^3 since they are only 2×1 matrices.
3. The statement is false. A two dimensional subspace of R^3 corresponds to a plane through the origin in 3-space. If S and T are two different two dimensional subspaces of R^3 then both correspond to planes through the origin and their intersection must correspond to a line through the origin. Thus the intersection cannot consist of just the zero vector.
4. The statement is false in general. See the solution to Exercise 19 of Section 2.
5. The statement is true. See the solution to Exercise 18 of Section 2.
6. The statement is true. See Theorem 3.4.3.
7. The statement is false in general. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ span a vector space V of dimension $k < n$, then they will be linearly dependent since there are more vectors than the dimension of the vector space. For example,

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are vectors that span R^2, but are not linearly independent. Since the dimension of R^2 is 2, any set of more than 2 vectors in R^2 must be linearly dependent.
8. The statement is true. If

$$\operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$$

then $\mathbf{x}_k$ must be in $\operatorname{Span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1})$. So $\mathbf{x}_k$ can be written as a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{k-1}$ and hence there is a dependency relation among the vectors. Specifically if

$$\mathbf{x}_k = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1}$$

then we have the dependency relation

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{k-1}\mathbf{x}_{k-1} - 1\mathbf{x}_k = \mathbf{0}$$
9. The statement is true. The rank of A is the dimension of the row space of A. The rank of A^T is the dimension of the row space of A^T. The independent rows of A^T correspond to the independent columns of A. So the rank of A^T equals the dimension of the column space of A. But the row space and

column space of A have the same dimension (Theorem 3.6.5). So A and A^T must have the same rank.

10. If $m \neq n$ then the statement is false since

$$\dim N(A) = n - r \qquad \text{and} \qquad \dim N(A^T) = m - r$$

where r is the rank of A.

CHAPTER TEST B

1. The vectors are linearly dependent since

$$0\mathbf{x}_1 + 0\mathbf{x}_2 + 1\mathbf{x}_3 = 0\mathbf{x}_1 + 0\mathbf{x}_2 + 1\mathbf{0} = \mathbf{0}$$

2. (a) S_1 consists of all vectors of the form

$$\mathbf{x} = \begin{pmatrix} -a \\ a \end{pmatrix}$$

so if

$$\mathbf{x} = \begin{pmatrix} -a \\ a \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} -b \\ b \end{pmatrix}$$

are arbitrary vectors in S_1 and c is any scalar then

$$c\mathbf{x} = \begin{pmatrix} -ca \\ ca \end{pmatrix} \in S_1$$

and

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} -a \\ a \end{pmatrix} + \begin{pmatrix} -b \\ b \end{pmatrix} = \begin{pmatrix} -a-b \\ a+b \end{pmatrix} \in S_1$$

Since S_1 is nonempty and closed under the operations of scalar multiplication and vector addition, it follow that S_1 is a subspace of R^2.

(b) S_2 is not a subspace of R^2 since it is not closed under addition. The vectors

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are both in S_2, however,

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is not in S_2.

3. (a)

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The free variables are x_2, x_4, and x_5. If we set $x_2 = a$, $x_4 = b$, and $x_5 = c$, then

$$x_1 = -3a - 2b - 3c \qquad \text{and} \qquad x_3 = -b - c$$

Thus $N(A)$ consists of all vectors of the form

$$\mathbf{x} = \begin{pmatrix} -3a-2b-3c \\ a \\ -b-c \\ b \\ c \end{pmatrix} = a\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

The vectors

$$\mathbf{x}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for $N(A)$.

(b) The lead 1's occur in the first and third columns of the echelon form. Therefore

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

form a basis for the column space of A.

4. The columns of the matrix that correspond to the lead variables are linearly independent and span the column space of the matrix. So the dimension of the column space is equal to the number of lead variables in any row echelon form of the matrix. If there are r lead variables then there are $n-r$ free variables. By the Rank-Nullity Theorem the dimension of the nullspace is $n-r$. So the dimension of the nullspace is equal to the number of free variables in any echelon form of the matrix.

5. (a) One dimensional subspaces correspond to lines through the origin in 3-space. If the first subspace U_1 is the span of a vector $\mathbf{u}_1$ and the second subspace U_2 is the span of a vector $\mathbf{u}_2$ and the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, then the two lines will only intersect at the origin and consequently we will have $U_1 \cap U_2 = \{\mathbf{0}\}$.

(b) Two dimensional subspaces correspond to planes through the origin in 3-space. Any two distinct planes through the origin will intersect in a line. So $V_1 \cap V_2$ must contain infinitely many vectors.

6. (a) If

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$$

are arbitrary symmetric matrices and α is any scalar, then

$$\alpha A = \begin{pmatrix} \alpha a & \alpha b \\ \alpha b & \alpha c \end{pmatrix} \qquad \text{and} \qquad A + B = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$$

are both symmetric. Therefore S is closed under the operations of scalar multiplication and vector addition and hence S is a subspace of $R^{2\times 2}$.

(b) The vectors

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are linearly independent and they span S. Therefore they form a basis for S.

7. (a) If A is 6×4 with rank 4, then by the Rank-Nullity Theorem $\dim N(A) = 0$ and consequently $N(A) = \{\mathbf{0}\}$.

(b) The column vectors of A are linearly independent since the rank of A is 4, however, they do not span R^6 since you need 6 linearly independent vectors to spanR^6.

(c) By the Consistency Theorem if $\mathbf{b}$ is in the column space of A then the system is consistent. The condition that the column vectors of A are linearly independent implies that there cannot be more than 1 solution. Therefore there must be exactly 1 solution.

8. (a) The dimension of R^3 is 3, so any collection of more than 3 vectors must be linearly dependent.

(b) Since $\dim R^3 = 3$, it takes 3 linearly independent vectors to span R^3. No 2 vectors can span, so $\mathbf{x}_1$ and $\mathbf{x}_2$ do not span R^3.

(c) The matrix

$$X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{pmatrix}$$

only has 2 linearly independent row vectors, so the dimension of the rowspace and dimension of the column space both must be equal to 2. Therefore $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly dependent and only span a 2-dimensional subspace of R^3. The vectors to not form a basis for R^3 since they are linearly dependent.

(d) If we set $A = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4)$, then

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1$$

Therefore $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent. Since $\dim R^3 = 3$, the three vectors will span and form a basis for R^3.

9. If

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \mathbf{0}$$

then

$$c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + c_3A\mathbf{x}_3 = A\mathbf{0} = \mathbf{0}$$

Multiplying through by A^{-1} we get

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$$

Since $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are linearly independent, it follows that $c_1 = c_2 = c_3 = 0$. Therefore $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are linearly independent.

10. (a) The rank of A equals the dimension of the column space of A which is 3. By the Rank-Nullity Theorem, $\dim N(A) = 5 - 3 = 2$.

(b) Since $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are linearly independent, the first three columns of the reduced row echelon form U will be

$$\mathbf{u}_1 = \mathbf{e}_1, \quad \mathbf{u}_2 = \mathbf{e}_2, \quad \mathbf{u}_3 = \mathbf{e}_3$$

The remaining columns of U satisfy the same dependency relations that the column vectors of A satisfy. Therefore

$$\begin{aligned} \mathbf{u}_4 &= \mathbf{u}_1 + 3\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{u}_5 &= 2\mathbf{u}_1 - \mathbf{u}_3 = 2\mathbf{e}_1 - \mathbf{e}_3 \end{aligned}$$

and it follows that

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

11. (a) If $U = (\mathbf{u}_1, \mathbf{u}_2)$, then the transition matrix corresponding to a change of basis from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is

$$U^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$$

(b) Let $V = (\mathbf{v}_1, \mathbf{v}_2)$. If $\mathbf{x} = V\mathbf{d} = U\mathbf{c}$ then $\mathbf{c} = U^{-1}V\mathbf{d}$ and hence the transition matrix corresponding to a change of basis from $[\mathbf{v}_1, \mathbf{v}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$ is

$$U^{-1}V = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix} = \begin{pmatrix} 31 & 10 \\ -13 & -3 \end{pmatrix}$$